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ORIGINAL RESEARCH

Two in One: A New Tool to Combine Two Rankings Based on the Voronoi Diagram

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Abstract
In this paper, we propose a novel method for ranking items such as countries, individuals, or firms based on two indices. This approach is particularly useful when constructing a composite indicator that combines both dimensions is not feasible. The proposed ranking approach involves an iterative scheme where the Voronoi algorithm is applied in a two-dimensional space at each step. To provide empirical evidence that our approach works satisfactorily, we applied the Voronoi-based iterative scheme to rank 34 European countries based on two dimensions: the *Human Development Index* (HDI) and the *Happiness Index* (HI). The correlation coefficient between the rankings based on HDI and HI is lower than the correlation coefficients between the Voronoi-based ranking and HDI, as well as between the Voronoi-based ranking and HI. These results suggest that the new method is capable of better capturing the information from both original indices.

Keywords Voronoi partition · Ranking · Composite indicator

1 Introduction

In many real-life situations, it is necessary to choose among alternatives and rank them based on certain criteria. For example, when we have a free day and want to go shopping in a mall, we may choose the mall based on its proximity to our home. Due to the large number of malls in our region, we may decide to go to the mall that is closest to our home to save time and have more time to visit shops. However, proximity to home is just one of many possible criteria. People may also be interested in finding the mall with the highest discounts, even if this means spending more time traveling to reach it. To choose the best mall, people need to rank them based on different criteria.

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Of course, the ranking based on the proximity criterion does not necessarily coincide with the ranking based on the discount criterion. Common sense suggests that we should sort the malls according to both criteria. Unfortunately, if we try to consider both criteria simultaneously, it is only possible to rank a few alternatives, since there may not be a clear dominance in both criteria for all options.
If there are options that can be ranked, they constitute the so-called "candidate set" (Chang et al., 2016). However, in most cases, one option may have a better value according to the first criterion, while another option may display a better value according to the second criterion. As a result, these two options cannot be ranked against each other. Sometimes, a good compromise is to sacrifice one criterion in favor of another. However, if we want to take both criteria into account, we could construct an indicator that summarizes the information into a single number. Specifically, when we have two or more criteria, scholars sometimes decide to combine them using a given function to collapse the information into a single number, which allows for comparison. Usually, the higher the value of the composite indicator, the better the option. This method is referred to as constructing composite indicators (OECD, 2008).

The construction of composite indicators involves making a series of choices that requires subjective judgments, such as selecting a model to aggregate or weighting indicators. These judgments, even if they are transparent and based on sound statistical principles, can be criticized because, as demonstrated, different aggregation functions can lead to different rankings.

Hence, it is desirable to sort options by considering both criteria without relying on the construction of a composite indicator. This is the main objective of our work.
We aim to develop a method for ranking items, including countries, individuals, or firms, based on two indices, without relying on any combination of the indices. This approach enables decision-makers to evaluate the two criteria independently and make decisions accordingly.

To achieve our goal, we propose an iterative procedure based on the *Voronoi* method. The "Dirichlet–Voronoi Diagram" was first introduced by Dirichlet (1850), and later generalized by Voronoi (1908). This method has been used in several different fields over time, including robotics, mobile sensor networks, and computational geometry, among others.[1]

The procedure starts from a "candidate set", made by the points ranked according to the Pareto dominance for the two criteria. At each iteration, first, we compute the Voronoi partition using as "reference pivots" the points in the current candidate set. Then, we update the latter inserting the non ranked points through a "mild" Pareto dominance criterium. The procedure stops when the candidate set no longer updates.

There are several advantages to utilizing a ranking based on the Voronoi partition. Firstly, our ranking process offers flexibility as it can be adjusted to different criteria or preferences by simply modifying the distance metric used to compute the Voronoi cells. The Voronoi partition proves especially valuable when working with datasets that exhibit nonlinear relationships. Furthermore, this method allows for local rankings and provides a visual representation, aiding in socio-economic interpretation and uncovering potential spatial patterns or clusters. Additionally, the use of the Voronoi partition enhances computational efficiency by reducing the number of iterations compared to linear ranking methods (Nedeljkovic et al., 2023). The major limitation of the proposed method lies in its dependency on the initial candidate set chosen.

The main merit of the Voronoi-based ranking is its versatility. In fact, since it is based uniquely on the proximity between points, this method can be applicable in various fields,

[1] See Bakolas and Tsiotras (2010).

including spatial analysis, optimization and decision-making in social sciences. The novel contribution of this paper is the introduction of a numerical scheme that, using iteratively the Voronoi partition, allows for a ranking in a mild Pareto sense. Moreover, the proposed methodology paves the way for new hybrid methods that combine objective criteria, such as spatial proximity, with subjective criteria based on preferences.

To illustrate the effectiveness of our proposal, we apply the Voronoi algorithm to rank European countries based on two dimensions: the Human Development Index and the Happiness Index. A concise description and discussion regarding these two indices are provided in Sect. 3 of the paper.

The remainder of the paper is organized as follows. In Sect. 2, we introduce the problem and define some notation. We also describe the sorting method that we propose. In Sect. 3, we illustrate the effectiveness of our method by ranking selected countries based on two dimensions. Finally, in Sect. 4, we summarize our results and discuss possible future extensions of our method.

2 The Problem

2.1 Notations and Basic Definitions

Let $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ be the two-dimensional real plane and $d(A, B)$ be the Euclidean distance between two points $A = (x_A, y_A)$ and $B = (x_B, y_B) \in \mathbb{R}^2$.

Definition 2.1 Given a subset U of the plane, $U \subset \mathbb{R}^2$, and a set of points $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ such that $A_i \in U$, $i = 1, 2, \ldots, n$, the Voronoi cell generated by A_i is the subset of U made up of the points in U that are closer to A_i than to any other point in $\mathcal{A}$. In other words, the Voronoi cell $V_i(U)$ is defined as:

$$V_i(U) = \{B \in U \ : \ d(B, A_i) \leq d(B, A_j) \text{ for } j = 1, 2, \ldots, n, \ \ j \neq i\}, \quad i = 1, 2, \ldots, n. \tag{1}$$

The set of Voronoi polygons $V_1(U)$, $V_2(U)$, ..., $V_n(U)$ partitions the set U into disjoint n regions, we refer to it as the *Voronoi partition* of U associated with the set $\mathcal{A}$ while denoting it with $V_{\mathcal{A}}(U)$. The points A_1, A_2, ..., A_n that generate the Voronoi partition of U are called pivots.

Definition 2.2 Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be two points in $\mathbb{R}^2$. We say that the point A dominates B and we write

$$A \succcurlyeq B \tag{2}$$

if only if

$$x_A \geq x_B \quad \textit{and} \quad y_A \geq y_B \quad \text{and} \quad A \neq B. \tag{3}$$

The relation defined in (3) is a partial ordering known as *component-wise dominance* or *Pareto dominance* (see Ehrgott, 2005).

Definition 2.3 Given a set of points $A_i \in \mathbb{R}^2$, $i = 1, 2, \ldots, n$, we say that the points A_1, A_2, ..., A_n are sorted in ascending order (according to the Pareto order) if only if

$$A_i \succcurlyeq A_j, \text{ for } i > j, \quad i, j = 1, 2, \ldots, n. \tag{4}$$

Now, let us suppose to be interested in ranking n points/units $I = \{1, 2, \ldots, n\}$ according to two indicators from the set $J = \{1, 2\}$. The ith unit is identified by the point of the plane A_i, $i \in I$ given by:

$$A_i = (x_i, y_i), \tag{5}$$

where x_i and y_i are the values assigned to that unit in the dimension from J. To ensure comparability, we assume that the indicators are normalized to the range of [0,1]. This guarantees that A_i represents a point within the unit square on the plane, i.e., $A_i \in [0, 1]^2 = [0, 1] \times [0, 1]$, where $i = 1, 2, \ldots, n$. We denote the set of points on the plane corresponding to the units in I as $\mathcal{S} = \{A_i \ : i \in I\}$.

We are interested in the problem of ranking the units in I based on the two dimensions in J. However, the Pareto order defined in (3) only allows for partial ordering.
To achieve a complete ranking of the units in I, we propose a procedure that involves sorting the units according to (3).

2.2 Sorting Method Based on an Iterative Voronoi Scheme

To fully rank the units in set I, we propose an iterative Voronoi scheme. The iterative procedure starts with a subset of $\mathcal{S}$ called the "candidate set," which consists of the points ranked according to the Pareto dominance.
After this initial step, the iterative procedure updates the candidate set by adding still non-ranked points. At each iteration, the Voronoi partition is first computed using the pivots, i.e., the points of the current candidate set. This assigns each point corresponding to non-ranked points to a cell of the partition. At each iteration of the iterative procedure, the candidate set is updated by considering each Voronoi cell and selecting at most one point to be added for any cell. The selected point is the one closest to the pivot based on the Euclidean distance, and its corresponding pivot is referred to as the "reference pivot". If the new point can be ranked according to the Pareto dominance, it is included in the candidate set. Otherwise, as mentioned in the Introduction, there are three cases.
First, the new point dominates its reference pivot but is not dominated by the pivot that follows its reference pivot in the ranking.
Second, the new point is dominated by its reference pivot but does not dominate the pivot that precedes its reference pivot in the ranking.
In both cases, the new point is added to the candidate set, and the pivot that follows (in the first case) or precedes (in the second case) its reference pivot is removed.
Third, if it is not possible to establish dominance between the new point and its reference pivot, the distances between the new point and the pivot that follows and precedes the "reference pivot" are computed. The new point is then included between its "reference pivot" and the pivot with the minimum distance from the new point. This is the mild Pareto dominance criterium mentioned in the Introduction.
The iterative procedure stops when the candidate set no longer update.

In summary, the method begins with a set of ranked units and proceeds to iteratively include the remaining non-sorted units into the set. This process is carried out at each iteration by utilizing the Voronoi partition of the set $[0, 1]^2$ generated from the current set of sorted units. The incorporation of new units into the sorted set is achieved through appropriate distance comparisons. The application of the Voronoi scheme, which updates the sorted set iteratively through distance comparisons, constitutes a novel contribution to the ranking literature.

2.2.1 The Initial Set of Pivots

Let's begin by explaining how to select the initial set of pivots used to construct the Voronoi partition. We refer to this subset of $\mathcal{S}$ as $\mathcal{O}_0$. We choose the points in $\mathcal{S}$ that are sorted in ascending order according to Definition 2.3 of Sect. 2 as our pivots. To build $\mathcal{O}_0$, we apply two procedures that we refer to as row-wise and column-wise sorting methods.

In the row-wise sorting method, we rank the points in $\mathcal{S}$ in ascending order based on their x-coordinate (by row). Let us denote this set by $\mathcal{S}_r = \{A_{r_1}, A_{r_2}, \ldots, A_{r_n}\}$, where A_{r_k} is such that $x_{r_k} \leq x_{r_{k+1}}$ for $k = 1, 2, \ldots, n-1$. Then, starting from A_{r_1} (i.e., the point with the smallest x-coordinate), we remove from $\mathcal{S}_r$ the points whose y-coordinate is smaller than the y-coordinate of the points that precede them in $\mathcal{S}_r$. The resulting set is a subset of $\mathcal{S}$ made up of points sorted row-wise in ascending order, which we denote by $\mathcal{O}_r$. Note that the set $\mathcal{S}_r \backslash \mathcal{O}_r$ consists of points that are not dominated by any others, i.e., a Pareto frontier for the set $\mathcal{S}$.

Analogously, in the column-wise sorting we rank in ascending order the points of $\mathcal{S}$ by their y-coordinate (by column). Let us denote this set by $\mathcal{S}_c = \{A_{c_1}, A_{c_2}, \ldots, A_{c_n}\}$, where A_{c_k} is such that $y_{c_k} \leq y_{c_{k+1}}$, $k = 1, 2, \ldots, n-1$.

Starting from the first point A_{c_1} in $\mathcal{S}_c$, we consider each subsequent point A_{c_k} and remove it from $\mathcal{S}_c$ if its x-coordinate is smaller than the x-coordinate of the previous point $A_{c_{k-1}}$ in $\mathcal{S}_c$. In other words, we only keep the points whose x-coordinates are non-decreasing as we move along the columns. This results in a final set of points that are sorted both by their x- and y-coordinates. The set obtained is a subset of $\mathcal{S}$ made by the points column-wise sorted in ascending order, we denote this subset with $\mathcal{O}_c$. Note that the set $\mathcal{S}_c \backslash \mathcal{O}_c$ is made by the points that are not dominated by any other, that is, a Pareto frontier for the set $\mathcal{S}$.

The initial set of pivots for the iterative Voronoi scheme consists of the largest set of ranked points from either $\mathcal{O}_r$ or $\mathcal{O}_c$, determined by Pareto dominance. Specifically, if $|\mathcal{O}_c| > |\mathcal{O}_r|$, then we choose $\mathcal{O}_0 = \mathcal{O}_c$, otherwise we choose $\mathcal{O}_0 = \mathcal{O}_r$. Here, $|\cdot|$ denotes the cardinality of the set $\cdot$. It is worth noting that this splits the original data set $\mathcal{S}$ into two disjoint subsets: the subset $\mathcal{O}_0$ containing the ranked points, and the subset $\mathcal{U}_0 = I \backslash \mathcal{O}_0$ containing the unranked points.

2.2.2 Iterative Scheme

Given the initial set of pivots $\mathcal{O}_0 = \{P_1, P_2, \ldots, P_m\}$, where $m = \max |\mathcal{O}_r|, |\mathcal{O}_c|$ and P_i is the ith sorted pivot of $\mathcal{O}_0$ (i.e., $P_{i+1} \succcurlyeq P_i$ for $i = 1, 2, \ldots, m-1$), we compute the Voronoi partition of $[0, 1]^2$ generated by the pivots. For $i = 1, 2, \ldots, m$, we denote by C_i the Voronoi cell associated with the pivot P_i. In other words, C_i is the subset of $[0, 1]^2$ consisting of points closer to P_i than to any other pivot P_j with $j \neq i$. It should be noted that the union of all Voronoi cells C_i covers the entire unit square $[0, 1]^2$, i.e., $[0, 1]^2 = \bigcup_{i=1}^{m} C_i$.

For each pivot P_i, we select, if possible, the point A_{k_i} among the remaining unranked points in $\mathcal{U}_0$ that is closest to the pivot P_i, where $i = 1, 2, ..., m$. Note that some pivots may have no points of $\mathcal{U}_0$ in their cell, in which case no point is selected. Next, we insert the point A_{k_i} among the ranked points using the following rule:

If the point A_{k_i} dominates P_i and is dominated by P_{i+1}, we place the point A_{k_i} after the pivot P_i in the ranking.
If the point A_{k_i} dominates P_{i-1} and is dominated by P_i, we place the point A_{k_i} before the pivot P_i in the ranking.

If the point A_{k_i} dominates P_i and is not dominated by P_{i+1}, we place the point A_{k_i} after the pivot P_i in the ranking and remove P_{i+1} from $\mathcal{O}_0$.
If the point A_{k_i} dominates P_{i-1} and is not dominated by P_i, we place the point A_{k_i} before the pivot P_i in the ranking and remove P_{i-1} from $\mathcal{O}_0$.
If A_{k_i} neither dominates nor is dominated by P_i, we place A_{k_i} after the pivot P_i in the ranking if the Euclidean distance between A_{k_i} and P_{i+1} is lower than the distance between A_{k_i} and P_{i-1}, and before the pivot P_i in the ranking otherwise (see Novak et al., 2011).

Note that in the first iteration, to ensure that all points in $\mathcal{S}$ are enclosed in the Voronoi cells of the actual pivots in $\mathcal{O}_0$, we add the fictitious pivots $P_0 = (0, 0)$ and $P_{m+1} = (1, 1)$. At each iteration, we update the set of pivots based on the positions of the selected points in the previous iteration. The new set of pivots $\mathcal{O}_k$ is used to compute the Voronoi partition for the next iteration. The algorithm stops when the set of pivots is no longer updated.
Finally, we update the set of ranked points $\mathcal{O}_0$ adding to it (in ascending order) the points A_{k_i} using the rule explained above. We denote by $\mathcal{O}_1 \subset \mathcal{S}$ the set of ordered points of $\mathcal{S}$ found at the first iteration and we splits $\mathcal{S}$ into the two disjoint sets $\mathcal{O}_1$ and $\mathcal{U}_1$. Note that, since $\{C_1, C_2, \ldots, C_m\}$ is a partition of $[0, 1]^2$ and $\mathcal{S} \subset [0, 1]^2$ then $\mathcal{O}_0 \subset \mathcal{O}_1$.
In the kth iteration, the method follows a similar procedure as the first iteration. It calculates the Voronoi partition linked to the set of pivots $\mathcal{O}_{k-1}$ determined in the previous $(k-1)$-th iteration and divides the set $\mathcal{S}$ into two subsets: $\mathcal{O}_k$ (ranked points) and $\mathcal{U}_k$ (unranked points). The procedure stops at iteration M when $\mathcal{O}_M = \mathcal{O}_{M-1}$. Table 1 summarizes the sorting algorithm.

3 An Illustrative Example

To provide an empirical evidence of the effectiveness of the ranking algorithm presented in Sect. 2, we apply it to the problem of ranking countries based on two indices: the *Happiness Index* and the *Human Development Index*. In the following paragraph, we will provide a brief overview of these indices before using the Voronoi method to compute the new ranking.

3.1 Happiness and Human Development: Two Faces of the Same Coin

For over half a century, a country's economic growth has been associated with an increase in Gross Domestic Product (GDP) per capita. As a result, GDP has become the most commonly used measure of a country's economic progress (Costanza et al., 2009).
Over the past two decades, there has been extensive research into the limitations of using GDP as a measure of a country's quality of life or societal well-being (see Stiglitz et al., 2018 and others). As a result, it is now widely accepted that per capita GDP or income alone are inadequate indicators of development, as they fail to account for the quality of life.
The idea that well-being is a multifaceted concept that cannot be captured by GDP alone (Stiglitz et al., 2009) has led to an increased interest in multidimensional indicators of development, such as the *Human Development Index* (HDI).
The HDI is an annual index computed since 1990 from the United Nation Development Programe (UNDP). It is defined as the geometric mean of three (normalized) dimensions: Health (Life Expectancy at birth), Education (that is the arithmetic mean between the Mean

Table 1 Sorting algorithm

determine the initial set of pivot $\mathcal{O}_0$

set $\mathcal{U}_0 = \mathcal{S} \backslash \mathcal{O}_0$

set k=0

while $\mathcal{U}_k \neq \emptyset$ **do**

set k=k+1

compute the Voronoi diagram associated with the set $\mathcal{O}_k$

for $i = 1, 2, \ldots, |\mathcal{O}_k|$ **do**

determine the point $A_{k_i} \in \mathcal{U}_k$ closest to the pivot P_i

if $A_{k_i} \succcurlyeq P_i$ and $A_{k_i} \preccurlyeq P_{i+1}$ **then**

$|\mathcal{O}_k| = |\mathcal{O}_k| + 1$

$P_{i+1} = A_{k_i}$ and $P_{i+2} = P_{i+1}$, for $i = 1, 2, \ldots, |\mathcal{O}_k| - 1$

else if $A_{k_i} \succcurlyeq P_i$ and $A_{k_i} \not\preccurlyeq P_{i+1}$ **then**

$P_{i+1} = A_{k_i}$

else if $A_{k_i} \preccurlyeq P_i$ and $A_{k_i} \succcurlyeq P_{i-1}$ **then**

$|\mathcal{O}_k| = |\mathcal{O}_k| + 1$

$P_i = A_{k_i}$ and $P_{i+1} = P_i$, for $i = 1, 2, \ldots, |\mathcal{O}_k|$

else if $A_{k_i} \preccurlyeq P_i$ and $A_{k_i} \not\succcurlyeq P_{i-1}$ **then**

$P_i = A_{k_i}$

else

if $d(A_{k_i}, P_{i+1}) \leq d(A_{k_i}, P_{i-1})$ **then**

$|\mathcal{O}_k| = |\mathcal{O}_k| + 1$

$P_{i+1} = A_{k_i}$ and $P_{i+2} = P_{i+1}$, for $i = 1, 2, \ldots, |\mathcal{O}_k|$

else

$|\mathcal{O}_k| = |\mathcal{O}_k| + 1$

$P_{i-1} = A_{k_i}$ and $P_{i+1} = P_i$, for $i = 1, 2, \ldots, |\mathcal{O}_k|$

end if

end if

end for

end while

return $\mathcal{O}_k$

Years of Schooling and the Expected Years of Schooling), and Economic (GNI per capita).[2] Deb (2015) found a positive correlation between the rankings of HDI and per capita GDP, with the highest correlation observed for the low-income group of countries. Despite these findings, there are several differences between the rankings of GDP and HDI.[3] For instance, Qatar is ranked 13th out of 203 countries according to HDI, whereas it is ranked 42nd out of 192 according to GDP. Similarly, Hong Kong is ranked 24th out of 203 according to HDI, but is ranked 4th out of 192 according to GDP.

[2] All editions are available at: https://hdr.undp.org/en/global-reports.

[3] Data for the GDP index is sourced from *The World Bank's DataBank* https://databank.worldbank.org/reports.aspx?source=2&series=NY.GDP.PCAP.CD&country=#. Data for HDI is obtained from UNDP https://hdr.undp.org/. Both indices refer to the year 2021.

Notably, the HDI has been criticized for its predominantly national focus and limited consideration of a global perspective. Moreover, data collection challenges impede cross-country comparisons, thereby reducing the practical applicability of the HDI in official statistics. Various authors have attempted to address these shortcomings.

Far from being exhaustive, Sagar and Najam (1998) proposed the Reformed HDI, which incorporates three modifications to the original index. One of these modifications involves using the logarithm of the unadjusted version of real GDP across all income ranges as a measure for estimating the standard of living. In another approach, Elvidge et al. (2012) introduced the Night Light Development Index (NLDI), derived from nighttime satellite imagery and population density, as an alternative to the HDI. Furthermore, Salvati et al. (2017) suggested a linear transformation of the NLDI to account for the fact that similar NLDI values may indicate vastly different levels of human development.
However, since our main focus is to illustrate the methodological approach, we utilize the classical HDI annually computed by UNDP.

In accordance with the recommendations of Stiglitz, Sen, and Fitoussi Stiglitz et al. (2009), the measurement of well-being should include both objective and subjective measures to capture the full spectrum of people's quality of life. This approach recognizes the importance of not only material wealth, but also of social and environmental factors that can impact people's overall well-being.

One of the most famous indices of happiness is the so-called *Happiness Index*[4] (HI), a subjective index of well-being annually computed by the Sustainable Development Solutions Network (SDNS) through the Gallup World Poll[5] (see Helliwell et al., 2022). The index is measured on a scale from 0 to 10, with 10 representing the best possible life for an individual and 0 representing the worst possible life. In order to evaluate subjective well-being and life satisfaction of the respondents, the Gallup advisory company uses the Cantrill ladder, also known as the Cantril Self-Anchoring Striving Scale or Cantril's ladder of life (Cantril, 1965). This tool provides individuals with a visual representation of a hypothetical ladder and asks the respondents to rate their current life satisfaction by selecting a step on the ladder that best represents their perception. Approximately 1000 responses are collected annually for each country, with sample weights used to ensure accuracy. As a result, the national happiness rankings are based on a three-year average, providing a larger sample size and more accurate estimates (Helliwell et al., 2022). The index provides a valuable insight into the subjective well-being of individuals, capturing the important context that GDP alone does not account for: how people feel about their lives.

The relationship between GDP and happiness has in-depth investigated in part motivated by the so-called *Easterlin Paradox*.[6]
To analyse this relationship, we compare per capita GDP and life satisfaction, in the long run (from 2003 to 2020) for world wide countries.[7] We focus on correlation, finding that it ranges from 0.6395 in 2007 and 0.831 in 2003. The average value is 0.7455. Moreover, the correlation is decreased in the last three years (from 0.8093 in 2018 to 0.7399 in 2020). Results

4 https://worldhappiness.report/.

5 https://www.gallup.com/.

6 The Easterlin paradox is an empirical relationship between income and measures of overall subjective well-being observed by Easterlin (1974) for USA. It claims that over time the long-term growth rates of happiness and income are not significantly correlated although in the short run or for low-income countries happiness varies directly with income, both among and within nations. See also Easterlin (1995).

7 Data come from https://ourworldindata.org/grapher/gdp-vs-happiness Retrieved on Febrary 8th March, 2023.

suggest that happiness, that is a proxy of life satisfaction, and GDP are not interchangeable since the two series are not moving in unison.

Thus, in one hand we find that there is a positive relationship between GDP and HDI, on the other hand there is a positive relationship between GDP and happiness, but both relationships are quite far from one. Following Hall and Helliwell (2014), we decide to investigate the relationship between happiness and HDI. Blanchflower and Oswald (2005) found that, despite Australia's high ranking according to the HDI, it performs relatively poorly on various happiness indicators. This suggests that there is still much to be understood in this important area.

Scholars who are interested in measuring happiness and human development have a shared interest in comprehending and quantifying well-being beyond the traditional economic metric, specifically without relying on GDP (Hall and Helliwell, 2014).

The research questions are: Do countries with higher HDI values truly experience higher life evaluations? Do European countries, which generally exhibit superior performance compared to the rest of the world, share a common ranking? To answer these questions, we analyze HDI and HI rankings separately for the whole sample and then concentrate in-depth on European countries.

3.2 European Countries Performance

Data for the HI comes from the 2022 edition of *World Happiness Report* (Helliwell et al., 2022), that ranks 146 countries according to their average values over 2019–2021. The index ranges from 2.40 to 7.82. Finland, Denmark (7.64) and Iceland (7.56) occupy the first positions, whereas in the bottom of the ranking there are Afghanistan, Malawi (3.75) and Tanzania (3.70).

The data for HDI come from the United Nations Development Programe (UNDP) dataset[8]. Data refer to 2021 and cover 191 countries around the world. The index range in [0, 1]. Switzerland (0.962), Norway (0.961) and Iceland (0.959) are in the top positions, whereas, the countries with the lower values are South Sudan (0.385), Chad (0.394) and Niger (0.400).

We restrict our analysis to 34 European Countries.[9] Table 2 reports summary statistics for the 34 countries included in the analysis. The values of the *Happiness Index* are reported in the range [0, 1][10] in order ensure the comparability with the results with the *Human Development Index*.

The Kendall's Tau coefficient (Kendall, 1938) is a non-parametric measure used to evaluate the relationship between two distributions of ranked data. In our case, we utilize it to examine the correlation between the HDI and HI rankings for European countries. The Kendall's rank correlation tau for European countries is 0.7172, indicating a statistically significant result with a very small probability value. However, it is important to note that the Kendall's Tau correlation for the entire sample of 118 countries with data for both indices is slightly lower, at 0.6772. This suggests that although there is a correlation between the two rankings, it is

[8] https://hdr.undp.org/content/human-development-report-2021-22, accessed on February 2021.

[9] Albania (ALB), Austria (AUT), Belgium (BEL), Bulgaria (BGR), Bosnia and Herzegovina (BIH), Belarus (BLR), Switzerland (CHE), Czechia (CZE), Germany (DEU), Denmark (DNK), Spain (ESP), Estonia (EST), Finland (FIN), France (FRA), Greece (GRC), Croatia (HRV), Hungary (HUN), Ireland (IRL), Iceland (ISL), Italy (ITA), Lithuania (LTU), Luxembourg (LUX), Latvia (LVA), Malta (MLT), Montenegro (MNE), Norway (NOR), Poland (POL), Portugal (PRT), Romania (ROU), Serbia (SRB), Slovakia (SVK), Slovenia (SVN), Sweden (SWE), Ukraine (UKR).

[10] Since the original scale is [0, 10], values are simply divided by 10.

Table 2 Summary statistics for European Countries

	HDI	HI
Min.	0.7730	0.5084
Max.	0.9620	0.7821
Mean	0.8833	0.6513
1st Qu.	0.8465	0.6095
Median	0.8895	0.6457
3rd Qu	0.9353	0.7039
Std.dev	0.0571	0.0717
Coef.var	0.0646	0.1101
CI.mean.0.95	0.0199	0.0250

Table 3 Top and bottom ten European Countries according to HDI and HI

Position	HDI Rank	HI Rank	Position	Rank	HI Rank
1	Switzerland	Finland	34	Ukraine	Ukraine
2	Norway	Denmark	33	Bosnia-Herz	Albania
3	Iceland	Iceland	32	Bulgaria	Bulgaria
4	Denmark	Switzerland	31	Albania	Montenegro
5	Sweden	Luxembourg	30	Serbia	Bosnia-Herz
6	Ireland	Sweden	29	Belarus	Belarus
7	Germany	Norway	28	Romania	Greece
8	Finland	Austria	27	Montenegro	Portugal
9	Belgium	Ireland	26	Hungary	Hungary
10	Luxembourg	Germany	25	Slovakia	Poland

not perfect. In other words, the HDI and HI rankings can produce different country rankings. This observation is supported by Table 3, which illustrates that not all countries in the top or bottom ten positions are the same in both rankings.

Therefore, by applying the Voronoi algorithm, we introduce a new ranking that combines the information from the two original rankings into a single comprehensive measure.

3.3 The Voronoi Ranking

We apply the sorting method developed in Sect. 2. The algorithm converges after 5 iterations, and all points are ordered.

Figure 1 displays the final Voronoi partition and the corresponding ranking obtained using the sorting algorithm of Sect. 2.

The Spearman's rank correlation coefficient between Voronoi and HDI is 0.9297, while for Voronoi and HI, it is 0.9875. Figure 2 illustrates the ranking differences between HDI and Voronoi (left panel) and HI and Voronoi (right panel). The points that deviate farther from the bisector of the plane indicate greater disparities between the two rankings. Additionally, these correlation values are significantly higher than the Spearman's rank correlation coefficient between HDI and HI (0.8943), indicating that the new method offers a more robust ranking.

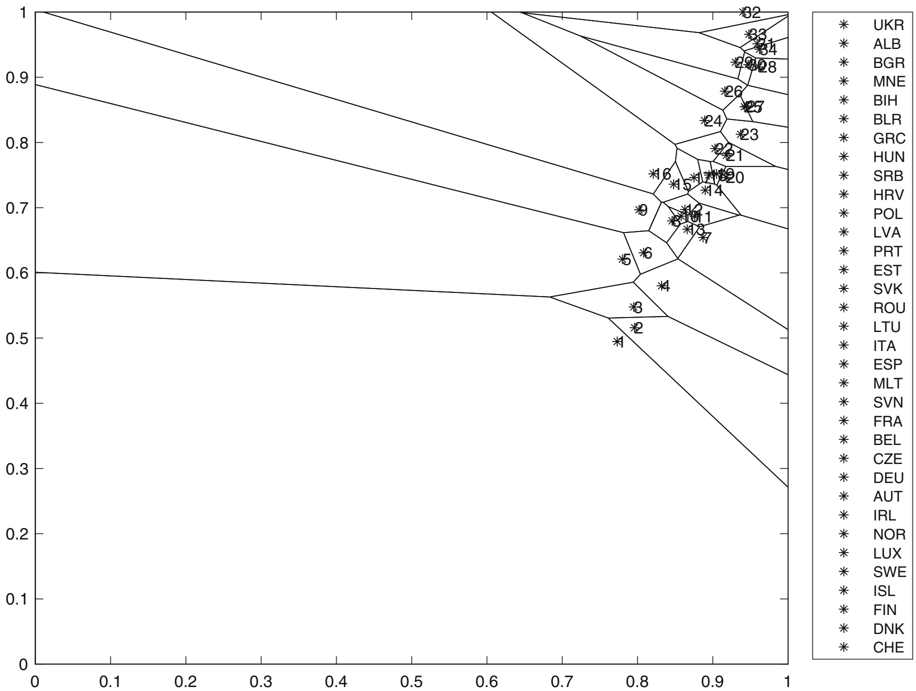


Fig. 1 Final Voronoi partition and corresponding ranking. Countries on the right are ordered from the last position in the ranking (UKR) to the first position (CHE)

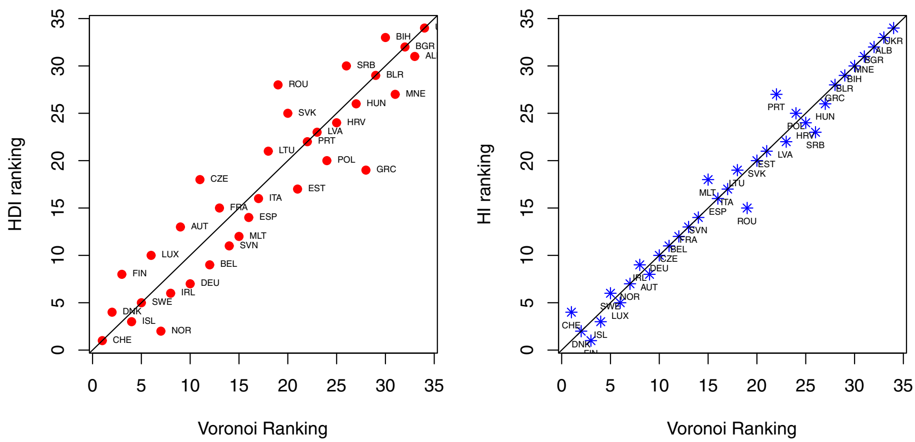


Fig. 2 HDI and HI rankings versus Voronoi

According to the Voronoi algorithm, Switzerland (CHE) occupies the first position, followed by Denmark (DNK) and Finland (FIN). On the bottom of the Voronoi ranking, we have Bulgaria (BGR, 32), Albania (ALB, 33), and Ukraine (UKR, 34).

Figure 3 reports the differences between the ranking produced by HDI and HI. Countries are ranked according to Voronoi algorithm.

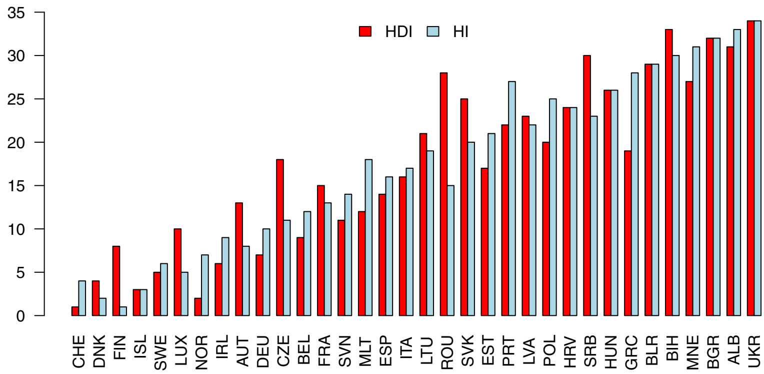


Fig. 3 Differences between HDI and HI

The ranking obtained with the new procedure (column V) as well as the rankings according to HDI and HI (columns HDI and HI, respectively) are reported in Table 4.

Additionally, we calculate the absolute differences in rankings between pairs of indices: Voronoi vs HDI, Voronoi vs HI, and HDI vs HI. The mean absolute difference between HDI and HI rankings is 3.471. On the other hand, the mean absolute differences are lower when comparing Voronoi (column V) with HDI (2.824) and Voronoi (column V) with HI (0.8824). These results indicate that the Voronoi ranking could better capture both aspects of human development and well-being, as it shows lower discrepancies with both HDI and HI rankings.

To further investigate the potential of our method, we analyse the changes in ranking. Among the different methods proposed in literature, we follow (Saisana et al., 2005) and (Karagiannis & Karagiannis, 2020). We apply five different methods: (i) the *Average shift in ranking*, (ii) the *k-Average shift in ranking*, (iii) the precision, (iv) the k-precision and (v) the quartiles precision. Thus, we compute the deviations of both the HDI and HI ranking with respect to Voronoi ranking.

The first index, the so-called *Average shift in ranking* (ASR), is defined as

$$ASR = \frac{\sum_{i=1}^{m} |rank_i^V - rank_i^H|}{m}.$$

Here, m represents the number of ranked units, $rank_i^V$ denotes the position occupied by the ith unit according to the Voronoi method, and $rank_i^H$ represents the position occupied by the ith unit according to HDI ($rank_i^{HDI}$) or Happiness ($rank_i^{HI}$), respectively.
The ASR provides an average measure of positional change observed across multiple rankings. A higher average shift indicates a greater level of inconsistency or variation between the rankings, while a lower average shift suggests a higher degree of agreement or similarity.

The ASR_k index is similar to ASR, but instead of considering the entire distribution, it focuses on a specific subgroup of items. In this case, we concentrate on k items. Specifically, we focus on the top 10 countries and the bottom 10 countries, ranked according to Voronoi. We denote the values computed in the top and bottom sub-samples as ASR^j_{T10} and ASR^j_{B10}, respectively, where j represents Voronoi, HDI, or Happiness.

Figure 4 reports a graphical representation of the first two methods.

The third method, the *precision*, measures the percentage of equal rankings. To capture the differences between ranking, instead of the precision, we compute the complement to 100.

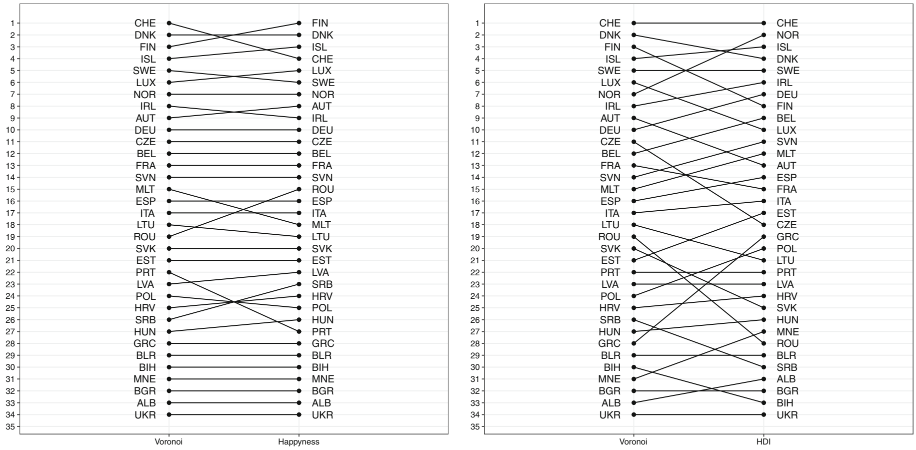
Table 4 Comparison of Voronoi ranking (column V) with HDI and HI rankings

ISO	Country	V	HDI	HI	V versus HDI	V versus HI	HDI versus HI
ALB	Albania	33	31	33	2	0	2
AUT	Austria	9	13	8	4	1	5
BEL	Belgium	12	9	12	3	0	3
BGR	Bulgaria	32	32	32	0	0	0
BIH	Bosnia and Herzegovina	30	33	30	3	0	3
BLR	Belarus	29	29	29	0	0	0
CHE	Switzerland	1	1	4	0	3	3
CZE	Czechia	11	18	11	7	0	7
DEU	Germany	10	7	10	3	0	3
DNK	Denmark	2	4	2	2	0	2
ESP	Spain	16	14	16	2	0	2
EST	Estonia	21	17	21	4	0	4
FIN	Finland	3	8	1	5	2	7
FRA	France	13	15	13	2	0	2
GRC	Greece	28	19	28	9	0	9
HRV	Croatia	25	24	24	1	1	0
HUN	Hungary	27	26	26	1	1	0
IRL	Ireland	8	6	9	2	1	3
ISL	Iceland	4	3	3	1	1	0
ITA	Italy	17	16	17	1	0	1
LTU	Lithuania	18	21	19	3	1	2
LUX	Luxembourg	6	10	5	4	1	5
LVA	Latvia	23	23	22	0	1	1
MLT	Malta	15	12	18	3	3	6
MNE	Montenegro	31	27	31	4	0	4
NOR	Norway	7	2	7	5	0	5
POL	Poland	24	20	25	4	1	5
PRT	Portugal	22	22	27	0	5	5
ROU	Romania	19	28	15	9	4	13
SRB	Serbia	26	30	23	4	3	7
SVK	Slovakia	20	25	20	5	0	5
SVN	Slovenia	14	11	14	3	0	3
SWE	Sweden	5	5	6	0	1	1
UKR	Ukraine	34	34	34	0	0	0

$$D = 100 - \frac{\sum_{i=1}^{m} \mathbb{1}_{\{rank_i^V = rank_i^H\}}}{m} \cdot 100.$$

In this way, a lower value of D indicates a higher level of concordance between the two indices.

The fourth method, known as the *k-precision*, is calculated in a similar manner to D, but it specifically examines the same subgroup of countries as defined in ASR_k. This measure



(a) Comparison of Voronoi and Happiness rankings.

(b) Comparison of Voronoi and HDI rankings.

Fig. 4 Ranking changes

Table 5 Five approaches for computing ranking variations

Method	Voronoi versus HDI	Voronoi versus Happiness	HDI versus Happiness
ASR	2.8235	0.8823	3.4706
ASR_{T10}	2.6000	1.0000	3.4000
ASR_{B10}	2.400	0.5000	2.5000
D	79.4118%	47.0588%	82.3529%
D_{T10}	94.1176%	91.1765%	97.0588%
D_{B10}	91.1765%	79.4118%	85.2941%
Quartile shift	0.1765	0.1765	0.3529

evaluates the accuracy of the top k items by calculating the percentage of common elements between a group of k elements in both rankings.

Finally, according to the last method, countries are classified into quartiles, and only ranking variations that result in a shift from one quartile to another are analysed.

Table 5 presents the results of the five ranking comparison methods. As expected, the ASR index produces smaller values for the Voronoi-HDI and Voronoi-HI pairs compared to the HDI and HI pair. The only exception to this pattern is the D_{B10} method for Voronoi versus HDI, which could be attributed to the small sample size. The higher values exhibited by Voronoi can be interpreted as a strength of the proposed algorithm.

4 Conclusions

In this paper, we propose a new method for ranking items, such as countries, individuals, or firms, according to two indices when constructing a composite indicator that combines the two dimensions is not suitable. Among its other advantages, the method can be effectively implemented in parallel. Furthermore, the proposed approach is highly versatile and applicable in diverse contexts. For example, traders in financial markets may be interested in ranking stocks based on return and volatility for asset allocation purposes. Medical directors could use this method to rank hospital wards according to complexity and efficiency in order to allocate funds effectively. Sales managers may find it useful to order clients based on two characteristics (dimensions) to make decisions regarding benefit allocation. Similarly, policy makers in well-being analysis could compare countries based on two indices. Additionally, the method can be applied to both micro and macro data, allowing for ranking of municipalities, regions, countries, as well as individuals or firms, among others.

In this paper, we showed the usefulness of the proposed ranking-method, based on the Voronoi algorithm, by ranking 34 European countries according to the HDI and HI. The correlation coefficients between the Voronoi-based ranking and both the HDI ranking, as well as the HI ranking, were found to be remarkably high. This observation strongly indicates that our new method has the potential to offer a more comprehensive and accurate summary of the information embedded within the original indices.

Further research can be pursued in two distinct directions. From a theoretical standpoint, it would be interesting to generalize the method in at least two ways. Firstly, by increasing the number of indicators and transitioning from a two-dimensional to a multidimensional approach, we can gain insights into the interplay of different factors in generating comprehensive rankings. Secondly, by extending the space from the square $[0, 1]^2$ to the positive bi-dimensional half-plane $\mathbb{R}^2_+$, we can explore how rankings change when considering increasingly broader ranges of values for the indices.
From an empirical perspective, it would be valuable to apply the method to different contexts and assess its performance across various datasets. By doing so, we could gain a better understanding of the generalizability of the method and its potential applications in different fields.

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Declarations

Competing interests The authors have no competing interests to declare that are relevant to the content of this article. The authors have no relevant financial or non-financial interests to disclose. All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors have no financial or proprietary interests in any material discussed in this article.

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