The Notion of Internal Dynamics and Its Impact on Modeling and Controlling Supply Chains With Goods Characterized by Uncertain Perishability Rate

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***Abstract*—For a deeper insight on the inventory control problem of periodically reviewed perishable supply chains (SCs) with perishable goods, we here introduce the notion of internal dynamics (ID) and propose a min-max model predictive control (MPC) of the inventory level based on this notion. The acronym ID denotes the actual scheduling of the usual management operations (OPs) executed inside each review period. We show that the information carried by the ID is fundamental to define an efficient inventory control policy (ICP). The notion of ID is particularly important in the case of SCs characterized by elements of complexity like: 1) perishable goods with large, not exactly known deterioration rate; 2) the length of the review period is such as to produce a high spoilage within it; 3) an uncertain future customer demand with oscillations that are not statistically modelable. Exploiting the notion of ID, we define an inventory model properly capturing these features. This model encompasses as particular cases other SC models proposed in the literature. We also define a coherent two criteria-based evaluation of the bullwhip effect (BE). We also provide key managerial insights for a proper scheduling of all the OPs inside each review period. The reported numerical simulations show the effectiveness of our approach.**

***Index Terms*—Inventory control, min-max model predictive control (MPC), perishable goods, supply chain (SC) management.**

I. Introduction

A. Perishable SC Context

It is well known that product perishability has a direct impact on many elements that characterize the dynamics of a supply chain (SC). These elements are: quantity of sales, inventory level, product availability, logistics, storage, and deterioration costs [1]. In turn these elements have a great influence on the SC profitability. This explains the great interest of both managers and academic researchers to understand, model and efficiently control the dynamics of perishable SCs from different points of view.

Among the topics considered in recent contributions we mention: the waste reduction problem [2], [3], defining a reliable decision support [4], [5], demand forecasting [6], [7], economic sustainability, logistic, and production management, [8], [9], [10], [11], [12].

Comprehensive reviews on the above topics can be found on the thorough surveys [13], [14].

Two different types of perishable products are recognized in the literature: 1) fixed life and 2) age-dependent. The former has a specified expiry date, beyond which they are no longer valid (e.g., pharmaceutical products). The latter loses value over to time according to a given decay factor (e.g., fruits and vegetables) and, in general, their expiry date is not predetermined.

In this article we focus on the inventory control problem for SCs affected by this second type of perishability. Accurate reviews on analysis of inventory models for perishable products can be found on the thorough surveys [1], [15], [16].

In general, developing an efficient ICP is not an easy task because opposite control requirements should be conciliated: minimize overstocking and maximize the satisfied customer demand.

In this regard, a large amount of research has been directed toward the application of optimal control methods based on the min-max MPC approach.

The main appealing features of MPC that strongly suggest its use are the capabilities of handling hardly constrained optimization problems and correcting the actual control action as a consequence of the receding horizon implementation [17].

Comprehensive surveys of the very numerous MPC-based techniques for SC control can be found in [18], [19], and [20].

The presence of perishable goods further complicates the problem. MPC of perishable SCs, has been considered in [21], [22], [23], [24], and [25]. Outside the MPC, alternative approaches have been proposed both in the discrete [26], [27], [28], [29], [30], [31], [32], [33] and continuous time domain [34], [35], [36], [37], [38].

All the aforementioned papers dealing with inventory control assume an exactly known deterioration rate. Unfortunately, this simplistic assumption is not satisfied in the most cases due to unstable and variable storage conditions [1]. The

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more realistic assumption of an inaccurate knowledge of this parameter has been considered in [39], [40], and [41] under different assumptions on the distribution of the uncertainty affecting deterioration factor.

Application of MPC to periodically reviewed perishable SCs with interval uncertainties on deterioration rate and on the future customer demand has been recently considered in [42], [43], [44], and [45]: using a min-max MPC approach, the inaccuracy on the deterioration rate is already considered in the controller design phase so as to prevent any significant loss of inventory management performance.

B. Our Contribution

The purpose of this article is to develop an effective ICP for perishable periodically reviewed SCs with uncertain deterioration rate and a given internal dynamic (ID). The notions of ID and effective ICP are defined beneath.

Definition of ID: given a periodically reviewed SC, the acronym ID shortly denotes the scheduled sequence of the usual OPs that take place inside each review period: OP1) updating the inventory value, OP2) receiving goods from manufacturer, OP3) shipping goods to the customer, and OP4) placing a replenishment order.

Definition of Effective ICP: We define effective an ICP that realizes a satisfactory tradeoff between the following conflicting control specifications (CSs): CS1) maximize the customer service level, CS2) avoid overstocking, and CS3) attenuate the BE.

In the literature on periodically reviewed SCs, OP1–OP4 are usually considered to be synchronized at the beginning of each review period (see the above mentioned papers and references therein). Nevertheless, in many practical situations, this simplifying hypothesis cannot be satisfied owing to organizational constraints that make it impossible for the manager a free scheduling of OP1–OP4. This is especially true for OP2 and OP3 that imply movements of goods. We refer to these situations with the acronym unmanageable ID (UID). The contrary case of a possible free scheduling is referred to as manageable ID (MID).

If the actual ID is not properly taken into account when modeling the inventory dynamics, an inadequate ICP will be obtained and a serious performance degradation of the SC performance will occur. This phenomenon is particularly accentuated in the case of perishable SC whose uncertain deterioration rate is large enough to yield a significant spoilage inside each review period. Neglecting the actual configuration of the ID for this class of systems would result in an insufficient customer service.

The purpose of this article is to fill this research gap.

On the basis of the above considerations and cited literature, we here deal with the following still open two-fold problem:

PR1) for any given UID, compute a corresponding effective ICP and

PR2) in the case of perishable SCs with MID, what is the ID that needs to be realized so that the relative ICP reduces both spoilage and BE, better than the ICPs corresponding to the other possible IDs?

Solving PR1 guarantees the effectiveness of the ICP independently of the “a priori” given UID. Solving PR2, automatically steers the managerial decision-making toward the ID that allows a further improvement of performance. Customer demand variability is another source of uncertainty. As in [42], [43], [44], and [45], here we deal with this problem in the robust modeling framework assuming that at any time instant and over a limited prediction horizon, the future customer demand is arbitrarily time varying inside a given compact set. This approach, alternative to statistical forecasting methods, has the following theoretical and practical motivations, (TM) and (PM) as follows.

1) TM, forecasting methods based on time series analysis are not able to adequately describe the turbulent statistical phenomena underlying the most part the demand generation process [46]. As a consequence they are potentially fragile with respect to rare or unanticipated factors [47].
2) PM, statistical methods are computationally expensive [48], [49], [50]. Their translation in a robust deterministic form can be handled in a numerically more efficient way.

Hence, we deal with PR1 considering a periodically reviewed SC characterized by a given UID, uncertain large perishability rate, and uncertain future customer demand.

As a first step we define a general dynamic inventory model capturing the assumed complexities and encompassing, as particular cases, several models proposed in the literature. Taking into account the information carried by the ID allows us to obtain a model that more carefully reflects the real dynamics of the SC and, as a consequence, to define a more effective ICP conciliating CS1–CS3. From this more general model we also derive a two-evaluation criteria (EC) of the well known BE [51]: 1) EC1) smoothness of the issued replenishment orders and 2) EC2) bounds on the interval over which the replenishment orders take values.

In accordance with EC1 we define a cost functional adaptively penalizing excessive differences between two consecutive orders. This is equivalent to take into account the smoothness requirement through a discrete version of the Lipschitz continuity condition. As better specified in Section IV, EC2 is directly involved in the solution of PR2.

The actual computation of the ICP conciliating CS1–CS3 is here set in the same general theoretical framework used in [42], [43], [44], and [45] where the concept of ID is not defined. The framework consists in a min-max MPC whose solution is parametrized using the universal optimal approximators given by polynomial B-splines, namely uniformly bounded functions parametrized as the convex combination of their control points. The min-max formulation of the MPC allows us to deal with the uncertainties on the perishable rate and on the future customer demand. The B-splines parametrization allows us: 1) to reformulate the min-max MPC problem as a parametric estimation problem and 2) to easily impose hard constraints on the replenishment order.

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TABLE I
ACRONYMS

BE	Bullwhip Effect	LS	Least Squares	RMPC	Robust MPC
CS	Control Specifications	MID	Manageable ID	SC	Supply Chain
EC	Evaluation Criteria	MPC	Model Predictive Control	TM	Theoretical Motivations
ICP	Inventory Control Policy	OP	Operation	UCD	Unsatisfied Customer Demand
ID	Internal Dynamics	PM	Practical Motivations	UID	Unmanageable ID

Compared to the latter references, we here provide more theoretical and implementation insights, i.e.,: more accurate description and prediction of the inventory dynamics, the definition of a new quadratic cost functional and new constraints on the control effort that take into account the knowledge of ID. Finally we show that, in the case of SCs with MID the best scheduling is to synchronize OP2 and OP3 with OP1 at the beginning of each review period. As OP4 does not concern an actual movement of goods it is required to be performed at any time after OP3.

This article is organized in the following way. The system model is described in Section II. The min-max MPC problem solving PR1 is stated in Section III: in Section III-A, we provide the general formulation; in Section III-B, we reformulate the MPC problem in a B-splines parametrized form that is more suitable for numerical solution; in Section III-C, we show that the parametrized MPC problem can be solved as a robust least squares (LSs) problem; and in Section III-D, we show how to compute the hard constraints on the replenishment policy. Section IV answers PR2. Numerical results on simulated and real data are reported on Section V. Concluding remarks and managerial insights are reported in Section VI. Acronyms and notation are summarized in Tables I and II, respectively.

II. Dynamic Uncertain Model

A. Stock Balance Equation

For ease of exposition (but without any loss of generality) we refer to a single-stage SC given by the series connection of a retailer with a manufacturer. This latter is modeled as a pure delay time.

Let $R_k \triangleq [kT, (k+1)T)$, $k \in Z^+$, denote the kth review period of length T. Let R'_k be an arbitrary subinterval of R_k of length T' such that $T = nT'$ for some integer $n > 0$.

Inside each R_k the retailer performs the OPs mentioned in the introduction.

We assume as follows.

A1) OP1 takes place at the beginning of R_k, OP2 and OP3 are performed inside R_k and may not be simultaneous, OP2 is not subsequent to OP3, and OP4 is executed final.

A2) any fixed UID satisfying A1) is exactly repeated over each R_k.

A3) each non null replenishment order issued by the retailer in R_k is realized inside R_{k+L}, for some integer $L > 0$.

A4) the goods arrive at the retailer new and deteriorate while kept in stock.

A5) over each R'_k, the percentage of spoiled stocked goods is uncertain and can take any value inside any arbitrary subset of the maximal interval (0%, 100%).

We model this situation defining the uncertain perishability rate $\alpha_{T'} \in [\alpha^-_{T'}, \alpha^+_{T'}] \subset (0, 1)$, $k \in Z^+$ for some a priori known $\alpha^-_{T'}, \alpha^+_{T'}$. Hence, over each R'_k the perishable goods is subject to the uncertain decay factor $\rho_{T'} \triangleq 1 - \alpha_{T'} \in [\rho^-_{T'}, \rho^+_{T'}] = [1 - \alpha^+_{T'}, 1 - \alpha^-_{T'}] \subset (0, 1)$. The above considerations imply that the stock level dynamics is described by the following inventory model

$$y(k+1) = \rho_l\big(\rho_y y(k) + \rho_u u(k-L) - h(k)\big) \tag{1}$$

where:

1) the explicit dependence on T has been omitted for simplicity;
2) $y(k)$ is the on hand stock level, i.e., the amount of remaining undamaged goods available at the beginning of R_k;
3) $u(k-L)$ is the replenishment order placed inside R_{k-L} and realized within R_k at time $kT + n_1T' < (k+1)T$ for some integer n_1 such that $0 \le n_1 < n$;
4) ρ_l, ρ_y, and ρ_u are the uncertain decay factors over the subintervals $R_{k,l}$, $R_{k,y}$ and $R_{k,u}$ of R_k of length $T_l = n_lT'$, $T_y = n_yT'$ and $T_u = n_uT'$, respectively. These intervals represent the time over which the corresponding quantity of goods are actually stocked in the warehouse inside each R_k (for details see point 6));
5) $h(k)$ is the fulfilled customer demand and, by A1), it is given by
$$h(k) = \min\big(\rho_y y(k) + \rho_u u(k-L), d(k)\big) \in [0, d(k)] \tag{2}$$
where the quantity
$$\rho_y y(k) + \rho_u u(k-L) \triangleq y_{av}(k) \tag{3}$$
is the actual amount of goods available for sale when OP3 is executed, while the quantity
$$\rho_y y(k) + \rho_u u(k-L) - h(k) \triangleq y_{\text{left}}(k) \tag{4}$$
is the amount of goods left in the stock inside R_k, just after OP3 has been performed. Also $y_{\text{left}}(k)$ is subject to a decaying factor ρ_l over the remaining subinterval $R_{k,l}$ of length $T_l = n_lT'$; and
6) in accordance with A5), the uncertain decay factors ρ_l, ρ_y, and ρ_u are defined as
$$\rho_l \triangleq \rho_{T'}^{n_l} \in \big[(\rho^-_{T'})^{n_l}, (\rho^+_{T'})^{n_l}\big] \subset (0, 1) \tag{5}$$
$$\rho_y \triangleq \rho_{T'}^{n_y} \in \big[(\rho^-_{T'})^{n_y}, (\rho^+_{T'})^{n_y}\big] \subset (0, 1) \tag{6}$$
$$\rho_u \triangleq \rho_{T'}^{n_u} \in \big[(\rho^-_{T'})^{n_u}, (\rho^+_{T'})^{n_u}\big] \subset (0, 1) \tag{7}$$
for some positive integers n_l, n_y, and n_u. The values of n_l, n_y, and n_u depend on the particular ID. A

TABLE II
NOTATION

$A_{k,i}$, $b_{k,i}$	known terms in the formulation of the predicted tracking error
$\underline{A}_k$	extended matrix containing the weighted terms $A_{k,i}$
$\underline{b}_k$	extended vector containing the weighted terms $b_{k,i}$
$B_d(j)$	time varying coefficients of the B-spline
c_k	vector of control points defining the B-spline over H_k
d	degree of the B-spline
$d(k)$	customer demand
$\mathcal{D}_k$	compact set containing $d(k)$ over P_k
$d^-(k+\ell)$, $d^+(k+\ell)$	lower and upper trajectories delimiting D_k
$d^+(k+L+l)$	time varying target inventory level
$d(k+j\|k)$	predicted demand
$e(k+L+l\|k)$	predicted tracking error
$h(k)$	amount of goods dispatched to customer
H_k	control horizon
$\{\hat{k}_i\}$	non decreasing sequence of knots points over H_k
L	lead time
M	length of $P_k \triangleq [k+1,\ k+M]$, $M > N$
N	length of $H_k \triangleq [k,\ k+N-1]$
P_k	demand prediction horizon
$q_i(k)$	weight on the predicted tracking error
R_k	k-th review period
R'_k	subinterval of R_k
$R_{k,l}$	subinterval of R_k of length $T_l = n_l T'$, $n_l \in Z^+$
$R_{k,u}$	subinterval of R_k of length $T_u = n_u T'$, $n_u \in Z^+$
$R_{k,y}$	subinterval of R_k of length $T_y = n_y T'$, $n_y \in Z^+$
T	length of R_k
T'	length of R'_k such that $T = nT'$, $n \in Z^+$
$u(k)$	order issued by the retailer
u_k^-, u_k^+	lower and upper bounds on $u(j\|k)$, $j \in H_k$
$u(j\|k) = B_d(j)c_k$	optimal predicted control sequence over H_k
$y(k)$	on hand stock level
$y_{av}(k)$	actual amount of goods available for sale
$y_{lost}(k)$	total amount of wasted goods
$y_{left}(k)$	amount of goods left in the stock, just after OP3
$y(k+L+l\|k)$	predicted inventory level
$z(k)$	amount of possibly unsatisfied demand
$\alpha_{T'} \in [\alpha^-_{T'},\ \alpha^+_{T'}]$	uncertain perishability rate over R'_k
β_k	upper bound on $\\|\underline{\delta A}_k\\|$
$\delta A_{k,i}$, $\delta b_{k,i}$	unknown terms in the formulation of the predicted tracking error
$\underline{\delta A}_k$	extended matrix containing the weighted terms $\delta A_{k,i}$
$\underline{\delta b}_k$	extended vector containing the weighted terms $\delta b_{k,i}$
$\delta d(k+j\|k)$	perturbation with respect to $d(k+j\|k)$
$\delta\bar{\rho}_{T'}$	perturbation with respect to $\bar{\rho}_{T'}$
$\lambda(k)$	weigth on control moves
ξ_k	upper bound on $\\|\underline{\delta b}_k\\|$
$\rho_{T'} \in [\rho^-_{T'},\ \rho^+_{T'}]$	uncertain decay factor over R'_k
$\bar{\rho}_{T'}$	nominal value of $\rho_{T'}$
$\rho_l = \rho_{T'}^{n_l}$	decay factor over $R_{k,l}$
$\rho_u = \rho_{T'}^{n_u}$	decay factor over $R_{k,u}$
$\rho_y = \rho_{T'}^{n_y}$	decay factor over $R_{k,y}$

clarifying example is shown in Fig. 1 where we suppose that $T = nT'$ with $n = 14$. In this case we have: $n_y = 10$ and $n_u = 6$ because $y(k)$ and $u(k-L)$ remain in the warehouse for subintervals $R_{k,y}$ and $R_{k,u}$ of length $T_y = 10T'$ and $T_u = 6T'$, respectively, before OP3 is executed. Analogously $n_l = 4$ because $y_{\text{left}}(k)$ remains in the warehouse for a subinterval $R_{k,l}$ of length $T_l = 4T'$ before the start of R_{k+1}. Note that for all the IDs respecting A1), it is always true that $n_y + n_l = n$.

By A5) and point 6), the quantity of already wasted goods when OP3 is executed is given by

$$\big(1-\rho_y\big)y(k) + (1-\rho_u)u(k-L) \triangleq y^1_{\text{lost}}(k) \tag{8}$$

so that, the total amount of wasted goods inside R_k is

$$(1-\rho_l)y_{\text{left}}(k) + y^1_{\text{lost}}(k) \triangleq y_{\text{lost}}(k). \tag{9}$$

Model (1) encompasses the following typologies of SC models.

1) Those ones with synchronized OP2 and OP3 are obtained for $\rho_u = 1$ $(n_u = 0)$, $\rho_y = \rho_{T'}^{n_y}$, and $\rho_l = \rho_{T'}^{n-n_y}$.
2) Those ones with synchronized OP1 and OP2 obtained for $\rho_y = \rho_u$, $(n_y = n_u)$, and $\rho_l = \rho_{T'}^{n-n_y}$.
3) Those ones with OP1–OP3 synchronized at the beginning of R_k are obtained for $\rho_y = \rho_u = 1$, $(n_y = n_u = 0)$, and $\rho_l = \rho^n_{T'}$.
4) Those ones with nonperishable goods are obtained as a limit case for $\rho_{T'} = 1$.

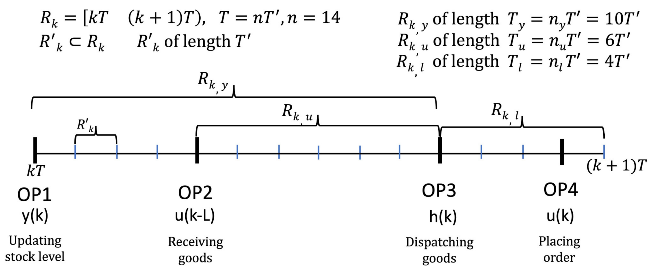

Fig. 1. Operations performed by the retailer inside each $R_k \triangleq [kT,\ (k+1)T)$ of length $T = nT'$ with $n = 14$.

For future developments we now rewrite (1) in a more convenient form where both $\rho_{T'}$ and $d(k)$ are made explicit.

By (2), an equivalent expression of $h(k)$ is

$$h(k) = d(k) - z(k) \tag{10}$$

for some $z(k) \in [0,\ d(k)]$ that represents the amount of possibly unsatisfied demand.

Definitions (5)–(7) and (10) imply that (1) can be rewritten as

$$y(k+1) = \rho_{T'}^{n_l}\left(\rho_{T'}^{n_y} y(k) + \rho_{T'}^{n_u} u(k-L) - d(k) + z(k)\right). \tag{11}$$

B. Demand Forecast Information

According to the robust modeling framework, we assume:

A6) $d(k), k \in Z^+$ is a uniformly bounded function and

A7) at any k and limitedly to a prediction horizon $P_k \triangleq [k+1, k+M]$, the future trajectory $d(k+j), j = 1, \ldots, M$, belongs to a compact set D_k limited below and above by two known trajectories: $d^-(k+j)$ and $d^+(k+j), j = 1, \ldots, M$.

Fig. 2 shows a typical example of a customer demand $d(k+j)$, $j > 0$ over a fixed D_k. The dashed line is the forecasted demand $d(k+j|k), j > 0$, that is assumed to coincide with the central trajectory of D_k.

Hence, the actual future values $d(k+j),\ j > 0$ can be expressed as

$$d(k+j) = d(k+j|k) + \delta d(k+j|k) \tag{12}$$

where $d(k+j|k) = (d^+(k+j) + d^-(k+j))/2$ is the predicted demand and $\delta d(k+j|k)$ is the corresponding estimation error.

III. Formulation and Solution of PR1

A. Min-Max MPC Approach

For any given UID satisfying A1, the control problem that we face consists in determining a corresponding ICP that matches the three conflictual CSs stated in the introduction. The intervallar nature of the uncertainties calls for a min-max MPC approach where the cost functional is defined so as to obtain the required tradeoff. This requires to solve a min-max constrained optimization problem over each control horizon $H_k = [k,\ \ k+N-1]$, $k \in Z^+$, (for some $N < M$) and, according to the receding horizon control policy, to only apply the first sample $u(k|k)$ of the computed predicted control sequence $u(k+i|k)$, $i = 0, \ldots, N-1$. In other words the actual ordering signal $u(k)$ issued by the retailer is given by $u(k|k)$.

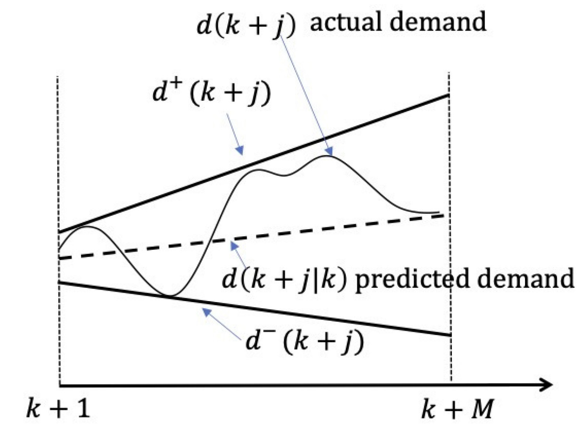

Fig. 2. Example of future customer demand $d(k+j)$, $j = 1, \ldots, M$, over a fixed D_k. The dashed trajectory is the predicted demand $d(k+j|k)$, $j = 1, \ldots, M$ coinciding with the central trajectory of $\mathcal{D}_k$.

The min-max MPC is formally defined as follows:

$$\min_{u(k+i|k)} \max_{\rho_{T'} \in \left[\rho_{T'}^-\ \rho_{T'}^+\right]} J_k \tag{13}$$

subject to: (11), (12), and:

$$0 \le u_k^- \le u(k+i|k) \le u_k^+ < \infty \tag{14}$$

where

$$J_k = \sum_{i=0}^{N-1} e^T(k+L+i|k) q_i(k) e(k+L+i|k) + \Delta u^T(k|k)\lambda(k)\Delta u(k|k) \tag{15}$$

with

$$e(k+L+i|k) \triangleq d^+(k+L+i) - y_{av}(k+L+i|k) \tag{16}$$

$$y_{av}(k+L+i|k) \triangleq \rho_{T'}^{n_y} y(k+L+i|k) + \rho_{T'}^{n_u} u(k+i|k) \tag{17}$$

$$\Delta u(k|k) \triangleq u(k|k) - u(k-1). \tag{18}$$

By (11) and (12), an equivalent expression of (17) is

$$\begin{aligned} y_{av}(k+L+i|k) = {} & \rho_{T'}^{(n_l+n_y)(L+i)+n_y} y(k) \\ & + \sum_{\ell=0}^{L-1} \rho_{T'}^{(n_l+n_y)(L+i-\ell)+n_u} u(k+\ell-L) \\ & + \sum_{\ell=0}^{i-1} \rho_{T'}^{(n_l+n_y)(i-\ell)+n_u} u(k+\ell|k) \\ & - \rho_{T'}^{(n_l+n_y)(L+i)} \big(d(k) - z(k)\big) \\ & - \sum_{\ell=1}^{L+i-1} \rho_{T'}^{(n_l+n_y)(L+i-\ell)} \\ & \times \big(d(k+\ell|k) + \delta d(k+\ell|k) - z(k+\ell|k)\big) \\ & + \rho_{T'}^{n_u} u(k+i|k). \end{aligned} \tag{19}$$

The following considerations explain why the form of cost functional (15) allows obtaining a satisfactory tradeoff between CS1–CS3.

1) By A6), A7), and (16), it can be seen that $M \ge L + N - 1$.

2) In the light of CS1 and CS2, requiring the actual amount of goods available for sale to track the maximum predicted customer demand is aimed at fulfilling CS1 without incurring the inconvenience of the overstocking usually owed to a constant reference level conservatively fixed a priori.
3) The hard constraints (14) need to guarantee the internal stability of the SC. Moreover, forcing the control input to take values within a pre-established amplitude range allows us to contain the BE according to EC2. How to calculate u_k^- and u_k^+ is explained in Section III-D.
4) The term $\Delta u^T(k|k)\lambda(k)\Delta u(k|k)$ has been included in order to meet EC1, i.e., defining a sufficiently smooth replenishment orders policy: penalizing large oscillations of control moves is useful to contain the negative aspect of the BE related to abrupt order quantity changes. The current quantitative measure of EC1 given by $\Delta u(k|k)^T\lambda(k)\Delta u(k|k)$, $k \in Z^+$, is a verification of the local smoothness of the replenishment orders signal according to the Lipschitz condition for discrete functions [52]: minimizing $\Delta u(k|k)^T\lambda(k)\Delta u(k|k)$ is equivalent to minimize $\lambda(k)^{1/2}|u(k) - u(k-1)|$, $k \in Z^+$. This in turn is equivalent to minimize the local one step Lipschitz constant and hence to maximize the local smoothness of $u(k)$.
5) Following [53], the weights $q_i(k)$, $i = 0, \dots, N-1$, and $\lambda(k)$ have been chosen inversely proportional to the square of the interval where the relative physical variables are allowed to vary.

B. B-Splines Parametrized Form of the Solution of the Min-Max MPC Problem

The parametrization of $u(k+i|k)$, $i = 0, 1, \dots, N-1$, in terms of a sampled smooth polynomial B-spline function of degree d is given by the following convex combination of control points:

$$u(j|k) \stackrel{\triangle}{=} \mathbf{B}_d(j)\mathbf{c}_k, \quad j \in [k,\ k+N-1] \stackrel{\triangle}{=} \left[\hat{k}_1,\ \hat{k}_{\ell+d+1}\right] \tag{20}$$

where according to [54],

1) $c_k \stackrel{\triangle}{=} [c_{k,1}, \dots, c_{k,\ell}]^T$ is the column vector of ℓ control points that uniquely defines the B spline function where an $\ell << N$ can be chosen;
2) $B_d(j) \stackrel{\triangle}{=} [B_{1,d}(j), \dots, B_{\ell,d}(j)]$ is the row vector containing the time varying coefficients $B_{i,d}(j)$, $i = 1, \dots, \ell$ of the convex combination; and
3) the sequence $(\hat{k}_i)_{i=1}^{\ell+d+1}$ represents the vector of knot points of the B-spline with $\hat{k}_1 = \dots = \hat{k}_{d+1} = k$, $\hat{k}_{\ell+1} = \dots = \hat{k}_{\ell+d+1} = k+N-1$, and the remaining $\ell-d-1$ knot points evenly distributed over $(k, k+N-1)$.

The advantage of parametrized form (20) concerns both theoretical and numerical aspects. Exploiting the convexity property of B-splines, the hard constraints (14) can be transferred to the control points of the B-spline. This allows us to prove the feasibility of the min-max MPC and stability of the controlled SC without any assumption on the length of the prediction horizon. Moreover:

1) the parametric parsimoniousness of (20) ($\ell << N$) reduces the number of unknowns to be computed and
2) as shown in the next section, the ℓ components of the column vector $\mathbf{c}_k$ can be computed as the solution of a constrained robust LS estimation problem, that can be efficiently solved using interior point methods.

Remark 1: Point 3) and the smoothness property of B splines imply that the first sample $u(k|k)$ of the B- spline $u(j|k)$ coincides with the first control point $c_{k,1}$ of the vector $\mathbf{c}_k$. $\triangle$

C. Numerical Computation of the Control Points Vector $\mathbf{c}_k$

In this section we show that the vector $\mathbf{c}_k$ uniquely defining $u(j|k)$ in (20), can be computed as the solution of the constrained robust LS estimation problem defined beneath.

As $\rho_{T'} \in [\rho_{T'}^-\ \ \rho_{T'}^+]$, an equivalent representation of $\rho_{T'}$ is

$$\rho_{T'} = \bar{\rho}_{T'} + \delta\rho_{T'}, \qquad \bar{\rho}_{T'} \stackrel{\triangle}{=} \left(\rho_{T'}^- + \rho_{T'}^+\right)/2 \tag{21}$$

where $\bar{\rho}_{T'}$ is the nominal value and $\delta\rho_{T'}$ is the perturbation with respect to $\bar{\rho}_{T'}$ satisfying $|\delta\rho_{T'}| \le (\rho_{T'}^+ - \rho_{T'}^-)/2$.

From (21) it follows that:

$$\rho_{T'}^k = (\bar{\rho}_{T'} + \delta\rho_{T'})^k = \bar{\rho}_{T'}^k + \Delta\rho_{T',k} \tag{22}$$

where

$$\Delta\rho_{T',k} \stackrel{\triangle}{=} (\bar{\rho}_{T'} + \delta\rho_{T'})^k - \bar{\rho}_{T'}^k \tag{23}$$

is the sum of all terms containing $\delta\rho_{T'}$ in the explicit expression of $(\bar{\rho}_{T'} + \delta\rho_{T'})^k$.

Exploiting (22) and (23) one has that the term $\rho_{T'}^{(n_l+n_y)(L+i)+n_y}y(k)$ of (19) can be rewritten as

$$\begin{aligned} &\rho_{T'}^{(n_l+n_y)(L+i)+ny}y(k) \\ &= \left(\bar{\rho}_{T'}^{(n_l+n_y)(L+i)+n_y} + \Delta\rho_{T',(n_l+n_y)(L+i)+n_y}\right)y(k). \end{aligned} \tag{24}$$

Analogously, the following terms of (19) can be rewritten as:

$$\begin{aligned} &\sum_{\ell=0}^{L-1}\rho_{T'}^{(n_l+n_y)(L+i-\ell)+n_u}u(k+\ell-L) \\ &= \sum_{\ell=0}^{L-1}\left(\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)+n_u} + \Delta\rho_{T',(n_l+n_y)(L+i-\ell)+n_u}\right) \\ &\quad\times u(k+\ell-L) \end{aligned} \tag{25}$$

$$\begin{aligned} &\sum_{\ell=0}^{i-1}\rho_{T'}^{(n_l+n_y)(i-\ell)+n_u}u(k+\ell|k) \\ &= \sum_{\ell=0}^{i-1}\left(\bar{\rho}_{T'}^{(n_l+n_y)(i-\ell)+n_u} + \Delta\rho_{T',(n_l+n_y)(i-\ell)+n_u}\right) \\ &\quad\times B_d(k+\ell)\mathbf{c}_k \end{aligned} \tag{26}$$

$$\begin{aligned} &\rho_{T'}^{(n_l+n_y)(L+i)}\big(d(k)-z(k)\big) \\ &= \left(\bar{\rho}_{T'}^{(n_l+n_y)(L+i)} + \Delta\rho_{T',(n_l+n_y)(L+i)}\right)\big(d(k)-z(k)\big) \end{aligned} \tag{27}$$

$$\begin{aligned} &\sum_{\ell=1}^{L+i-1}\rho_{T'}^{(n_l+n_y)(L+i-\ell)}d(k+\ell|k) \\ &= \sum_{\ell=1}^{L+i-1}\left(\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)} + \Delta\rho_{T',(n_l+n_y)(L+i-\ell)}\right) \\ &\quad\times d(k+\ell|k) \end{aligned} \tag{28}$$

$$\begin{aligned}\sum_{\ell=1}^{L+i-1} \rho_{T'}^{(n_l+n_y)(L+i-\ell)}\big(\delta d(k+\ell|k) - z(k+\ell|k)\big) &= \sum_{\ell=1}^{L+i-1}\Big(\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)} + \Delta\rho_{T',(n_l+n_y)(L+i-\ell)}\Big)\\ &\quad\times\big(\delta d(k+\ell|k) - z(k+\ell|k)\big) \qquad (29)\end{aligned}$$

$$\rho_{T'}^{n_u} u(k+i|k) = \big(\bar{\rho}_{T'}^{n_u} + \Delta\rho_{T',n_u}\big)B_d(k+i)\mathbf{c}_k. \qquad (30)$$

Equations (24)–(30) allow us: 1) to separate the terms depending on $u(k+\ell|k) = B_d(k+\ell)\mathbf{c}_k$ from the independent ones and 2) to separate, in either groups of terms, the known quantities from the unknown ones. Using (24)–(30), an equivalent representation of the predicted tracking error given by (16) is

$$e(k+L+i|k) = \big(b_{k,i} + \delta b_{k,i}\big) - \big(A_{k,i} + \delta A_{k,i}\big)\mathbf{c}_k \qquad (31)$$

where

$$\begin{aligned} b_{k,i} &= d^{+}(k+L+i|k) - \bar{\rho}_{T'}^{(n_l+n_y)(L+i)+n_y} y(k)\\ &\quad - \sum_{\ell=0}^{L-1}\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)+n_u} u(k+\ell-L)\\ &\quad + \bar{\rho}_{T'}^{(n_l+n_y)(L+i)}\big(d(k)-z(k)\big)\\ &\quad + \sum_{\ell=1}^{L+i-1}\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)} d(k+\ell|k) \qquad (32)\\ \delta b_{k,i} &= -\Delta\rho_{T',(n_l+n_y)(L+i)+n_y} y(k)\\ &\quad - \sum_{\ell=0}^{L-1}\Delta\rho_{T',(n_l+n_y)(L+i-\ell)+n_u} u(k+\ell-L)\\ &\quad + \Delta\rho_{T',(n_l+n_y)(L+i)}\big(d(k)-z(k)\big)\\ &\quad + \sum_{\ell=1}^{L+i-1}\Delta\rho_{T',(n_l+n_y)(L+i-\ell)} d(k+\ell|k)\\ &\quad + \sum_{\ell=1}^{L+i-1}\Big(\bar{\rho}_{T'}^{(n_l+n_y)(L+i-\ell)} + \Delta\rho_{T',(n_l+n_y)(L+i-\ell)}\Big)\\ &\quad \times\big(\delta d(k+\ell|k) - z(k+\ell|k)\big) \qquad (33)\\ A_{k,i} &= \sum_{\ell=0}^{i-1}\bar{\rho}_{T'}^{(n_l+n_y)(i-\ell)+n_u} B_d(k+\ell) + \bar{\rho}_{T'}^{n_u} B_d(k+i) \qquad (34)\\ \delta A_{k,i} &= \sum_{\ell=0}^{i-1}\Delta\rho_{T',(n_l+n_y)(i-\ell)+n_u} B_d(k+\ell)\\ &\quad + \Delta\rho_{T',n_u} B_d(k+i). \qquad (35)\end{aligned}$$

By Remark 1 also the term

$$\Delta u(k|k) = u(k|k) - u(k-1) = c_{k,1} - u(k-1)$$

in (15) can be rewritten as

$$\Delta u(k|k) = \big(b_{u_k} + \delta b_{u_k}\big) - \big(A_{u_k} + \delta A_{u_k}\big)\mathbf{c}_k \qquad (36)$$

where $b_{u_k} = -u(k-1)$, $\delta b_{u_k} = 0$, $A_{u_k} = -\big[1\ 0 \cdots 0\big]$, and δA_{u_k} is a null row vector.

Defining the following augmented column vectors $\underline{e}_k$, $\underline{b}_k$, and $\underline{\delta b}_k$ of $N+1$ elements:

$$\underline{e}_k = \begin{bmatrix} q_0^{1/2}(k)e(k+L|k)\\ \vdots\\ q_{N-1}^{1/2}(k)e(k+L+N-1|k)\\ \lambda^{1/2}(k)\Delta u(k|k)\end{bmatrix} \qquad (37)$$

$$\underline{b}_k = \begin{bmatrix} q_0^{1/2}(k)b_{k,0}\\ \vdots\\ q_{N-1}^{1/2}(k)b_{k,N-1}\\ \lambda^{1/2}(k)b_{u_k}\end{bmatrix} \quad \underline{\delta b}_k = \begin{bmatrix} q_0^{1/2}(k)\delta b_{k,0}\\ \vdots\\ q_{N-1}^{1/2}(k)\delta b_{k,N-1}\\ \lambda^{1/2}(k)\delta b_{u_k}\end{bmatrix} \qquad (38)$$

and the extended matrices $\underline{A}_k$ and $\underline{\delta A}_k$ of dimensions $((N+1)\times\ell)$

$$\underline{A}_k = \begin{bmatrix} q_0^{1/2}(k)A_{k,0}\\ \vdots\\ q_{N-1}^{1/2}(k)A_{k,N-1}\\ \lambda^{1/2}(k)A_{u_k}\end{bmatrix} \quad \underline{\delta A}_k = \begin{bmatrix} q_0^{1/2}(k)\delta A_{k,0}\\ \vdots\\ q_{N-1}^{1/2}(k)\delta A_{k,N-1}\\ \lambda^{1/2}(k)\delta A_{u_k}\end{bmatrix} \qquad (39)$$

allow us to reformulate the min-max MPC (13) and (14) as the following constrained robust LS estimation problem:

$$\min_{\mathbf{c}_k}\ \max_{\|\underline{\delta A}_k\|\le\beta_k\ \|\underline{\delta b}_k\|\le\xi_k} \big\|\big(\underline{b}_k + \underline{\delta b}_k\big) - \big(\underline{A}_k + \underline{\delta A}_k\big)\mathbf{c}_k\big\|^2 \qquad (40)$$

$$\text{subject to } u_k^- \le c_{k,i} \le u_k^+, \quad i = 1,\ldots,\ell. \qquad (41)$$

Constraints (41) derive from (20) and the convex hull property of B splines.

The constrained min-max LS problem (40) and (41) can be numerically solved exploiting the following well-known result on robust LS [55]: the min-max problem

$$\min_x\ \max_{\|\delta A\|\le\beta,\ \|\delta b\|\le\xi} \|(b+\delta b) - (A+\delta A)x\| \qquad (42)$$

$$\text{subject to } \underline{x} \le x \le \bar{x} \qquad (43)$$

(where the matrix norm is the spectral norm) is equivalent to

$$\min_x \|b - Ax\| + \beta\|x\| + \xi \qquad (44)$$

$$\text{subject to } \underline{x} \le x \le \bar{x}. \qquad (45)$$

As

$$\arg\min_x \sum_i \|f_i(x)\| \equiv \arg\min_x \Big(\sum_i \|f_i(x)\|\Big)^2$$

is seen that min-max problem (40) and (41) defines a problem of the kind (44) and (43).

Hence, at any k, the solution $\mathbf{c}_k$ of the constrained robust LS estimation problem (40) and (41) can be determined solving

$$\min_{\mathbf{c}_k} \|\underline{b}_k - \underline{A}_k\mathbf{c}_k\| + \beta_k\|\mathbf{c}_k\| + \xi_k \qquad (46)$$

where the components of $\mathbf{c}_k$ must satisfy (41).

This is a semi-definite programming problem that can be numerically solved using SeDuMi, a commonly used open-source interior-point method [56].

Remark 2: The maximum Euclidean norm of the term $\underline{\delta b}_k$ given by (38) corresponds to the term ξ_k of (46). As ξ_k is independent of $\mathbf{c}_k$, its value can be minimized with respect to

$\Delta\rho_{T'}$ and $\delta d(k+\ell|k)$ assuming a nominal $\rho_{T'}$ given by the central value of $[\rho_{T'}^-, \rho_{T'}^+]$ and a predicted customer demand coinciding with the central trajectory of $\mathcal{D}_k$. △

The recursive feasibility of the proposed RMPC and the internal asymptotic stability of the controlled SC can be proved following the lines of [44].

D. Computing the Bounds u_k^- and u_k^+

In this section we provide an explicit expression of the hard constraints (14) on $u(k+i|k)$ [namely (41) on the components of $\mathbf{c}_k$]. Coherently with the robust worst-case approach we refer to the following hypothetical scenario:

1) The retailer is able to fully satisfy a customer demand that, over each control horizon H_k, is a constant signal with value $d_k \in [d_k^-, d_k^+]$, where, by A7, d_k^- and d_k^+ are the minimum of $d^-(k)$ and the maximum of $d^+(k)$ over $P_k = [k+1, k+M]$, respectively. This implies $z(k) = 0$ in (11).
2) Each control horizon H_k is long enough to allow the output (the on hand stock level), to practically attain the steady-state value y_k under the forcing action of constant inputs u_k and d_k.

The existence of an output steady-state response is assured by the asymptotic stability of (11) (consequence of $\rho_{T'} \in (0, 1)$).

Using classical z-transform methods and applying the final value theorem [57], the steady state y_k is

$$y_k = \left.\frac{\rho_{T'}^{n_l+n_u}}{z^L\left(z-\rho_{T'}^{n}\right)}\right|_{z=1} u_k - \left.\frac{\rho_{T'}^{n_l}}{\left(z-\rho_{T'}^{n}\right)}\right|_{z=1} d_k. \quad (47)$$

The problem we now consider is: for any given constant demand $d_k \in [d_k^- \quad d_k^+]$, it is required to find the interval $\mathcal{C}_k \triangleq [u_k^-, u_k^+]$, where the corresponding constant control input u_k takes values, so that, in accordance with (16) and (17), the following condition holds:

$$\rho_{T'}^{n_y} y_k + \rho_{T'}^{n_u} u_k \geq d_k^+ \quad \forall \rho_{T'} \in \left[\rho_{T'}^-, \rho_{T'}^+\right]. \quad (48)$$

By (47) and recalling that $n_l + n_y = n$, (48) can be rewritten as

$$\left(\frac{\rho_{T'}^{n_u}}{1-\rho_{T'}^{n}}\right) u_k - \frac{\rho_{T'}^{n}}{1-\rho_{T'}^{n}} d_k \geq d_k^+ \quad \forall \rho_{T'} \in \left[\rho_{T'}^-, \rho_{T'}^+\right]. \quad (49)$$

The minimum u_k guaranteeing a robust solution to (49) $\forall \rho_{T'} \in [\rho_{T'}^-, \rho_{T'}^+]$, is

$$u_k = \frac{\left[d_k^+ - \left(\rho_{T'}^-\right)^n \left(d_k^+ - d_k\right)\right]}{\left(\rho_{T'}^-\right)^{n_u}}.$$

The interval $\mathcal{C}_k$ can be directly found considering the two limit situations $d_k = d_k^-$ and $d_k = d_k^+$. We obtain

$$u_k^- = \frac{\left[d_k^+ - \left(\rho_{T'}^-\right)^n \left(d_k^+ - d_k^-\right)\right]}{\left(\rho_{T'}^-\right)^{n_u}} \quad (50)$$

$$u_k^+ = \frac{d_k^+}{\left(\rho_{T'}^-\right)^{n_u}}. \quad (51)$$

The amplitude A_k of $\mathcal{C}_k$ is

$$A_k = \left(\rho_{T'}^-\right)^{n-n_u}\left(d_k^+ - d_k^-\right). \quad (52)$$

Equations (50)–(52) give the a priori estimate of the BE corresponding to EC2.

1) Conditions (50) and (51) show that the presence of perishable goods is another source of BE as regards the required amount of control effort: $u(k)$ takes value over an interval whose lower and upper bounds increase as $(1/[(\rho_{T'}^-)^{n_u}])$. This needs to compensate the loss of goods due to deterioration. This source of amplification disappears if $n_u = 0$.
2) Condition (52) evidences another effect: the reduced amplitude A_k of $\mathcal{C}_k$ with respect to the interval $[d_k^-, d_k^+]$. This is a positive effect because reduces the possibility of an ICP with uneconomic large oscillations.

IV. Solution of PR2

In the previous section we have determined the ICP on the basis of a stock balance equation model whose structure depends on a priori fixed UID. On the basis of these results we are now in a position to also answer the second issue of the twofold problem posed in the Introduction about the most convenient ID that the SC manager should realize in the case of an MID.

By (8) it follows that $\rho_y = \rho_u = 1$ yields a null $y_{\text{lost}}^1(k)$. Moreover $\rho_u = 1$ is also useful to contain the BE according to the measurement criterion EC2. In fact by (7) the $\rho_u = 1$ is obtained for $n_u = 0$, so that by (52) it is evident that $\rho_u = 1$ reduces the amplitude A_k of the interval where the ICP takes values. Therefore, recalling A1), it follows that in the case of MID the most convenient ID is obtained if both OP2 and OP3 are synchronized with OP1 at the beginning of each R_k.

Analogous considerations lead to the conclusion that the worst ID is obtained for $\rho_y \to 0$ and $\rho_u \to 0$, namely if OP1 and OP2 are synchronized at the beginning of each long R_k and OP3 is executed at the end.

V. Numerical Results

A. Simulation Experiment

We consider a perishable single echelon SC with the same UID shown in Fig. 1 ($n = 14$, $n_l = 4$, $n_y = 10$, and $n_u = 6$). The other parameters of the model are reported in Table III.

We perform two different simulations. In the first one the ICP is computed exploiting the information carried by the actual UID. In the second one the ICP is computed neglecting this information.

The following operative assumptions are common to both simulations.

1) At each $k \in Z^+$ the min-max MPC is solved parametrizing $u(j|k)$ in (20) as a B-spline function of degree $d = 1$ with $\ell = 3$ control points. Other parameters defining the control algorithm are $N = 5$ (length of each H_k) and the weights of (15): $q_i(k) = (1/[(0.01 \cdot d^+(k+L+i))^2])e^{-i}$, $\lambda(k) = (1/[(0.01 \cdot u(k-1))^2])$ for $k \geq 1$, $\lambda(0) = 1$.

TABLE III
PARAMETERS OF THE ACTUAL SC MODEL WITH UID

Lead time	deterioration rate over T'	decay factor over T'	decay factors ρ_l, ρ_y and ρ_u in (11)	initial stock level
$L = 2$	$\alpha_{T'} \in [\alpha^-_{T'}, \alpha^+_{T'}] = [0.06, 0.1]$	$\rho_{T'} \in [\rho^-_{T'}, \rho^+_{T'}] = [0.9, 0.94]$	$\rho_l = \rho^{n_l}_{T'} \in [0.6561, 0.7807]$ $\rho_y = \rho^{n_y}_{T'} \in [0.3487, 0.5386]$ $\rho_u = \rho^{n_u}_{T'} \in [0.5314, 0.6899]$	$y(0) = 0$

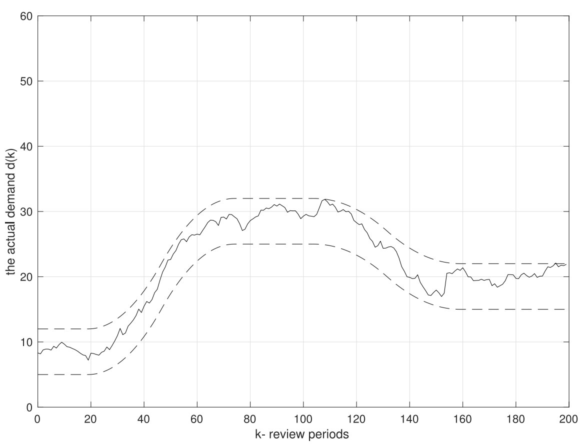

Fig. 3. (Simulated data) The actual customer demand $d(k)$ (solid line). The dashed lines delimit the compact set $\mathcal{D}$ given by the consecutive contiguous overlapping of all the a priori given sets D_k's.

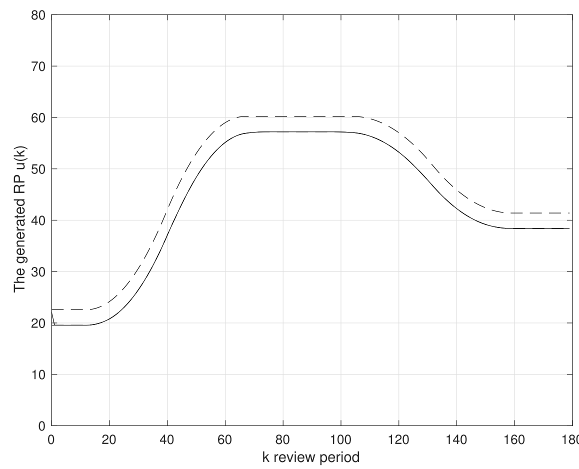

Fig. 4. (Simulated data) The ICP based on the UID: the generated ordering signal $u(k)$ (solid line) and the boundaries trajectories u^-_k and u^+_k (dashed lines) computed by (50) and (51).

2) At each $k \in Z^+$, the future customer demand $d(k + \ell)$ is known to belong to a given compact set D_k (of length $M = 6$ according to $M \geq N + L - 1 = 6$). The profile of the whole, assumed, and actual customer demand $d(k)$ is the irregular continuous line shown in Fig. 3. The dashed lines are boundaries of the compact set $\mathcal{D}$ enclosing $d(k)$.
3) The dynamic equation (11) of the actual SC is implemented assuming an actual decay factor $\rho_{T'} = 0.91$, namely a perishability rate $\alpha_{T'} = 0.09$.
4) Both simulations are stopped at time $k = 180$.

For a quantitative comparison of the two ICPs we use the percentage of unsatisfied customer demand defined as

$$\text{UCD} = \frac{\sum_{k=1}^{N_s} |d(k) - h(k)|}{\sum_{k=1}^{N_s} d(k)} \times 100 \tag{53}$$

where Ns is the length of the simulation.

1) ICP Based on the Actual UID: The ordering signal $u(k)$ obtained implementing the algorithm described in Section III is reported in Fig. 4. This figure shows a $u(k)$ that satisfies EC1 and EC2: it has a smoother waveform than $d(k)$ and belongs to the narrow interval delimited by (50) and (51). The evident enhanced degree of smoothness of $u(k)$ with respect to $d(k)$ is due to the introduction of the penalizing term $\Delta u^T(k|k)\lambda(k)\Delta u(k|k)$ in the cost functional (15). The actual ($d(k)$) and fulfilled ($h(k)$) customer demands are shown in Fig. 5. The almost total overlap of the two curves evidences the effectiveness of the proposed method: the customer demand is not satisfied only for $k \leq L = 2$, as a consequence of lead time and null initial stock. The UCD resulted to be 0.43 %.

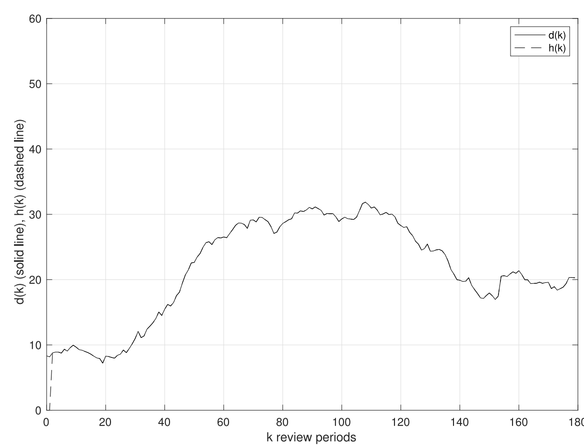

Fig. 5. (Simulated data) The ICP based on the UID: the customer demand $d(k)$ (solid line) and the fulfilled customer demand $h(k)$ (dashed line). The two trajectories are overlapped for $k > L = 2$.

2) ICP Neglecting the Actual UID: The actual dynamics of SC over the review period R_k is not taken into account and it is erroneously supposed that all the OPs are synchronized at the time kT. This means that the ICP is designed on the basis of the following misleading stock balance equation:

$$y(k + 1) = \rho(y(k) + u(k - L) - d(k) + z(k)) \tag{54}$$

for some $\rho \in [\rho^-, \rho^+]$, that represents the decay factor over the whole review period R_k.

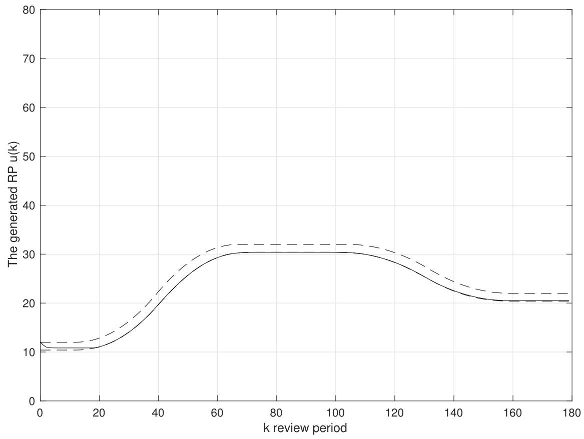

Fig. 6. (Simulated data) The ICP neglecting the actual UID: the generated $u(k)$ (solid line) and the boundaries trajectories u_k^- and u_k^+(dashed lines) computed by (50) and (51) putting $n_y = n_u = 0$ and $n_l = n$.

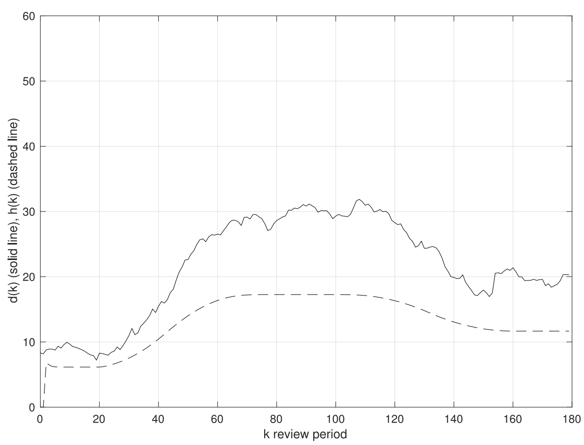

Fig. 7. (Simulated data) The ICP neglecting the actual UID: the customer demand $d(k)$ (solid line) and the fulfilled customer demand $h(k)$ (dashed line).

In accordance with (54) the RMPC algorithm, described in Section III, requires the following change: replace $\rho_{T'}^{n_l}$, $\rho_{T'}^{n_u}$, and $\rho_{T'}^{n_y}$ with ρ,1,1, respectively.

To implement the modified controller algorithm we need an estimate of the interval, where ρ takes values. In this case we assumed $\rho = (\rho_{T'})^{14}$ and, as a consequence: $\rho \in [(\rho_{T'}^-)^{14}, (\rho_{T'}^+)^{14}]$ and the nominal $\bar{\rho}$ is $\bar{\rho} = (\bar{\rho}_{T'})^{14}$.

The obtained ordering signal $u(k)$ is reported in Fig. 6. As a consequence of using the erroneous value $n_u = 0$ in (50), (51), and (52), it is constrained to vary inside a lower and narrower range with respect to the case of exact ID (compare with Fig. 4). The consequent significant performance degradation of the SC in terms of a greatly reduced amount of fulfilled demand emerges from the comparison between Figs. 7 and 5 and of the two UCDs, that in this case resulted to be 39.5 %.

B. Applications to Real Data

We have also tested our method on the real data provided by a medium size wholesaler of fresh fruit and vegetables from the Marche region in Italy. The data used concern the daily supply of fresh strawberries to a retailer over a period of 48

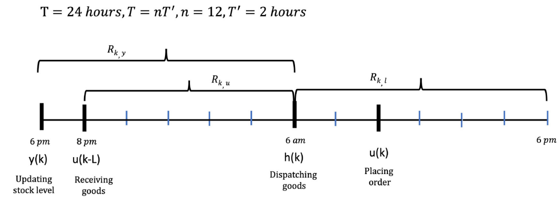

Fig. 8. UID relative to the daily supply of fresh strawberries.

days; because of their high perishability, strawberries can last for about 3–7 days when stored properly in the refrigerator.

On the basis of the information provided by the wholesaler, the UID can be well described by a model (1) with a review period R_k of length $T = 24$ h, a time delay $L = 1$ day, and an ID, such as the one shown in Fig. 8. Assuming $T' = 2$ h, we have that $n_y = 6$, $n_u = 5$, $n_l = 6$, and $n = n_y + n_l = 12$.

The whole set of model parameters are reported in Table IV.

Analogously to the simulation results, we compare the performances of the two ICPs computed first exploiting and then neglecting the information carried by the actual UID.

The following operative assumptions are common to both cases.

1) At each $k \in Z^+$ the min-max MPC is solved parametrizing $u(j|k)$ in (20) as a B-spline function of degree $d = 1$ with $\ell = 3$ control points. Other parameters defining the control algorithm are $N = 5$ (length of each H_k) and the weights of (15): $q_i(k) = (1/[(0.01 \cdot d^+(k+L+i))^2])e^{-i}$, $\lambda(k) = (1/[(0.1 \cdot u(k-1))^2])$ for $k \geq 1$, $\lambda(0) = 1$.
2) At each $k \in Z^+$, the future customer demand $d(k+\ell)$ is known to belong to a given compact set D_k (of length $M = 5$ according to $M \geq N + L - 1 = 5$). The data provided by the wholesaler are shown in Fig. 9 (solid line). Analogously to Fig. 5, the dashed lines are the limits of the whole compact set D given by the consecutive contiguous overlapping of all the a priori given sets D_ks.
3) The dynamic equation (11) of the actual SC is implemented assuming an actual decay factor $\rho_{T'} = 0.93$, namely a perishability rate $\alpha_{T'} = 0.07$.
4) Both simulations are stopped at time $k = 40$.

1) ICP Based on the Actual UID: The ordering signal $u(k)$ obtained implementing the algorithm described in Sections III and IV is reported in Fig. 10. The actual $(d(k))$ and fulfilled $(h(k))$ customer demands are shown in Fig. 11. The almost total overlap of the two curves evidences the effectiveness of the proposed method: the customer demand is not satisfied only for $k \leq L = 1$. The UCD resulted to be 1.74 %.

2) ICP Neglecting the Actual UID: Arguing as in Section V-A2, the ICP is designed on the basis of the misleading stock balance (54) assuming $\rho = (\rho_{T'})^{n_l}$, $n_l = 12$, and as a consequence: $\rho \in [(\rho_{T'}^-)^{12}, (\rho_{T'}^+)^{12}]$ and the nominal $\bar{\rho}$ is $\bar{\rho} = (\bar{\rho}_{T'})^{12}$.

The resulting ordering signal $u(k)$ is reported in Fig. 12. The consequent performance degradation of the SC in terms of a reduced amount of fulfilled demand emerges from the comparison between Figs. 13 and 11 and of the two UCDs that in this case resulted to be 12.96 %.

TABLE IV
PARAMETERS OF THE FRESH STRAWBERRIES SC MODEL WITH UID

Lead time	deterioration rate over T'	decay factor over T'	decay factors ρ_l, ρ_y and ρ_u in (11)	initial stock level
$L = 1$	$\alpha_{T'} \in [\alpha_{T'}^-, \alpha_{T'}^+] = [0.05, 0.07]$	$\rho_{T'} \in [\rho_{T'}^-, \rho_{T'}^+] = [0.93, 0.95]$	$\rho_l = \rho_{T'}^{n_l} \in [0.6470, 0.7351]$ $\rho_y = \rho_{T'}^{n_y} \in [0.6470, 0.7351]$ $\rho_u = \rho_{T'}^{n_u} \in [0.6957, 0.7738]$	$y(0) = 0$

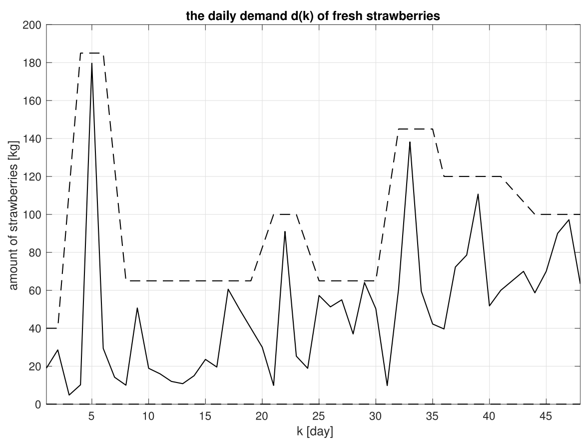

Fig. 9. (Real data) The daily demand of fresh strawberries (solid line). The dashed lines are the limits of the whole compact set D. The lower trajectory delimiting D is the abscissa axis.

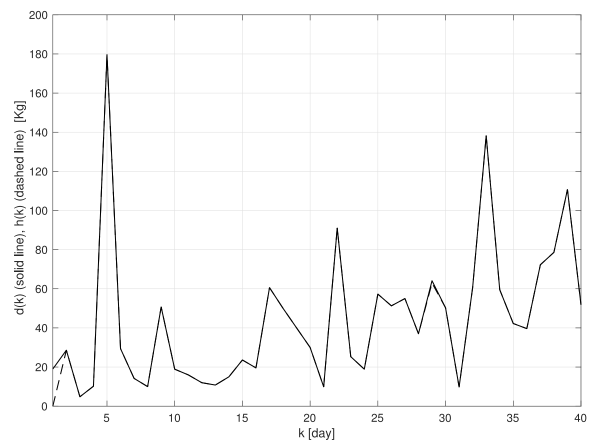

Fig. 11. (Real data) The ICP based on the actual UID: the actual $d(k)$ (solid line) and fulfilled $h(k)$ (dashed line) demands. The two trajectories are overlapped for $k > L = 1$.

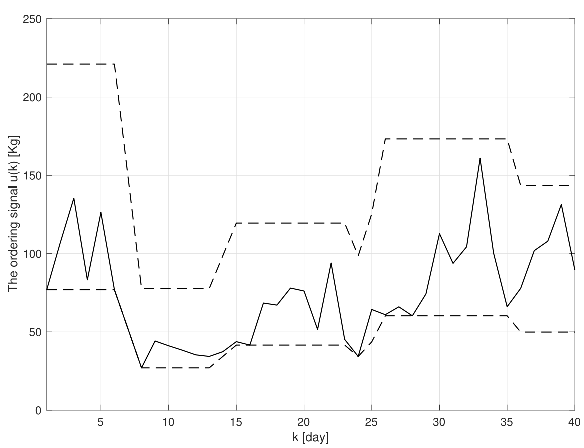

Fig. 10. (Real data) The ICP based on the actual UID: the generated $u(k)$ and the boundary trajectories delimiting $u(k)$.

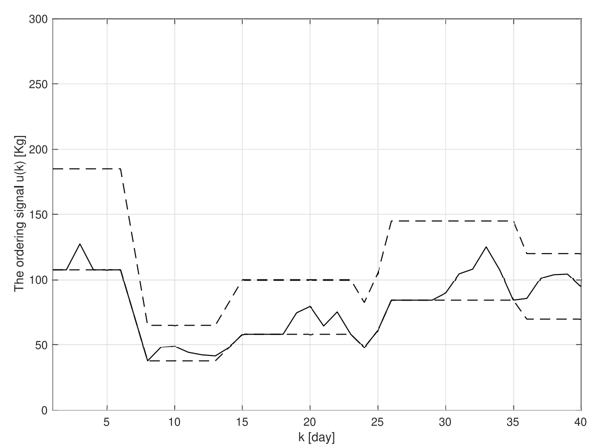

Fig. 12. (Real data) The ICP neglecting the actual UID: The generated $u(k)$ and the boundary trajectories delimiting $u(k)$

VI. CONCLUSION

Through theoretical considerations, numerical simulations, and application to real data we have shown that a proper description of a periodically reviewed SC dynamics can not be limited to the time instants $kT, k \in Z^+$ delimiting the review period.

Going into the details of the ID allows the ICP designer to take into account what really happens inside each interval $[kT, (k+1)T)$, $k \in Z^+$. Exploiting this information is fundamental to define an effective ICP, especially in the case of deterioration rate and review period length that result in significant spoilage within the review period. Equipping a properly defined RMPC approach with the ID notion, has allowed us to define an ICP actually conciliating the three CSs defined in the introduction.

Managerial and practical insights deriving by our approach can be summarized as follows.

1) The robust approach to customer demand forecasting avoids the use of complicated parameter estimation methods and cumbersome numerical procedures. An experience-based analysis of historical data and seasonal trends more easily provides information on upper and lower limits of the predicted demand rather than on its actual value. By (16), this, in turn, facilitates the choice of the time-varying target inventory level.

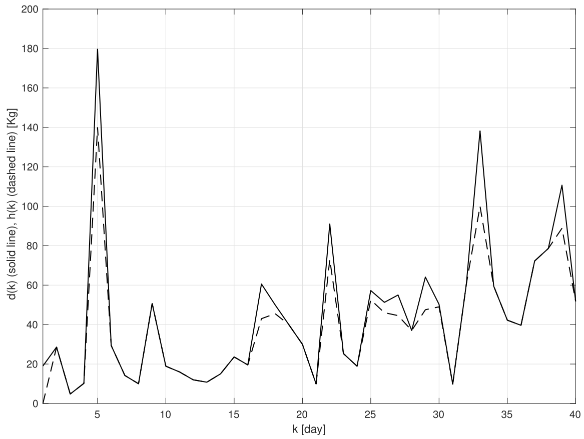

Fig. 13. (Real data) The ICP neglecting the actual UID: The actual $d(k)$ (solid line) and fulfilled $h(k)$ (dashed line) demands

2) As stability and feasibility are guaranteed independently of the length of the prediction horizon, the demand prediction can be limited to short intervals. This greatly simplifies the consequent forecasting and demand management problems [58].
3) In the case of MID, the managerial decision on the best way to organize the ID directly follows from the solution of PR2 reported in Section IV.

Future developments of this line of research will be addressed to the extension of the ID notion to the case of uncertain perishability rates time-varying over large uncertainty intervals.

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