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Calculus of Variations

Nonlinear scalar field (p_1, p_2)-Laplacian equations in $\mathbb{R}^N$: existence and multiplicity

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Abstract
In this paper, we deal with the following class of (p_1, p_2)-Laplacian problems:

$$\begin{cases} -\Delta_{p_1} u - \Delta_{p_2} u = g(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p_1}(\mathbb{R}^N) \cap W^{1,p_2}(\mathbb{R}^N), \end{cases}$$

where $N \geq 2, 1 < p_1 < p_2 \leq N$, Δ_{p_i} is the p_i-Laplacian operator, for $i = 1, 2$, and $g : \mathbb{R} \to \mathbb{R}$ is a Berestycki-Lions type nonlinearity. Using appropriate variational arguments, we obtain the existence of a ground state solution. In particular, we provide three different approaches to deduce this result. Finally, we prove the existence of infinitely many radially symmetric solutions. Our results improve and complement those that have appeared in the literature for this class of problems. Furthermore, the arguments performed throughout the paper are rather flexible and can be also applied to study other p-Laplacian and (p_1, p_2)-Laplacian equations with general nonlinearities.

Mathematics Subject Classification 35A15 · 35J92 · 35J60

1 Introduction

In this paper, we consider the following class of (p_1, p_2)-Laplacian problems:

$$\begin{cases} -\Delta_{p_1} u - \Delta_{p_2} u = g(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p_1}(\mathbb{R}^N) \cap W^{1,p_2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$, $1 < p_1 < p_2 \leq N$, $\Delta_{p_i} u := \mathrm{div}(|\nabla u|^{p_i-2}\nabla u)$ is the p_i-Laplacian operator, for $i = 1, 2$, and $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function satisfying the following assumptions:

$(g1)$ $-\infty < \liminf_{t\to 0^+} \frac{g(t)}{t^{p_1-1}} \leq \limsup_{t\to 0^+} \frac{g(t)}{t^{p_1-1}} < 0$ when $p_2 < N$, and $\lim_{t\to 0^+} \frac{g(t)}{t^{p_1-1}} \in (-\infty, 0)$ when $p_2 = N$.

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($g2$) $-\infty \leq \limsup_{t\to+\infty} \frac{g(t)}{t^{p_2^*-1}} \leq 0$ when $p_2 < N$, and $\limsup_{t\to+\infty} \frac{g(t)}{\exp\left(\alpha t^{\frac{N}{N-1}}\right)} \leq 0$ for all $\alpha > 0$ when $p_2 = N$.
($g3$) There exists $\xi > 0$ such that $G(\xi) > 0$, where $G(t) := \int_0^t g(\tau)\,d\tau$.

As pointed out in [16], problem (1.1) comes from the research of stationary solutions for the general reaction-diffusion system

$$u_t = \mathrm{div}(D(u)\nabla u) + c(x,u) \quad \text{where } D(u) := |\nabla u|^{p_1-2} + |\nabla u|^{p_2-2},$$

which finds applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. In such situations, u denotes a concentration, $\mathrm{div}(D(u)\nabla u)$ represents the diffusion with diffusion coefficient $D(u)$, whereas the reaction term $c(x,u)$ relates to source and loss processes. Usually, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to u. Another important example where Eq. (1.1) emerges is the study of soliton-like solutions of the following nonlinear Schrödinger equation

$$\imath \psi_t = -\Delta\psi + V(x)\psi - \Delta_{p_2}\psi + W'(x,\psi)$$

proposed by Derrick as a model for elementary particles. We also observe that the (p_1, p_2)-Laplacian operator $\Delta_{p_1} + \Delta_{p_2}$ is a particular case of the well-known double-phase operator $\mathrm{div}(|\nabla u|^{p_1-2}\nabla u + a(x)|\nabla u|^{p_2-2}\nabla u)$, with $a \geq 0$ and bounded, whose corresponding energy functional was analyzed in the context of problems of homogenization and elasticity [48, 49], and of the calculus of variations [34, 38]. For some interesting existence and multiplicity results for (p_1, p_2)-Laplacian problems in $\mathbb{R}^N$ and in bounded domains, we refer to [5, 7, 8, 10, 21–23, 26, 31, 41, 42] and the references therein.

When $p_1 = p_2 = 2$, problem (1.1) boils down to the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u = g(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2}$$

In the seminal paper [12, Theorem 1], Berestycki and Lions used a constrained minimization argument to prove that, under assumptions ($g1$)–($g3$) with $p_1 = p_2 = 2$ and $N \geq 3$, there exists a ground state solution (or least energy solution) which is positive and radially symmetric. In [13, Theorem 6] the authors obtained infinitely many radially symmetric solutions. In [11, Theorem 1] Berestycki, Gallouët and Kavian extended the result in [12] for the case $N = 2$. Subsequently, Jeanjean and Tanaka [29, Theorem 0.2] provided a mountain pass characterization of ground state solutions to (1.2). In [25, Theorem 1.3] Hirata, Ikoma and Tanaka developed mountain pass and symmetric mountain pass approaches to generalize the results in [11–13]. The authors in [25] employed the auxiliary functional introduced in [27, Section 2] and constructed a Pohozaev-Palais-Smale sequence in the radial subspace $H^1_{\mathrm{r}}(\mathbb{R}^N)$, that is, a Palais-Smale sequence in $H^1_{\mathrm{r}}(\mathbb{R}^N)$ satisfying asymptotically the Pohozaev identity

$$\left(\frac{N-2}{2}\right) \|\nabla u\|^2_{L^2(\mathbb{R}^N)} = N \int_{\mathbb{R}^N} G(u)\,dx,$$

which has the advantage to be bounded in $H^1_{\mathrm{r}}(\mathbb{R}^N)$ and that, up to a subsequence, strongly converges to a weak solution to (1.2) (thanks to the compactness of the embedding $H^1_{\mathrm{r}}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$). Recently, Mederski [35, Theorem 1.3] gave a new proof of the existence of a ground state solution to (1.2) by using a variational approach based on a critical point theory built on the Pohozaev manifold, and combining a concentration-compactness approach with profile decompositions. In [35] the author also examined the existence and multiplicity of nonradial solutions to (1.2). Motivated by [35], Jeanjean and Lu [30] proposed an alternative and more elementary approach to recover the results in [35]. In particular, in [30, Theorems 1.1 and 1.2] the authors reestablished the results in [12, 13] by means of the monotonicity trick [28, 44] and a decomposition result for bounded Palais-Smale sequences (see also [35, Theorem 1.4] and [46, Chapter 8]). This decomposition result is useful to recover compactness, up to subsequences and translations, of bounded Palais-Smale sequences in $H^1(\mathbb{R}^N)$ for C^1-functionals with general subcritical nonlinearities.

When $p_1 = p_2 = p \in (1, +\infty)$, problem (1.1) becomes the following quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = g(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

In [17, Theorem 1.1] Citti investigated the existence of a positive radially symmetric ground state solution to (1.3) with $p \in (1, N)$ in the spirit of [12]; see also [20, Theorem 1]. Later, do Ó and Medeiros [19, Theorems 1.4, 1.6 and 1.8] generalized the results in [17, 29] considering even the case $p = N$. Finally, Byeon et al. [15, Section 3] showed that every ground state solution to (1.3) has a constant sign and, if it tends to zero at infinity, then it is, up to a translation, radially symmetric and monotone with respect to the radial variable.

For what concerns the (p_1, p_2)-Laplacian problem (1.1) under general assumptions $(g1)$–$(g3)$, only two results appeared in the literature and both supposed $1 < p_1 < p_2 < N$ and $N \geq 3$. More precisely, Pomponio and Watanabe [42, Theorem 1.2] obtained the existence of a positive radially symmetric ground state solution to (1.1) applying the monotonicity trick, and in [7, Theorem 3.1] the author proved the existence of a positive ground state solution to (1.1) utilizing Pohozaev-Palais-Smale sequences.

Motivated by the above-mentioned papers for the Laplacian and the p-Laplacian case with $p \in (1, N] \setminus \{2\}$, the purpose of this work is to improve and complement the results in [7, 42]. More precisely, the main results of this paper can be stated as follows.

Theorem 1.1 *Let $N \geq 2$ and $1 < p_1 < p_2 \leq N$. Assume that $(g1)$–$(g3)$ hold. Then (1.1) has a ground state solution.*

Theorem 1.2 *Let $N \geq 2$ and $1 < p_1 < p_2 \leq N$. Assume that $(g1)$–$(g3)$ hold. Then (1.1) has infinitely many radially symmetric solutions.*

The proofs of Theorems 1.1 and 1.2 rely on suitable variational arguments. First we show that every weak solution of (1.1) belongs to $L^\infty(\mathbb{R}^N) \cap C^{1,\sigma}_{loc}(\mathbb{R}^N)$, for some $\sigma \in (0, 1)$, and fulfills a Pohozaev type identity; see Theorem 3.1. We also get an exponential decay estimate at infinity; see Theorem 3.2. As in [7, Section 3], we introduce the energy functional associated with (1.1), namely,

$$L(u) := \sum_{i=1}^{2} \frac{1}{p_i} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(u)\,dx \quad \text{for all } u \in \mathcal{W} := W^{1,p_1}(\mathbb{R}^N) \cap W^{1,p_2}(\mathbb{R}^N),$$

and we demonstrate that L has a mountain pass geometry [6]; see Lemma 4.1. Thanks to an auxiliary functional and the general minimax principle [46, Theorem 2.8], we produce a Pohozaev-Palais-Smale sequence $(u_n) \subset \mathcal{W}$ for L at the mountain pass level c_{MP}; see Proposition 4.1. In addition, we are able to prove that (u_n) is bounded in $\mathcal{W}$; see Lemma 4.2. After that, we establish an almost everywhere convergence of the gradients of Pohozaev-Palais-Smale sequences; see Lemma 4.3. This result will be also convenient to apply the

Brezis-Lieb lemma [14, Theorem 1] to the gradients. Then we develop a concentration-compactness type argument to show that, up to translations and extraction of a subsequence, (u_n) strongly converges in $\mathcal{W}$ to a weak solution u to (1.1); see Proposition 4.2. To verify that u is indeed a ground state solution to (1.1), we exploit the fact that every weak solution to (1.1) satisfies a Pohozaev type identity and we construct an optimal path in the spirit of [29, Lemma 2.1] (note that the construction when $p_2 = N$ is much more elaborate with respect to $p_2 < N$); see Proposition 4.4. Hence we conclude that the mountain pass level c_{MP} coincides with the ground state energy level c_{LE}. Moreover, we derive the compactness, modulo translations, of the set of ground state solutions to (1.1); see Proposition 4.3. As in [30], we also establish a new decomposition result for bounded Palais-Smale sequences in the (p_1, p_2)-Laplacian framework; see Theorem 5.1. We recall that decomposition results for Palais-Smale sequences associated with quasilinear problems in bounded and unbounded domains can be found in [4, Theorem 2], [9, Proposition 1], [36, Theorem 1.1], and [37, Theorem 1.2]. However, in such papers, no general nonlinearities were considered. Our decomposition result allows us to exhibit a second proof of Theorem 1.1, whereas a third proof of Theorem 1.1 will be obtained by combining the monotonicity trick and the aforesaid decomposition result. Finally, by virtue of a symmetric mountain pass approach, we give the proof of Theorem 1.2. We stress that our proofs are much more difficult and intriguing with respect to the Laplacian and p-Laplacian cases. In fact, due to the presence of the (p_1, p_2)-Laplacian operator, which is nonlinear and not homogeneous in scaling, our calculations are much more complicated and an accurate analysis will be carried out to handle the combination of two different p-Laplacians. Furthermore, we are able to treat in a unified way the cases $p_2 < N$ and $p_2 = N$. We emphasize that our proofs are rather flexible and also work, with slight modifications, for the p-Laplacian problem (1.3). In this manner, we deduce alternative proofs of the results in [17, 19] and extend [30, Theorems 1.1 and 1.2] to the p-Laplacian setting (notice that in [30] the authors studied (1.2) for $N \geq 3$ and without considering general subcritical exponential nonlinearities). Moreover, the multiplicity result in Theorem 1.2 turns out to be completely new even in the p-Laplacian framework. We believe that the approaches developed along this paper can be applied to investigate other various p-Laplacian and (p_1, p_2)-Laplacian problems with general nonlinearities.

The paper is organized as follows. In Sect. 2, we collect some notations and definitions, and we establish some useful lemmas. In Sect. 3, we explore the regularity of solutions to (1.1), we prove a Pohozaev type identity and an exponential decay estimate. In Sect. 4, we present a first proof of Theorem 1.1. In Sect. 5, we provide a second proof of Theorem 1.1. The last section is devoted to the third proof of Theorem 1.1 and the proof of Theorem 1.2.

2 Notations and some preliminaries

For any real valued function $u : \mathbb{R}^N \to \mathbb{R}$, we put $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. Let $p \in [1, +\infty)$. The Sobolev space $W^{1,p}(\mathbb{R}^N)$ given by

$$W^{1,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\right\}$$

is equipped with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} := \left(\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

or sometimes with the equivalent norm $\|\nabla u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}$. It is well-known that $W^{1,p}(\mathbb{R}^N)$ is a separable reflexive Banach space for all $p \in (1, +\infty)$ (see [2, Theorems 3.3 and 3.6]), and that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$ (see [2, Corollary 3.23]). Set

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ +\infty & \text{if } N = p. \end{cases}$$

Theorem 2.1 [2, Theorems 4.12, 4.31 and 6.3] *Let $p \in [1, +\infty)$ and $N \geq 2$. If $p \in [1, N)$, then the following Sobolev inequality holds:*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq S_*(N, p)\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \tag{2.1}$$

where $S_*(N, p) > 0$ denotes the best Sobolev constant and $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to $\|\nabla \cdot \|_{L^p(\mathbb{R}^N)}$, or equivalently,

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N \right\}.$$

Furthermore, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p^*]$ and compactly embedded in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p^*)$. If $p = N$, then $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $q \in [p, p^*)$ and compactly embedded in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p^*)$. Finally, if $p > N$, then $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$ and compactly embedded in $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ for all $\alpha \in (0, 1 - \frac{N}{p})$.

When $p = N$, we have the following Trudinger-Moser inequality.

Theorem 2.2 [1, Theorem 0.1] *Let $N \geq 2$ and $\alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. Then for every $\alpha \in (0, \alpha_N)$ there exists $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_{L^N(\mathbb{R}^N)}} \right)^{\frac{N}{N-1}} \right) dx \leq C_\alpha \frac{\|u\|^N_{L^N(\mathbb{R}^N)}}{\|\nabla u\|^N_{L^N(\mathbb{R}^N)}} \quad \text{for all } u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}, \tag{2.2}$$

where

$$\Phi_N(t) := \exp(t) - \sum_{k=0}^{N-2} \frac{t^k}{k!} = \sum_{k=N-1}^{+\infty} \frac{t^k}{k!}.$$

From Theorem 2.2 we easily derive that, for all fixed $\alpha \in (0, \alpha_N)$ and $K > 0$, it holds

$$\int_{\mathbb{R}^N} \Phi_N \left(\alpha \left(\frac{|u(x)|}{K} \right)^{\frac{N}{N-1}} \right) dx \leq C_\alpha \frac{\|u\|^N_{L^N(\mathbb{R}^N)}}{K^N} \quad \text{for all } u \in W^{1,N}(\mathbb{R}^N) : \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq K. \tag{2.3}$$

In fact, if $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$ is such that $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leq K$, then

$$K^N \Phi_N \left(\alpha \left(\frac{|u|}{K} \right)^{\frac{N}{N-1}} \right) = K^N \sum_{j=N-1}^{+\infty} \frac{1}{j!} \left[\alpha \left(\frac{|u|}{K} \right)^{\frac{N}{N-1}} \right]^j$$

$$
\begin{aligned}
&= \sum_{j=N-1}^{+\infty} \frac{\alpha^j}{j!} \frac{|u|^{\frac{Nj}{N-1}}}{K^{\frac{Nj}{N-1}-N}} \\
&\leq \sum_{j=N-1}^{+\infty} \frac{\alpha^j}{j!} \frac{|u|^{\frac{Nj}{N-1}}}{\|\nabla u\|_{L^N(\mathbb{R}^N)}^{\frac{Nj}{N-1}-N}} \\
&= \|\nabla u\|_{L^N(\mathbb{R}^N)}^N \Phi_N\left(\alpha\left(\frac{|u|}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}\right)^{\frac{N}{N-1}}\right),
\end{aligned}
$$

and using (2.2) we obtain that (2.3) is true.

Remark 2.1 The function Φ_N possesses the following useful properties (see [47, Lemmas 2.1 and 2.2]):

$$
(\Phi_N(t))^\rho \leq \Phi_N(\rho t) \quad \text{for all } t \geq 0 \text{ and } \rho \geq 1, \tag{2.4}
$$

$$
\Phi_N(s+t) \leq \frac{1}{\mu_1}\Phi_N(\mu_1 s) + \frac{1}{\mu_2}\Phi_N(\mu_2 t) \quad \text{for all } s, t \geq 0 \text{ and } \mu_1, \mu_2 > 1 : \frac{1}{\mu_1} + \frac{1}{\mu_2} = 1. \tag{2.5}
$$

The next inequality will be used later.

Lemma 2.1 *Let $t \in [1, +\infty)$, $N \geq 2$ and $s \in [t, t^*)$. Then there exists $C > 0$ such that*

$$
\|u\|_{L^s(\mathbb{R}^N)}^s \leq C\left(\sup_{x_0\in\mathbb{R}^N} \|u\|_{L^s(B_1(x_0))}^s\right)^{1-\frac{t}{s}} \|u\|_{W^{1,t}(\mathbb{R}^N)}^t \quad \textit{for all } u \in W^{1,t}(\mathbb{R}^N).
$$

Proof The assertion is clear when $s = t$. Let now $s \in (t, t^*)$. From [2, Theorem 4.12], we learn that, for all fixed $x_0 \in \mathbb{R}^N$,

$$
\|u\|_{L^s(B_1(x_0))} \leq \tilde{C}\|u\|_{W^{1,t}(B_1(x_0))} \quad \text{for all } u \in W^{1,t}(B_1(x_0)), \tag{2.6}
$$

for some $\tilde{C} > 0$ depending on N, s, and t, but independent of x_0. Applying the Hölder inequality with exponents $\frac{s}{s-t}$ and $\frac{s}{t}$, and using (2.6), we obtain

$$
\begin{aligned}
\|u\|_{L^s(B_1(x_0))}^s &\leq \left(\int_{B_1(x_0)} |u|^s\, dx\right)^{1-\frac{t}{s}} \left(\int_{B_1(x_0)} |u|^s\, dx\right)^{\frac{t}{s}} \\
&\leq \tilde{C}^t \left(\sup_{x_0\in\mathbb{R}^N} \|u\|_{L^s(B_1(x_0))}^s\right)^{1-\frac{t}{s}} \|u\|_{W^{1,t}(B_1(x_0))}^t.
\end{aligned}
$$

Covering $\mathbb{R}^N$ by balls with radius 1 in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce

$$
\|u\|_{L^s(\mathbb{R}^N)}^s \leq (N+1)\tilde{C}^t \left(\sup_{x_0\in\mathbb{R}^N} \|u\|_{L^s(B_1(x_0))}^s\right)^{1-\frac{t}{s}} \|u\|_{W^{1,t}(\mathbb{R}^N)}^t.
$$

The proof of the lemma is now complete. □

The vanishing Lions lemma below is well-known [33, Lemma I.1].

Lemma 2.2 [33, Lemma I.1] *Let $p \in (1, +\infty)$ and $s \in [p, p^*)$. Let $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^s \, dx = 0,$$

for some $R > 0$. Then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (p, p^*)$.

Let us consider the radial Sobolev space

$$W_{\mathrm{r}}^{1,p}(\mathbb{R}^N) := \left\{ u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|) \right\}.$$

We have the following compact embedding.

Theorem 2.3 [32, Theorem II.1] *Let $N \geq 2$ and $p \in [1, +\infty)$. Then $W_{\mathrm{r}}^{1,p}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for all $q \in (p, p^*)$.*

We recall that the proof of Theorem 2.3 in [32] is based on the next useful result.

Lemma 2.3 [32, Lemma II.1] *Let $N \geq 2$, $p \in [1, +\infty)$ and $u \in W_{\mathrm{r}}^{1,p}(\mathbb{R}^N)$. Then it holds*

$$|u(x)| \leq C(N, p) |x|^{-\frac{N-1}{p}} \|u\|_{L^p(\mathbb{R}^N)}^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}^{\frac{1}{p}} \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Moreover, $u(x)$ can be identified with a function $\tilde{u}(|x|)$ such that $\tilde{u} \in C^{0, \frac{p-1}{p}}((0, +\infty))$.

Since we aim to deal with (p_1, p_2)-Laplacian problems, with $1 < p_1 < p_2 \leq N$, we introduce the Sobolev space

$$\mathcal{W} := W^{1,p_1}(\mathbb{R}^N) \cap W^{1,p_2}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_{\mathcal{W}} := \|u\|_{W^{1,p_1}(\mathbb{R}^N)} + \|u\|_{W^{1,p_2}(\mathbb{R}^N)}.$$

When $p_1 = p_2 = p \in (1, N]$, we identify $\mathcal{W}$ with $W^{1,p}(\mathbb{R}^N)$, endowed with the standard norm. Let us observe that $\mathcal{W}$ is a separable reflexive Banach space. By $\mathcal{W}'$ we denote the dual space of $\mathcal{W}$. We also define the radial subspace of $\mathcal{W}$, namely

$$\mathcal{W}_{\mathrm{r}} := \{u \in \mathcal{W} : u(x) = u(|x|)\}.$$

Next we establish two useful lemmas that extend and improve [35, Lemma 3.1 and equation (3.12)] (see also [30, Lemmas 3.2 and 3.3]). The first one is a variant of Lemma 2.2.

Lemma 2.4 *Let $(u_n) \subset \mathcal{W}$ be a bounded sequence such that*

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^s \, dx = 0, \tag{2.7}$$

for some $R > 0$ and $s \in [p_2, p_2^*)$. Then, as $n \to +\infty$,

$$\left| \int_{\mathbb{R}^N} \Psi(u_n) \, dx \right| \leq \int_{\mathbb{R}^N} |\Psi(u_n)| \, dx \to 0,$$

for every continuous function $\Psi:\mathbb{R}\to\mathbb{R}$ *such that*

$$\lim_{|t|\to 0}\frac{\Psi(t)}{|t|^{p_1}}=\lim_{|t|\to+\infty}\frac{\Psi(t)}{|t|^{p_2^*}}=0\quad \textit{if } 1<p_1\le p_2<N,$$
$$\lim_{|t|\to 0}\frac{\Psi(t)}{|t|^{p_1}}=\lim_{|t|\to+\infty}\frac{\Psi(t)}{e^{\alpha|t|^{\frac{N}{N-1}}}}=0\quad \textit{for all } \alpha>0\quad \textit{if } 1<p_1<p_2=N\ \vee\ 1<p_1=p_2=N.$$

Proof First we suppose that $1<p_1\le p_2<N$. Fix $\varepsilon>0$ and $q\in(p_2,p_2^*)$. Then there exist $0<\delta_\varepsilon<M_\varepsilon$ and $c_{\varepsilon,q}>0$ such that

$$|\Psi(t)|\le\varepsilon\,|t|^{p_1}\quad\text{for all } |t|\le\delta_\varepsilon,$$
$$|\Psi(t)|\le\varepsilon\,|t|^{p_2^*}\quad\text{for all } |t|\ge M_\varepsilon,$$
$$|\Psi(t)|\le c_{\varepsilon,q}|t|^q\quad\text{for all } \delta_\varepsilon\le|t|\le M_\varepsilon.$$

Hence,

$$|\Psi(t)|\le\varepsilon(|t|^{p_1}+|t|^{p_2^*})+c_{\varepsilon,q}|t|^q\quad\text{for all } t\in\mathbb{R}. \tag{2.8}$$

In view of (2.7), it follows from Lemma 2.2 that $u_n\to 0$ in $L^q(\mathbb{R}^N)$. This fact combined with (2.8) and the boundedness of (u_n) in $L^{p_1}(\mathbb{R}^N)\cap L^{p_2^*}(\mathbb{R}^N)$ yields

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}|\Psi(u_n)|\,dx\le\varepsilon\left(\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)}+\|u_n\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)}\right)\le C\,\varepsilon\,.$$

Since $\varepsilon>0$ is arbitrary, we obtain the assertion. Now we assume that $1<p_1<p_2=N$ or $1<p_1=p_2=N$. Take $\varepsilon>0$, $\alpha>0$, and $q\in(p_2,+\infty)$. Then there exists $C_{\varepsilon,\alpha,q}>0$ such that

$$|\Psi(t)|\le\varepsilon\left(|t|^{p_1}+\Phi_N\left(\alpha|t|^{\frac{N}{N-1}}\right)\right)+C_{\varepsilon,\alpha,q}|t|^q\quad\text{for all } t\in\mathbb{R}. \tag{2.9}$$

Because (u_n) is bounded in $W^{1,N}(\mathbb{R}^N)$, there exists $K>0$ such that $\|u_n\|_{W^{1,N}(\mathbb{R}^N)}\le K$ for all $n\in\mathbb{N}$. Choosing $\alpha>0$ such that $\alpha K^{\frac{N}{N-1}}<\alpha_N$, we can see that (2.3) gives

$$\int_{\mathbb{R}^N}\Phi_N\left(\alpha|u_n|^{\frac{N}{N-1}}\right)dx=\int_{\mathbb{R}^N}\Phi_N\left(\alpha K^{\frac{N}{N-1}}\left(\frac{|u_n|}{K}\right)^{\frac{N}{N-1}}\right)dx\le C'\quad\text{for all } n\in\mathbb{N}. \tag{2.10}$$

Exploiting $u_n\to 0$ in $L^q(\mathbb{R}^N)$, the boundedness of (u_n) in $L^{p_1}(\mathbb{R}^N)$, (2.9), and (2.10), we find

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}|\Psi(u_n)|\,dx\le C''\,\varepsilon,$$

and so the assertion follows from the arbitrariness of $\varepsilon>0$. □

Remark 2.2 Clearly, the conclusion of Lemma 2.4 still holds if we replace (2.7) by $u_n\to 0$ in $L^q(\mathbb{R}^N)$ for all $q\in(p_2,p_2^*)$.

The second lemma is a Brezis-Lieb type result [14, Theorem 1].

Lemma 2.5 *Let* $\Psi:\mathbb{R}\to\mathbb{R}$ *be a* C^1 *function such that* $\Psi(0)=0$. *When* $1<p_1\le p_2<N$, *we assume that there exists* $C>0$ *such that*

$$|\Psi'(t)|\le C\left(|t|^{p_1-1}+|t|^{p_2^*-1}\right)\quad \textit{for all } t\in\mathbb{R}, \tag{2.11}$$

while if $1 < p_1 < p_2 = N$ *or* $1 < p_1 = p_2 = N$ *then we assume that for every* $\alpha > 0$ *and* $q \geq 1$ *there exists* $\bar{C} > 0$ *such that*

$$|\Psi'(t)| \leq \bar{C}\left(|t|^{p_1-1} + |t|^{q-1}\Phi_N\left(\alpha|t|^{\frac{N}{N-1}}\right)\right) \quad \textit{for all } t \in \mathbb{R}. \tag{2.12}$$

Let $(u_n) \subset \mathcal{W}$ *be a bounded sequence such that* $u_n \to u$ *a.e. in* $\mathbb{R}^N$ *for some* $u \in \mathcal{W}$. *Then we have*

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}\Psi(u_n)\,dx = \int_{\mathbb{R}^N}\Psi(u)\,dx + \limsup_{n\to+\infty}\int_{\mathbb{R}^N}\Psi(u_n-u)\,dx.$$

Proof We aim to apply the Vitali convergence theorem to show that

$$\int_{\mathbb{R}^N}[\Psi(u_n)-\Psi(u_n-u)]\,dx = \int_{\mathbb{R}^N}\Psi(u)\,dx + o_n(1). \tag{2.13}$$

In fact, once proved (2.13), we deduce

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}\Psi(u_n)\,dx = \int_{\mathbb{R}^N}\Psi(u)\,dx + \limsup_{n\to+\infty}\int_{\mathbb{R}^N}\Psi(u_n-u)\,dx.$$

Next we demonstrate (2.13). The mean value theorem ensures that

$$\Psi(u_n) - \Psi(u_n-u) = \Psi'(u_n - u + \theta_n u)u,$$

with $\theta_n = \theta_n(x) \in [0, 1]$. Assume that $1 < p_1 \leq p_2 < N$. From (2.11), we derive

$$\begin{aligned}|\Psi'(u_n - u + \theta_n u)u| &\leq C\left(|u_n - u + \theta_n u|^{p_1-1} + |u_n - u + \theta_n u|^{p_2^*-1}\right)|u| \\ &\leq C_1(|u_n| + |u|)^{p_1-1}|u| + C_2(|u_n| + |u|)^{p_2^*-1}|u|.\end{aligned}$$

Thus, utilizing the Hölder inequality and the boundedness of (u_n) in $L^{p_1}(\mathbb{R}^N) \cap L^{p_2^*}(\mathbb{R}^N)$, we see that for every $\Omega \subset \mathbb{R}^N$ measurable set,

$$\begin{aligned}&\int_\Omega |\Psi'(u_n - u + \theta_n u)u|\,dx \\ &\leq C_3\left[\|u_n\|^{p_1-1}_{L^{p_1}(\mathbb{R}^N)}\|u\|_{L^{p_1}(\Omega)} + \|u\|^{p_1}_{L^{p_1}(\Omega)} + \|u_n\|^{p_2^*-1}_{L^{p_2^*}(\mathbb{R}^N)}\|u\|_{L^{p_2^*}(\Omega)} + \|u\|^{p_2^*}_{L^{p_2^*}(\Omega)}\right] \\ &\leq C_4\left(\|u\|_{L^{p_1}(\Omega)} + \|u\|^{p_1}_{L^{p_1}(\Omega)} + \|u\|_{L^{p_2^*}(\Omega)} + \|u\|^{p_2^*}_{L^{p_2^*}(\Omega)}\right) \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Now, if $|\Omega| \to 0$, then

$$\int_\Omega |\Psi'(u_n - u + \theta_n u)u|\,dx \to 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

On the other hand, since $u \in L^{p_1}(\mathbb{R}^N) \cap L^{p_2^*}(\mathbb{R}^N)$, fixed $\varepsilon > 0$ we can find $R > 0$ such that

$$C_4\left(\|u\|_{L^{p_1}(B_R^c(0))} + \|u\|^{p_1}_{L^{p_1}(B_R^c(0))} + \|u\|_{L^{p_2^*}(B_R^c(0))} + \|u\|^{p_2^*}_{L^{p_2^*}(B_R^c(0))}\right) < \varepsilon\,.$$

Hence, if $\Omega = B_R^c(0)$, then

$$\int_{B_R^c(0)} |\Psi'(u_n - u + \theta_n u)u|\,dx < \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $(\Psi(u_n)-\Psi(u_n-u))$ satisfies the assumptions of the Vitali convergence theorem. Noting that $\Psi(0)=0$, we conclude that (2.13) holds.

Next we consider the case $1<p_1<p_2=N$. Using (2.12) with $q>N+1$, the elementary estimate

$$\alpha|u_n-u+\theta_n u|^{\frac{N}{N-1}}\leq \alpha c_0\left(|u_n|^{\frac{N}{N-1}}+|u|^{\frac{N}{N-1}}\right),$$

with $c_0:=2^{\frac{1}{N-1}}$, and the fact that Φ_N is nondecreasing in $[0,+\infty)$, we have

$$\begin{aligned}
&|\Psi'(u_n-u+\theta_n u)u|\\
&\leq \bar{C}\left(|u_n-u+\theta_n u|^{p_1-1}+|u_n-u+\theta_n u|^{q-1}\Phi_N\left(\alpha|u_n-u+\theta_n u|^{\frac{N}{N-1}}\right)\right)|u|\\
&\leq C_5(|u_n|+|u|)^{p_1-1}|u|+C_6(|u_n|+|u|)^{q-1}\Phi_N\left(\alpha c_0\left(|u_n|^{\frac{N}{N-1}}+|u|^{\frac{N}{N-1}}\right)\right)|u|.
\end{aligned}$$

Let now $\Omega\subset\mathbb{R}^N$ be any measurable set. Exploiting the Hölder inequality and the boundedness of (u_n) in $L^{p_1}(\mathbb{R}^N)$, we see

$$\begin{aligned}
\int_\Omega(|u_n|+|u|)^{p_1-1}|u|\,dx&\leq C_7(\|u_n\|^{p_1-1}_{L^{p_1}(\mathbb{R}^N)}\|u\|_{L^{p_1}(\Omega)}+\|u\|^{p_1}_{L^{p_1}(\Omega)})\\
&\leq C_8\left(\|u\|_{L^{p_1}(\Omega)}+\|u\|^{p_1}_{L^{p_1}(\Omega)}\right).
\end{aligned}$$

Fix $\tau_1>1$, $\tau_2>1$ and $\tau_3>N$ such that $\sum_{i=1}^3\frac{1}{\tau_i}=1$. Note that $\tau_1(q-1)>N$. Using the generalized Hölder inequality, (2.4), and (2.5) with $\mu_1=\mu_2=2$, we obtain

$$\begin{aligned}
&\int_\Omega(|u_n|+|u|)^{q-1}\Phi_N\left(\alpha c_0\left(|u_n|^{\frac{N}{N-1}}+|u|^{\frac{N}{N-1}}\right)\right)|u|\,dx\\
&\leq \||u_n|+|u|\|^{q-1}_{L^{\tau_1(q-1)}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\Phi_N(\alpha\tau_2 c_0(|u_n|^{\frac{N}{N-1}}+|u|^{\frac{N}{N-1}}))\,dx\right)^{\frac{1}{\tau_2}}\|u\|_{L^{\tau_3}(\Omega)}\\
&\leq \||u_n|+|u|\|^{q-1}_{L^{\tau_1(q-1)}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left[\frac{1}{2}\Phi_N(2\alpha\tau_2c_0|u_n|^{\frac{N}{N-1}})+\frac{1}{2}\Phi_N(2\alpha\tau_2c_0|u|^{\frac{N}{N-1}})\right]dx\right)^{\frac{1}{\tau_2}}\\
&\quad\|u\|_{L^{\tau_3}(\Omega)}.
\end{aligned}$$

Since (u_n) is bounded in $W^{1,N}(\mathbb{R}^N)$, there exists $K>0$ such that $\|u_n\|_{W^{1,N}(\mathbb{R}^N)}\leq K$ for all $n\in\mathbb{N}$. Select $\alpha>0$ such that $2\alpha\tau_2c_0K^{\frac{N}{N-1}}<\alpha_N$. Then, invoking (2.3), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Phi_N(2\alpha\tau_2c_0|u_n|^{\frac{N}{N-1}})\,dx\\
&=\int_{\mathbb{R}^N}\Phi_N\left(2\alpha\tau_2c_0K^{\frac{N}{N-1}}\left(\frac{|u_n|}{K}\right)^{\frac{N}{N-1}}\right)dx\leq C_9\quad\text{for all } n\in\mathbb{N}.
\end{aligned}$$

In a similar manner, choosing $\alpha>0$ sufficiently small, we infer

$$\int_{\mathbb{R}^N}\Phi_N(2\alpha\tau_2c_0|u|^{\frac{N}{N-1}})\,dx\leq C_{10}.$$

Therefore, for $\alpha>0$ small enough, we arrive at

$$\int_{\mathbb{R}^N}\left[\frac{1}{2}\Phi_N\left(2\alpha\tau_2c_0|u_n|^{\frac{N}{N-1}}\right)+\frac{1}{2}\Phi_N(2\alpha\tau_2c_0|u|^{\frac{N}{N-1}})\right]dx\leq C_{11}\quad\text{for all } n\in\mathbb{N}.$$

Observing that (u_n) is bounded in $L^{\tau_1(q-1)}(\mathbb{R}^N)$, we deduce

$$\int_\Omega (|u_n|+|u|)^{q-1}\Phi_N\left(\alpha c_0\left(|u_n|^{\frac{N}{N-1}}+|u|^{\frac{N}{N-1}}\right)\right)|u|\,dx \le C_{12}\|u\|_{L^{\tau_3}(\Omega)} \quad \text{for all } n\in\mathbb{N},$$

and so, for every $\Omega\subset\mathbb{R}^N$ measurable set,

$$\int_\Omega |\Psi'(u_n-u+\theta_n u)u|\,dx \le C_{13}(\|u\|_{L^{p_1}(\Omega)}+\|u\|^{p_1}_{L^{p_1}(\Omega)}+\|u\|_{L^{\tau_3}(\Omega)}) \quad \text{for all } n\in\mathbb{N}.$$

Arguing as in the case $p_2 < N$, it follows from the above estimate and $u \in L^{p_1}(\mathbb{R}^N)\cap L^{\tau_3}(\mathbb{R}^N)$ that $(\Psi(u_n)-\Psi(u_n-u))$ satisfies the assumptions of the Vitali convergence theorem. Because $\Psi(0)=0$, we obtain that (2.13) is still valid. □

Finally, we prove a suitable compactness result in the spirit of the celebrated compactness lemma due to Strauss (see [12, Theorem A.I]).

Lemma 2.6 *Let $(u_n)\subset\mathcal{W}$ be a bounded sequence such that $u_n\to u$ a.e. in $\mathbb{R}^N$ and $u_n\to u$ in $L^q(\mathbb{R}^N)$ for all $q\in(p_2,p_2^*)$. Let $\Psi:\mathbb{R}\to\mathbb{R}$ be a continuous function such that*

$$\lim_{|t|\to 0}\frac{\Psi(t)}{|t|^{r_1}}=0 \ \text{ for some } r_1\in[p_1,p_2^*), \tag{2.14}$$

$$\lim_{|t|\to+\infty}\frac{\Psi(t)}{|t|^{p_2^*}}=0 \ \text{ if } p_2<N, \quad \lim_{|t|\to+\infty}\frac{\Psi(t)}{e^{\alpha|t|^{\frac{N}{N-1}}}}=0 \ \text{ for all } \alpha>0 \ \text{ if } p_2=N. \tag{2.15}$$

Then,

$$\lim_{n\to+\infty}\|\Psi(u_n)-\Psi(u)\|_{L^1(\mathbb{R}^N)}=0.$$

Proof Since (u_n) is bounded in $\mathcal{W}$, there exists $M>0$ such that $\|u_n\|_{\mathcal{W}}\le M$ for all $n\in\mathbb{N}$. Pick $q_0\in(p_2,p_2^*)$. Define $Q(t):=|t|^{r_1}+R(t)$, where

$$R(t):=\begin{cases}|t|^{p_2^*} & \text{if } p_2<N,\\ \Phi_N(\alpha|t|^{\frac{N}{N-1}}) & \text{if } p_2=N,\end{cases}$$

with $\alpha>0$ such that $\alpha M^{\frac{N}{N-1}}<\alpha_N$. From (2.14) and (2.15), we derive that for every fixed $\varepsilon>0$, there exists $C_\varepsilon>0$ (here C_ε depends on ε and q_0 when $p_2<N$, while it depends on ε, q_0, and α when $p_2=N$) such that

$$|\Psi(t)|\le\varepsilon\, Q(t)+C_\varepsilon|t|^{q_0} \quad \text{for all } t\in\mathbb{R}.$$

Using the boundedness of (u_n) in $\mathcal{W}$, Theorem 2.1, and (2.3), we see

$$\int_{\mathbb{R}^N}|\Psi(u_n)|\,dx\le\varepsilon\int_{\mathbb{R}^N}Q(u_n)\,dx+C_\varepsilon\int_{\mathbb{R}^N}|u_n|^{q_0}\,dx\le\varepsilon\, C_1+C_\varepsilon C_2 \quad \text{for all } n\in\mathbb{N}.$$

Thus, by the continuity of Ψ and Fatou's lemma, we have

$$\int_{\mathbb{R}^N}|\Psi(u)|\,dx\le\liminf_{n\to+\infty}\int_{\mathbb{R}^N}|\Psi(u_n)|\,dx\le\varepsilon\, C_1+C_\varepsilon C_2,$$

that is, $\Psi(u)\in L^1(\mathbb{R}^N)$. Let us now consider

$$S_{\varepsilon,n}(x):=(|\Psi(u_n(x))-\Psi(u(x))|-\varepsilon\, Q(u_n(x)))^+ \quad \text{for a.e. } x\in\mathbb{R}^N \text{ and for all } n\in\mathbb{N}.$$

Clearly,

$$0 \le S_{\varepsilon,n}(x) \le C_\varepsilon |u_n(x)|^{q_0} + |\Psi(u(x))| \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } n \in \mathbb{N}.$$

Because $u_n \to u$ in $L^{q_0}(\mathbb{R}^N)$, there exists $h_0 \in L^{q_0}(\mathbb{R}^N)$ such that, up to a subsequence, $|u_n(x)| \le h_0(x)$ for a.e. $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. Hence, $S_{\varepsilon,n}(x) \le C_\varepsilon h_0^{q_0}(x) + |\Psi(u(x))|$ for a.e. $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$, with $C_\varepsilon h_0^{q_0} + |\Psi(u)| \in L^1(\mathbb{R}^N)$. Moreover, exploiting the continuity of Ψ and Q, we know that $S_{\varepsilon,n} \to 0$ a.e. in $\mathbb{R}^N$ as $n \to +\infty$. Then, applying the dominated convergence theorem, we deduce

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} S_{\varepsilon,n}\, dx = 0.$$

Consequently,

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} |\Psi(u_n) - \Psi(u)|\, dx \le \limsup_{n\to+\infty} \int_{\mathbb{R}^N} S_{\varepsilon,n}\, dx + \varepsilon \limsup_{n\to+\infty} \int_{\mathbb{R}^N} Q(u_n)\, dx \le \varepsilon\, C_1.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the assertion. □

Remark 2.3 In view of Lemma 2.2, the conclusion of Lemma 2.6 is still valid with $u = 0$ if we replace $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (p_2, p_2^*)$ by (2.7).

3 Regularity and Pohozaev identity for solutions to (1.1)

As in [12], we modify the nonlinearity g by considering a new function $\hat{g} : \mathbb{R} \to \mathbb{R}$ defined as follows:

(i) If $g(t) \ge 0$ for all $t \ge \xi$, then we put $\hat{g}(t) := g(t)$.
(ii) If there exists $\xi_0 \ge \xi$ such that $g(\xi_0) = 0$, then we put

$$\hat{g}(t) := \begin{cases} g(t) & \text{for } t \in [0, \xi_0], \\ 0 & \text{for } t \ge \xi_0, \\ -g(-t) & \text{for } t < 0. \end{cases}$$

Note that $\hat{g}$ satisfies $(g1)$, $(g3)$ and

$(g2)'$ $\lim_{|t|\to+\infty} \frac{|\hat{g}(t)|}{|t|^{p_2^*-1}} = 0$ when $p_2 < N$, and $\lim_{|t|\to+\infty} \frac{|\hat{g}(t)|}{\exp\left(\alpha |t|^{\frac{N}{N-1}}\right)} = 0$ for all $\alpha > 0$ when $p_2 = N$.

Furthermore, if (ii) occurs and u is a solution to (1.1) with $\hat{g}(t)$ in place of $g(t)$, then we can see that $|u| \le \xi_0$ in $\mathbb{R}^N$, that is, u is a solution to (1.1). Hereafter, we replace g by $\hat{g}$ and keep the same notation $g(t)$. With this modification, we will assume that g fulfills $(g1)$, $(g2)'$, and $(g3)$. Set

$$\nu := -\frac{1}{2} \limsup_{t\to 0^+} \frac{g(t)}{t^{p_1-1}} \in (0, +\infty).$$

Define $g_1(t) := (g(t) + 2\nu(t^{p_1-1} + t^{p_2-1}))^+$ and $g_2(t) := g_1(t) - g(t)$ for $t \ge 0$. Extend $g_1(t)$ and $g_2(t)$ as odd functions for $t \le 0$. Thus, $g = g_1 - g_2$ with $g_1, g_2 \ge 0$ in $[0, +\infty)$, and we see

$$g_1(t) = o(t^{p_1-1}) \text{ as } t \to 0^+, \tag{3.1}$$

$$g_1(t) = o(t^{p_2^*-1}) \text{ as } t \to +\infty \text{ if } p_2 < N, \; g_1(t) = o(e^{\alpha t^{\frac{N}{N-1}}}) \\ \text{for all } \alpha > 0 \text{ as } t \to +\infty \text{ if } p_2 = N, \tag{3.2}$$

$$g_2(t) \geq 2\nu \left(t^{p_1-1} + t^{p_2-1}\right) \quad \text{for all } t \geq 0. \tag{3.3}$$

Put $G_i(t) := \int_0^t g(\tau)\, d\tau$ for all $i = 1, 2$. When $p_2 < N$, thanks to (3.1)–(3.3), we deduce that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $g_1(t) \leq \varepsilon\, g_2(t) + C_\varepsilon t^{p_2^*-1}$ for all $t \geq 0$, and so

$$G_1(t) \leq \varepsilon\, G_2(t) + C'_\varepsilon |t|^{p_2^*} \quad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

In the result below, we focus on the regularity of solutions to (1.1), and we establish a Pohozaev type identity.

Theorem 3.1 *Assume that* $(g1)$, $(g2)'$, *and* $(g3)$ *hold. Let* $u \in \mathcal{W}$ *be a weak solution to (1.1). Then* $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\sigma}_{loc}(\mathbb{R}^N)$ *for some* $\sigma \in (0, 1)$. *Moreover, u obeys the following Pohozaev type identity:*

$$P(u) := \sum_{i=1}^{2} \left(\frac{N-p_i}{p_i}\right) \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - N \int_{\mathbb{R}^N} G(u)\, dx = 0. \tag{3.5}$$

Proof We start by observing that u solves

$$-\Delta_{p_1} u - \Delta_{p_2} u + g_2(u) = g_1(u) \text{ in } \mathbb{R}^N.$$

Let $z := |u|$ and $z_\varepsilon := \sqrt{u^2 + \varepsilon^2} - \varepsilon$ for $\varepsilon > 0$. Note that $z_\varepsilon \to z$ in $\mathcal{W}$ as $\varepsilon \to 0^+$. Let us now show that z satisfies

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2} \nabla z \nabla \phi\, dx + 2\nu \int_{\mathbb{R}^N} z^{p_1-1} \phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2} \nabla z \nabla \phi\, dx + 2\nu \int_{\mathbb{R}^N} z^{p_2-1} \phi\, dx \\ \leq \int_{\mathbb{R}^N} g_1(z) \phi\, dx \tag{3.6}$$

for all $\phi \in \mathcal{W}$ such that $\phi \geq 0$. Take $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi \geq 0$. Then, for all $i = 1, 2$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla z|^{p_i-2} \nabla z_\varepsilon \nabla \phi\, dx &= \int_{\mathbb{R}^N} |\nabla u|^{p_i-2} \nabla u \frac{u}{\sqrt{u^2+\varepsilon^2}} \nabla \phi\, dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^{p_i-2} \nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx - \int_{\mathbb{R}^N} |\nabla u|^{p_i} \frac{\varepsilon^2}{(u^2+\varepsilon^2)^{3/2}} \phi\, dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^{p_i-2} \nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z|^{p_1-2} \nabla z_\varepsilon \nabla \phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2} \nabla z_\varepsilon \nabla \phi\, dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^{p_1-2} \nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx + \int_{\mathbb{R}^N} |\nabla u|^{p_2-2} \nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx \\ &= \int_{\mathbb{R}^N} g_1(u) \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx - \int_{\mathbb{R}^N} g_2(u) \left(\frac{u}{\sqrt{u^2+\varepsilon^2}} \phi\right) dx, \end{aligned}$$

which, combined with $g_i(t)t = g_i(|t|)|t|$ for all $t \in \mathbb{R}$ and $i = 1, 2$, $\phi \geq 0$, and $0 \leq \frac{|t|}{\sqrt{t^2+\varepsilon^2}} \leq 1$ for all $t \in \mathbb{R}$, implies

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2}\nabla z_\varepsilon \nabla\phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2}\nabla z_\varepsilon \nabla\phi\, dx + \int_{\mathbb{R}^N} g_2(|u|)\left(\frac{|u|}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx. \\ \leq \int_{\mathbb{R}^N} |g_1(u)|\phi\, dx.$$

Taking the limit as $\varepsilon \to 0^+$ in the above relation, and exploiting all (3.3)), the dominated convergence theorem, and Fatou's lemma, we conclude that (3.6) is valid for every $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi \geq 0$. By density, (3.6) is true for all $\phi \in \mathcal{W}$ such that $\phi \geq 0$.

In what follows, we prove that $u \in L^\infty(\mathbb{R}^N)$. We first assume that $p_2 < N$. Thanks to (3.1) and (3.2), we have that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$0 \leq g_1(t) \leq \varepsilon\, t^{p_1-1} + C_\varepsilon t^{p_2^*-1} \quad \text{for all } t \geq 0.$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2}\nabla z \nabla\phi\, dx + 2\nu\int_{\mathbb{R}^N} z^{p_1-1}\phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2}\nabla z\nabla\phi\, dx + 2\nu\int_{\mathbb{R}^N} z^{p_2-1}\phi\, dx \\ \leq \int_{\mathbb{R}^N} [\varepsilon\, z^{p_1-1} + C_\varepsilon z^{p_2^*-1}]\phi\, dx$$

for all $\phi \in \mathcal{W}$ such that $\phi \geq 0$. Taking $\varepsilon \in (0, 2\nu)$, we obtain that, for some $C > 0$,

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2}\nabla z\nabla\phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2}\nabla z\nabla\phi\, dx \leq C\int_{\mathbb{R}^N} z^{p_2^*-1}\phi\, dx \tag{3.7}$$

for all $\phi \in \mathcal{W}$ such that $\phi \geq 0$. Now we show that $z \in L^\infty(\mathbb{R}^N)$ by means of a Moser iteration scheme [39]. For all $L > 0$ and $\beta > 1$, we define $z_L := \min\{z, L\}$, $\tilde{z}_L := z z_L^{p_2(\beta-1)}$ and $w_L := z z_L^{\beta-1}$. Suppose that $z \in L^{p_2\beta}(\mathbb{R}^N)$ and verify that $z \in L^{p_2^*\beta}(\mathbb{R}^N)$. Inserting $\phi = \tilde{z}_L \in \mathcal{W}$ into (3.7), we find

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2}\nabla z\nabla\tilde{z}_L\, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2}\nabla z\nabla\tilde{z}_L\, dx \leq C\int_{\mathbb{R}^N} z^{p_2^*-1}\tilde{z}_L\, dx. \tag{3.8}$$

Let us observe that

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2}\nabla z\nabla\tilde{z}_L\, dx = \int_{\mathbb{R}^N} |\nabla z|^{p_1} z_L^{p_2(\beta-1)}\, dx + p_2(\beta-1)\int_{\{z\leq L\}} |\nabla z|^{p_1} z^{p_2(\beta-1)}\, dx \geq 0, \tag{3.9}$$

and

$$\int_{\mathbb{R}^N} |\nabla z|^{p_2-2}\nabla z\nabla\tilde{z}_L\, dx = \int_{\mathbb{R}^N} |\nabla z|^{p_2} z_L^{p_2(\beta-1)}\, dx + p_2(\beta-1)\int_{\{z\leq L\}} |\nabla z|^{p_2} z^{p_2(\beta-1)}\, dx \\ \geq \int_{\mathbb{R}^N} |\nabla z|^{p_2} z_L^{p_2(\beta-1)}\, dx. \tag{3.10}$$

In view of (3.8)–(3.10), we get

$$\int_{\mathbb{R}^N} |\nabla z|^{p_2} z_L^{p_2(\beta-1)}\, dx \leq C\int_{\mathbb{R}^N} z^{p_2^*} z_L^{p_2(\beta-1)}\, dx.$$

Invoking the Sobolev inequality (2.1), and using the fact that

$$\left(\left(\frac{\beta-1}{\beta}\right)^{p_2}+\frac{1}{\beta^{p_2}}\right)<2 \quad \text{for all } \beta>1,$$

we deduce

$$\begin{aligned}\|w_L\|^{p_2}_{L^{p_2^*}(\mathbb{R}^N)} &\le S_*^{p_2}(N,p_2)\|\nabla w_L\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}\\ &\le 2^{p_2-1}S_*^{p_2}(N,p_2)\left((\beta-1)^{p_2}\int_{\{z\le L\}}|\nabla z|^{p_2}z_L^{p_2(\beta-1)}\,dx+\int_{\mathbb{R}^N}|\nabla z|^{p_2}z_L^{p_2(\beta-1)}\,dx\right)\\ &\le 2^{p_2-1}S_*^{p_2}(N,p_2)((\beta-1)^{p_2}+1)\int_{\mathbb{R}^N}|\nabla z|^{p_2}z_L^{p_2(\beta-1)}\,dx\\ &\le 2^{p_2}S_*^{p_2}(N,p_2)\beta^{p_2}\int_{\mathbb{R}^N}|\nabla z|^{p_2}z_L^{p_2(\beta-1)}\,dx.\end{aligned} \tag{3.11}$$

Hence (3.11) yields

$$\|w_L\|^{p_2}_{L^{p_2^*}(\mathbb{R}^N)} \le C_0\beta^{p_2}\int_{\mathbb{R}^N} z^{p_2^*}z_L^{p_2(\beta-1)}\,dx, \tag{3.12}$$

where $C_0 := 2^{p_2}S_*^{p_2}(N,p_2)C > 0$. Since $z^{p_2^*}z_L^{p_2(\beta-1)} = hw_L^{p_2}$ with $h := z^{p_2^*-p_2} \in L^{\frac{N}{p_2}}(\mathbb{R}^N)$, (3.12) becomes

$$\|w_L\|^{p_2}_{L^{p_2^*}(\mathbb{R}^N)} \le C_0\beta^{p_2}\int_{\mathbb{R}^N} hw_L^{p_2}\,dx. \tag{3.13}$$

Let $M>0$ to be fixed later and put $A_M := \{x\in\mathbb{R}^N : h(x)\le M\}$ and $B_M := \{x\in\mathbb{R}^N : h(x)>M\}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^N} hw_L^{p_2}\,dx &= \int_{A_M} hw_L^{p_2}\,dx+\int_{B_M} hw_L^{p_2}\,dx\\ &\le M\|w_L\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}+\left(\int_{B_M} h^{\frac{N}{p_2}}\,dx\right)^{\frac{p_2}{N}}\|w_L\|^{p_2}_{L^{p_2^*}(\mathbb{R}^N)}.\end{aligned} \tag{3.14}$$

By virtue of $h\in L^{\frac{N}{p_2}}(\mathbb{R}^N)$, we can choose $M=M_\beta>0$ sufficiently large such that

$$\left(\int_{B_M} h^{\frac{N}{p_2}}\,dx\right)^{\frac{p_2}{N}} \le \frac{1}{2C_0\beta^{p_2}}.$$

Thus, using (3.13), (3.14), and that $w_L\le z^\beta$, we obtain

$$\|w_L\|^{p_2}_{L^{p_2^*}(\mathbb{R}^N)} \le 2C_0M_\beta\beta^{p_2}\|z\|^{p_2\beta}_{L^{p_2\beta}(\mathbb{R}^N)},$$

and passing to the limit as $L\to+\infty$, Fatou's lemma ensures that

$$\|z\|^{p_2\beta}_{L^{p_2^*\beta}(\mathbb{R}^N)} \le 2C_0M_\beta\beta^{p_2}\|z\|^{p_2\beta}_{L^{p_2\beta}(\mathbb{R}^N)}. \tag{3.15}$$

Now we start a bootstrap argument. Since $z\in L^{p_2^*}(\mathbb{R}^N)$, we can apply (3.15) with $\beta=\frac{p_2^*}{p_2}$ to deduce that $z\in L^{\left(\frac{p_2^*}{p_2}\right)p_2^*}(\mathbb{R}^N)$. Employing (3.15) once again, after k iterations, we learn that $z \in L^{\left(\frac{p_2^*}{p_2}\right)^k p_2^*}(\mathbb{R}^N)$, and so $z \in L^t(\mathbb{R}^N)$ for all $t \in [p_2^*, +\infty)$. Now we return to consider (3.12). Observing that $0 \leq z_L \leq z$, and sending $L \to +\infty$ in (3.12), we see

$$\left(\int_{\mathbb{R}^N} z^{p_2^*\beta}\, dx\right)^{\frac{p_2}{p_2^*}} \leq C_0\beta^{p_2} \int_{\mathbb{R}^N} z^{p_2^*+p_2(\beta-1)}\, dx,$$

which implies

$$\left(\int_{\mathbb{R}^N} z^{p_2^*\beta}\, dx\right)^{\frac{1}{p_2^*(\beta-1)}} \leq (C_0^{\frac{1}{p_2}}\beta)^{\frac{1}{\beta-1}} \left(\int_{\mathbb{R}^N} z^{p_2^*+p_2(\beta-1)}\, dx\right)^{\frac{1}{p_2(\beta-1)}}. \tag{3.16}$$

Set $\beta_1 := \frac{p_2^*}{p_2} > 1$ and define β_m inductively so that $p_2^* + p_2(\beta_{m+1} - 1) = p_2^*\beta_m$ for $m \in \mathbb{N}$. Therefore,

$$\beta_m = \beta_1^{m-1}(\beta_1 - 1) + 1 \quad \text{for } m \in \mathbb{N}, \quad \text{and} \quad \lim_{m\to+\infty} \beta_m = +\infty.$$

Put

$$\Psi_m := \left(\int_{\mathbb{R}^N} z^{p_2^*\beta_m}\, dx\right)^{\frac{1}{p_2^*(\beta_m-1)}} \quad \text{for all } m \in \mathbb{N}.$$

Then (3.16) can be written as

$$\Psi_{m+1} \leq C_{m+1}^{\frac{1}{\beta_{m+1}-1}}\Psi_m \quad \text{for all } m \in \mathbb{N},$$

where $C_{m+1} := C_0^{\frac{1}{p_2}}\beta_{m+1}$. Iterating the above relation, we arrive at

$$\Psi_{m+1} \leq \left(\prod_{j=2}^{m+1} C_j^{\frac{1}{\beta_j-1}}\right)\Psi_1 \quad \text{for all } m \in \mathbb{N}. \tag{3.17}$$

Because $z \in L^{p_2^*}(\mathbb{R}^N)$, from (3.15) with $\beta = \beta_1 = \frac{p_2^*}{p_2}$ we derive

$$\Psi_1 \leq \left(2C_0 M_{\frac{p_2^*}{p_2}} \left(\frac{p_2^*}{p_2}\right)^{p_2}\right)^{\frac{1}{p_2^*-p_2}} \|z\|_{L^{p_2^*}(\mathbb{R}^N)}^{\frac{p_2^*}{p_2^*-p_2}}.$$

On the other hand, for some $C'' > 0$,

$$\prod_{j=2}^{m+1} C_j^{\frac{1}{\beta_j-1}} = (C_0^{\frac{1}{p_2}})^{\sum_{j=2}^{m+1}\frac{1}{\beta_1^{j-1}(\beta_1-1)}} \prod_{j=2}^{m+1}\left(\beta_1^{j-1}(\beta_1-1)+1\right)^{\frac{1}{\beta_1^{j-1}(\beta_1-1)}} \leq C'' \quad \text{for all } m \in \mathbb{N},$$

where we have used that $\log(\beta_1^{j-1}(\beta_1 - 1) + 1) \leq \log(\beta_1^{j-1}(\beta_1 - 1) + \beta_1^{j-1}) = j \log \beta_1$ for all $j \in \mathbb{N}$ implies

$$\prod_{j=2}^{m+1}\left(\beta_1^{j-1}(\beta_1-1)+1\right)^{\frac{1}{\beta_1^{j-1}(\beta_1-1)}} = e^{\sum_{j=2}^{m+1}\frac{\log(\beta_1^{j-1}(\beta_1-1)+1)}{\beta_1^{j-1}(\beta_1-1)}} \leq e^{\frac{\log\beta_1}{\beta_1-1}\sum_{j=2}^{m+1}\frac{j}{\beta_1^{j-1}}},$$

and that

$$\sum_{j=2}^{+\infty}\frac{1}{\beta_1^{j-1}} < +\infty \quad \text{and} \quad \sum_{j=2}^{+\infty}\frac{j}{\beta_1^{j-1}} < +\infty.$$

Combining the above estimates with (3.17) and taking the limit as $m \to +\infty$, we discover that $z \in L^\infty(\mathbb{R}^N)$, that is, $u \in L^\infty(\mathbb{R}^N)$. From $(g1)$ and $(g2)'$, we deduce that $g(u) \in L^\infty(\mathbb{R}^N)$. Thanks to [24, Theorem 1], we infer that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$. Next, we deal with the case $p_2 = N$. In this context, it suffices to show that $u \in L^\infty(\mathbb{R}^N)$. In fact, once proved this, we can argue as in the case $p_2 < N$ to conclude that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\sigma}_{loc}(\mathbb{R}^N)$, for some $\sigma \in (0, 1)$. Even in this situation, we perform a convenient Moser iteration for z. In light of (3.1) and (3.2), for all $\varepsilon > 0$ and $\alpha > 0$ there exists $C_{\varepsilon,\alpha} > 0$ such that

$$0 \le g_1(t) \le \varepsilon\, t^{p_1-1} + C_\varepsilon t^{N-1} \Phi_N(\alpha t^{\frac{N}{N-1}}) \quad \text{for all } t \ge 0.$$

Hence, choosing $\varepsilon \in (0, 2\nu)$, (3.7) changes into

$$\int_{\mathbb{R}^N} |\nabla z|^{p_1-2} \nabla z \nabla \phi \, dx + \int_{\mathbb{R}^N} |\nabla z|^{p_2-2} \nabla z \nabla \phi \, dx \le C \int_{\mathbb{R}^N} z^{N-1} \Phi_N \left(\beta z^{\frac{N}{N-1}}\right) \phi \, dx, \tag{3.18}$$

for all $\phi \in \mathcal{W}$ such that $\phi \ge 0$. As before, for all $L > 0$ and $\gamma \ge 1$, we consider $z_L := \min\{z, L\}$, $\tilde{z}_L := z z_L^{N(\gamma-1)}$ and $w_L := z z_L^{\gamma-1}$. Inserting $\phi = \tilde{z}_L$ into (3.18), and utilizing (3.9)–(3.10), we obtain

$$\int_{\mathbb{R}^N} |\nabla z|^N z_L^{N(\gamma-1)} \, dx \le C \int_{\mathbb{R}^N} w_L^N \, \Phi_N \left(\alpha z^{\frac{N}{N-1}}\right) dx.$$

Reasoning as in (3.11), we find

$$\int_{\mathbb{R}^N} |\nabla w_L|^N \, dx \le C_1 \gamma^N \int_{\mathbb{R}^N} |\nabla z|^N z_L^{N(\gamma-1)} \, dx,$$

and so

$$\int_{\mathbb{R}^N} |\nabla w_L|^N \, dx \le C_2 \gamma^N \int_{\mathbb{R}^N} w_L^N \, \Phi_N \left(\alpha z^{\frac{N}{N-1}}\right) dx.$$

Pick $t > N$ and select $\alpha > 0$ such that $\frac{\alpha t}{t-N} \|\nabla z\|^{\frac{N}{N-1}}_{L^N(\mathbb{R}^N)} < \alpha_N$. Exploiting the Hölder inequality, (2.4), and (2.3), we see

$$\begin{aligned} \int_{\mathbb{R}^N} w_L^N \, \Phi_N(\alpha z^{\frac{N}{N-1}}) \, dx &\le \|w_L\|^N_{L^t(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |\Phi_N(\alpha z^{\frac{N}{N-1}})|^{\frac{t}{t-N}} \, dx \right)^{\frac{t-N}{t}} \le \|w_L\|^N_{L^t(\mathbb{R}^N)} \\ &\quad \left(\int_{\mathbb{R}^N} \left| \Phi_N \left(\frac{\alpha t}{t-N} \|\nabla z\|^{\frac{N}{N-1}}_{L^N(\mathbb{R}^N)} \left(\frac{z}{\|\nabla z\|_{L^N(\mathbb{R}^N)}} \right)^{\frac{N}{N-1}} \right) \right|^{\frac{t}{t-N}} dx \right)^{\frac{t-N}{t}} \\ &\le C_3 \|w_L\|^N_{L^t(\mathbb{R}^N)}, \end{aligned}$$

where $C_3 = C_3(\beta, t, N, \|\nabla z\|_{L^N(\mathbb{R}^N)}) > 0$. Accordingly,

$$\|\nabla w_L\|_{L^N(\mathbb{R}^N)} \le C_4 \gamma \|w_L\|_{L^t(\mathbb{R}^N)}. \tag{3.19}$$

Invoking the Gagliardo-Nirenberg interpolation inequality [2, Theorem 5.8], there is a $s > t$ such that

$$\|u\|_{L^s(\mathbb{R}^N)} \le C_5 \|u\|^{1-\theta}_{L^t(\mathbb{R}^N)} \|\nabla u\|^\theta_{L^N(\mathbb{R}^N)} \quad \text{for all } u \in W^{1,N}(\mathbb{R}^N),$$

where $\theta := \frac{s-t}{s} \in (0, 1)$. Thus, due to (3.19), we obtain

$$\|w_L\|_{L^s(\mathbb{R}^N)} \leq C_6\gamma \|w_L\|_{L^t(\mathbb{R}^N)},$$

and sending $L \to +\infty$ we arrive at

$$\|z\|_{L^{s\gamma}(\mathbb{R}^N)} \leq C_6^{\frac{1}{\gamma}} \gamma^{\frac{1}{\gamma}} \|z\|_{L^{t\gamma}(\mathbb{R}^N)}. \tag{3.20}$$

Set $\varsigma := \frac{s}{t} > 1$ and $\gamma := \varsigma^m$ with $m \in \mathbb{N} \cup \{0\}$. Then (3.20) becomes

$$\|z\|_{L^{t\varsigma^{m+1}}(\mathbb{R}^N)} \leq C_6^{\varsigma^{-m}} \varsigma^{m\varsigma^{-m}} \|z\|_{L^{t\varsigma^m}(\mathbb{R}^N)} \quad \text{for all } m \in \mathbb{N} \cup \{0\}. \tag{3.21}$$

Iterating (3.21), we find

$$\|z\|_{L^{t\varsigma^{m+1}}(\mathbb{R}^N)} \leq C_6^{\sum_{i=0}^m \varsigma^{-i}} \varsigma^{\sum_{i=0}^m i\varsigma^{-i}} \|z\|_{L^t(\mathbb{R}^N)} \quad \text{for all } m \in \mathbb{N} \cup \{0\}.$$

Letting $m \to +\infty$, we deduce that $z \in L^\infty(\mathbb{R}^N)$. Therefore, $u \in L^\infty(\mathbb{R}^N)$ even in the case $p_2 = N$.

Finally, we prove that u fulfills the Pohozaev type identity (3.5) arguing as in [7, Lemma 3.1]. Because u is locally Lipschitz (recall that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\sigma}_{loc}(\mathbb{R}^N)$), we can apply [18, Lemma 1] with $\mathcal{L}(s, \xi) := \frac{1}{p_1}|\xi|^{p_1} + \frac{1}{p_2}|\xi|^{p_2} - G(s)$, $f := 0$, $h(x) := \varphi_k(x)x$ for $k \in \mathbb{N}$, where $\varphi_k(x) := \varphi(\frac{x}{k})$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$ is such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$, to see

$$\begin{aligned} &\sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i\varphi_k\, x_j\, D_{\xi_i}\mathcal{L}(u, \nabla u) D_j u\, dx + \int_{\mathbb{R}^N} \varphi_k D_\xi \mathcal{L}(u, \nabla u)\nabla u\, dx \\ &\quad - \int_{\mathbb{R}^N} \nabla\varphi_k\, x\, \mathcal{L}(u, \nabla u)\, dx - N \int_{\mathbb{R}^N} \varphi_k \mathcal{L}(u, \nabla u)\, dx = 0. \end{aligned} \tag{3.22}$$

In order to pass to the limit as $k \to +\infty$ in (3.22), we note that $0 \leq \varphi_k(x) \leq 1$ and $|x\nabla\varphi_k(x)| \leq C$ for all $x \in \mathbb{R}^N$ and $k \in \mathbb{N}$, $\varphi_k \to 1$ and $\nabla\varphi_k \to 0$ as $k \to +\infty$, and $\mathcal{L}(u, \nabla u), D_\xi\mathcal{L}(u, \nabla u)\nabla u \in L^1(\mathbb{R}^N)$ (in view of $u \in \mathcal{W}$ and the growth assumptions on g). Thus the dominated convergence theorem yields

$$\int_{\mathbb{R}^N} (|\nabla u|^{p_1} + |\nabla u|^{p_2})\, dx - N \int_{\mathbb{R}^N} \left(\frac{1}{p_1}|\nabla u|^{p_1} + \frac{1}{p_2}|\nabla u|^{p_2} - G(u)\right) dx = 0,$$

that is, (3.5) is valid. The proof of the theorem is now complete. □

Remark 3.1 When $p_2 < N$, the proof of the fact that $z \in L^\infty(\mathbb{R}^N)$ can also be obtained by adopting the strategy in [24, Theorem 3-(i)]. However, here we prefer to give a different approach which allows us to consider even subcritical exponential nonlinearities.

Finally, we show that every solution to (1.1) has an exponential decay at infinity.

Theorem 3.2 *Let $u \in \mathcal{W}$ be a weak solution to (1.1). Then there exist $C, c > 0$ such that $|u(x)| \leq Ce^{-c|x|}$ for all $x \in \mathbb{R}^N$.*

Proof By virtue of $u \in L^\infty(\mathbb{R}^N)$ and $g(u) \in L^\infty(\mathbb{R}^N)$, we derive from [24, Theorem 1-(i)] that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leq C,$$

and so u is Lipschitz continuous in $\mathbb{R}^N$. Hence, u is uniformly continuous in $\mathbb{R}^N$, and because $u \in L^{p_1}(\mathbb{R}^N)$, we can infer that $|u(x)| \to 0$ as $|x| \to +\infty$. Let us now focus on the exponential decay estimate for u. The proof of Theorem 3.1 reveals that $|u|$ satisfies

$$-\Delta_{p_1}|u| - \Delta_{p_2}|u| + 2\nu(|u|^{p_1-1} + |u|^{p_2-1}) \leq g_1(|u|) \text{ in } \mathbb{R}^N,$$

Since $|u(x)| \to 0$ as $|x| \to +\infty$, it follows from (3.1) and $p_1 < p_2$ that there exists $R > 0$ such that

$$g_1(u) \leq \nu(|u|^{p_1-1} + |u|^{p_2-1}) \text{ in } \overline{B_R(0)}^c.$$

Consequently,

$$-\Delta_{p_1}|u| - \Delta_{p_2}|u| + \nu(|u|^{p_1-1} + |u|^{p_2-1}) \leq 0 \text{ in } \overline{B_R(0)}^c. \tag{3.23}$$

Define $\phi(x) := M' e^{\kappa R} e^{-\kappa|x|}$, where $\kappa, M' > 0$ are such that

$$\kappa < \min\left\{\left(\frac{\nu}{(p_1-1)}\right)^{\frac{1}{p_1}}, \left(\frac{\nu}{(p_2-1)}\right)^{\frac{1}{p_2}}\right\}$$

and $\|u\|_{L^\infty(\mathbb{R}^N)} \leq M'$. Obviously, $|u(x)| \leq \phi(x)$ for all $|x| \leq R$. It is easy to check that

$$\begin{aligned}
&-\Delta_{p_1}\phi - \Delta_{p_2}\phi + \nu\left(\phi^{p_1-1} + \phi^{p_2-1}\right) \\
&\quad = \phi^{p_1-1}\left[\nu - \kappa^{p_1}(p_1-1) + \frac{N-1}{|x|}\kappa^{p_1-1}\right] \\
&+ \phi^{p_2-1}\left[\nu - \kappa^{p_2}(p_2-1) + \frac{N-1}{|x|}\kappa^{p_2-1}\right] > 0 \text{ in } \overline{B_R(0)}^c.
\end{aligned} \tag{3.24}$$

Subtracting (3.24) from (3.23), and taking $\eta := (|u| - \phi)_+ \in W_0^{1,p_1}(\overline{B_R(0)}^c) \cap W_0^{1,p_2}(\overline{B_R(0)}^c)$ as test function, we discover

$$\begin{aligned}
0 &\geq \int_{\{|x|>R\,:\,|u(x)|>\phi(x)\}} \Big(\big((|\nabla|u||^{p_1-2}\nabla|u| - |\nabla\phi|^{p_1-2}\nabla\phi)\nabla\eta + (|\nabla|u||^{p_2-2}\nabla|u| - |\nabla\phi|^{p_2-2}\nabla\phi)\nabla\eta\big) \\
&\quad + \nu\left(\left(|u|^{p_1-1} - \phi^{p_1-1}\right) + \left(|u|^{p_2-1} - \phi^{p_2-1}\right)\right)\eta\Big)\,dx \\
&\geq \nu\int_{\{|x|>R\,:\,|u(x)|>\phi(x)\}} (|u|^{p_1-1} - \phi^{p_1-1})(|u| - \phi)\,dx \geq 0,
\end{aligned}$$

where we have used the fact that for all $r > 1$ it holds

$$(|\eta_1|^{r-2}\eta_1 - |\eta_2|^{r-2}\eta_2)(\eta_1 - \eta_2) > 0 \text{ for all } \eta_1, \eta_2 \in \mathbb{R}^N \text{ such that } \eta_1 \neq \eta_2. \tag{3.25}$$

Hence, $(|u|^{p_1-1} - \phi^{p_1-1})(|u| - \phi) = 0$ a.e. in $\{|x| > R : |u(x)| > \phi(x)\}$. Considering that $|u|$ and ϕ are continuous in $\mathbb{R}^N$, we conclude that $\{|x| > R : |u(x)| > \phi(x)\}$ is empty. Thus, $|u(x)| \leq \phi(x)$ for all $x \in \mathbb{R}^N$, and so the required estimate is true. □

Remark 3.2 When $p_2 < N$, because $z = |u|$ is a weak subsolution to $-\Delta_{p_1}z - \Delta_{p_2}z + 2\nu(z^{p_1-1} + z^{p_2-1}) = g_1(u)$ in $\mathbb{R}^N$ and $g_1(u) \leq \varepsilon z^{p_1-1} + C_\varepsilon z^{p_2^*-1}$, we can confirm that $z(x) = |u(x)| \to 0$ as $|x| \to +\infty$ by following the lines of the proof in [24, Theorem 3-(i)].

4 A first proof of Theorem 1.1

In this section, we provide a first approach to obtain the existence of a ground solution to (1.1). Since we are interested in weak solutions to (1.1), we seek critical points of the energy functional $L : \mathcal{W} \to \mathbb{R}$ given by

$$L(u) := \sum_{i=1}^{2} \frac{1}{p_i} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(u)\, dx.$$

From $(g1)$–$(g2)'$, it is readily seen that $L \in C^1(\mathcal{W}, \mathbb{R})$. Next we prove that L possesses a mountain-pass structure [6].

Lemma 4.1 *Assume that $(g1)$, $(g2)'$, and $(g3)$ hold. Then, L has a mountain pass geometry, that is:*

$(MP1)$ $L(0) = 0$.
$(MP2)$ *There exist $\rho, \delta > 0$ such that $L(u) \geq \delta$ for all $u \in \mathcal{W}$ such that $\|u\|_{\mathcal{W}} = \rho$.*
$(MP3)$ *There exists $w \in \mathcal{W}$ such that $\|w\|_{\mathcal{W}} > \rho$ and $L(w) < 0$.*

Proof $(MP1)$ is trivial. Let us verify $(MP2)$. We first assume that $p_2 < N$. Exploiting $(g1)$, $p_1 < p_2$, and that if $\lim_{n\to+\infty} a_n = a > 0$ then

$$\liminf_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} a_n \liminf_{n\to+\infty} b_n \text{ and } \limsup_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} a_n \limsup_{n\to+\infty} b_n,$$

we deduce

$$-\infty < \liminf_{t\to 0^+} \frac{g(t)}{t^{p_1-1} + t^{p_2-1}} \leq \limsup_{t\to 0^+} \frac{g(t)}{t^{p_1-1} + t^{p_2-1}} = -2\nu < 0.$$

In light of this fact, $(g2)'$, and that g is odd, we see that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$-G(t) \geq (2\nu - \varepsilon) \left(\frac{1}{p_1}|t|^{p_1} + \frac{1}{p_2}|t|^{p_2} \right) - C_\varepsilon |t|^{p_2^*} \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

Pick $\varepsilon \in (0, 2\nu)$. Thus (4.1) implies that, for all $u \in \mathcal{W}$,

$$\begin{aligned} L(u) &\geq \frac{1}{p_1}\|\nabla u\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{1}{p_2}\|\nabla u\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} + \frac{2\nu - \varepsilon}{p_1}\|u\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{2\nu - \varepsilon}{p_2}\|u\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} \\ &\quad - C_\varepsilon \|u\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} \geq \frac{1}{p_1}\min\{1, 2\nu - \varepsilon\}\|u\|^{p_1}_{W^{1,p_1}(\mathbb{R}^N)} + \frac{1}{p_2}\min\{1, 2\nu - \varepsilon\}\|u\|^{p_2}_{W^{1,p_2}(\mathbb{R}^N)} \\ &\quad - C_\varepsilon \|u\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)}. \end{aligned}$$

Let $u \in \mathcal{W}$ be such that $\|u\|_{\mathcal{W}} = \rho \in (0, 1)$. Since $1 < p_1 < p_2$ and $\|u\|_{W^{1,p_1}(\mathbb{R}^N)} < 1$, we know that $\|u\|^{p_1}_{W^{1,p_1}(\mathbb{R}^N)} \geq \|u\|^{p_2}_{W^{1,p_1}(\mathbb{R}^N)}$. Therefore, recalling that

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \quad \text{for all } a, b \geq 0 \text{ with } r \geq 1, \tag{4.2}$$

and utilizing the continuous embedding $\mathcal{W} \subset L^{p_2^*}(\mathbb{R}^N)$, we get

$$\begin{aligned} L(u) &\geq C_1(\|u\|^{p_2}_{W^{1,p_1}(\mathbb{R}^N)} + \|u\|^{p_2}_{W^{1,p_2}(\mathbb{R}^N)}) - C_\varepsilon \|u\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} \\ &\geq C_2(\|u\|_{W^{1,p_1}(\mathbb{R}^N)} + \|u\|_{W^{1,p_2}(\mathbb{R}^N)})^{p_2} - C_\varepsilon \|u\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} \end{aligned}$$

$$\geq C_2\|u\|_{\mathcal{W}}^{p_2} - C_3\|u\|_{\mathcal{W}}^{p_2^*}.$$

Taking

$$\rho \in \left(0, \min\left\{1, \left(\frac{C_2}{C_3}\right)^{\frac{1}{p_2^*-p_2}}\right\}\right),$$

we obtain

$$\inf_{\|u\|_{\mathcal{W}}=\rho} L(u) \geq \rho^{p_2}(C_2 - C_3\rho^{p_2^*-p_2}) =: \delta > 0.$$

Now we suppose $p_2 = N$. In view of

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p_1-1} + t^{N-1}} = -2\nu,$$

and $(g2)'$, we have that fixed $\varepsilon > 0$, $q > N$, and $\alpha \in (0, \alpha_N)$, we can find $C_{\varepsilon,q,\alpha} > 0$ such that

$$G(t) \leq (\varepsilon - 2\nu)\left(\frac{1}{p_1}|t|^{p_1} + \frac{1}{N}|t|^N\right) + C_{\varepsilon,q,\alpha}|t|^q\,\Phi_N\left(\alpha|t|^{\frac{N}{N-1}}\right) \quad \text{for all } t \in \mathbb{R}. \tag{4.3}$$

Thanks to (4.3), we deduce that, for all $u \in \mathcal{W}$,

$$\begin{aligned} L(u) &\geq \frac{1}{p_1}\|\nabla u\|_{L^{p_1}(\mathbb{R}^N)}^{p_1} + \frac{1}{N}\|\nabla u\|_{L^N(\mathbb{R}^N)}^N + \frac{2\nu-\varepsilon}{p_1}\|u\|_{L^{p_1}(\mathbb{R}^N)}^{p_1} + \frac{2\nu-\varepsilon}{N}\|u\|_{L^N(\mathbb{R}^N)}^N \\ &\quad - C_{\varepsilon,q,\alpha}\int_{\mathbb{R}^N}|u|^q\Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)dx \\ &\geq C_1\|u\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} + C_2\|u\|_{W^{1,N}(\mathbb{R}^N)}^N - C_{\varepsilon,q,\alpha}\int_{\mathbb{R}^N}|u|^q\Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)dx. \end{aligned}$$

Select $\sigma > 1$ such that $\alpha\sigma < \alpha_N$. Using the Hölder inequality with exponents σ and $\sigma' := \frac{\sigma}{\sigma-1}$, and employing (2.4), we see

$$\begin{aligned} \int_{\mathbb{R}^N}|u|^q\Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)dx &\leq \|u\|_{L^{\sigma' q}(\mathbb{R}^N)}^q\left(\int_{\mathbb{R}^N}|\Phi_N(\alpha|u|^{\frac{N}{N-1}})|^\sigma\,dx\right)^{\frac{1}{\sigma}} \\ &\leq \|u\|_{L^{\sigma' q}(\mathbb{R}^N)}^q\left(\int_{\mathbb{R}^N}\Phi_N(\alpha\sigma|u|^{\frac{N}{N-1}})\,dx\right)^{\frac{1}{\sigma}}. \end{aligned}$$

Then, invoking (2.3) and the continuous embedding $W^{1,N}(\mathbb{R}^N) \subset L^{\sigma' q}(\mathbb{R}^N)$ (note that $\sigma' q > N$), we can infer that for all $u \in W^{1,N}(\mathbb{R}^N)$ such that $\|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1$, it holds

$$\int_{\mathbb{R}^N}|u|^q\Phi_N\left(\alpha|u|^{\frac{N}{N-1}}\right)dx \leq C_3\|u\|_{W^{1,N}(\mathbb{R}^N)}^q. \tag{4.4}$$

Pick $u \in \mathcal{W}$ such that $\|u\|_{\mathcal{W}} = \rho \in (0, 1)$. Thus, $\|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1$ and $\|u\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} \geq \|u\|_{W^{1,p_1}(\mathbb{R}^N)}^N$. Hence, (4.2) and (4.4) yield

$$\begin{aligned} L(u) &\geq C_4(\|u\|_{W^{1,p_1}(\mathbb{R}^N)}^N + \|u\|_{W^{1,N}(\mathbb{R}^N)}^N) - C_5\|u\|_{W^{1,N}(\mathbb{R}^N)}^q \\ &\geq C_6\|u\|_{\mathcal{W}}^N - C_5\|u\|_{\mathcal{W}}^q. \end{aligned}$$

Choosing

$$\rho \in \left(0, \min\left\{1, \left(\frac{C_6}{C_5}\right)^{\frac{1}{q-N}}\right\}\right),$$

we find

$$\inf_{\|u\|_{\mathcal{W}}=\rho} L(u) \geq \rho^N (C_6 - C_5 \rho^{q-N}) =: \delta > 0.$$

Finally, we check $(MP3)$. For all $R > 0$, we consider

$$w_R(x) := \begin{cases} \xi & \text{if } |x| \leq R, \\ \xi\,(R+1-|x|) & \text{if } R \leq |x| \leq R+1, \\ 0 & \text{if } |x| \geq R+1. \end{cases}$$

It is clear that $w_R \in \mathcal{W}_{\mathrm{r}}$. Using $(g3)$, we can see that, for $R > 0$ large enough,

$$\int_{\mathbb{R}^N} G(w_R)\, dx \geq 1.$$

Fix such an $R > 0$ and set $w_{R,\theta}(x) := w_R(x/e^\theta)$. Then we have

$$\begin{aligned} L(w_{R,\theta}) &= \sum_{i=1}^{2} \frac{1}{p_i} e^{(N-p_i)\theta} \|\nabla w_R\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - e^{N\theta} \int_{\mathbb{R}^N} G(w_R)\, dx \\ &\leq \sum_{i=1}^{2} \frac{1}{p_i} e^{(N-p_i)\theta} \|\nabla w_R\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - e^{N\theta} \to -\infty \quad \text{as } \theta \to +\infty. \end{aligned}$$

The proof of the lemma is now complete. □

Remark 4.1 From the proof of Lemma 4.1-$(MP2)$, it follows that

$$L(u) > 0 \quad \text{for all } u \in \mathcal{W} \text{ such that } 0 < \|u\|_{\mathcal{W}} \leq \rho.$$

Remark 4.2 If $u \in \mathcal{W}\setminus\{0\}$ is such that $L'(u) = 0$, then we can prove that there exists $C > 0$, independent of u, such that

$$\|u\|_{\mathcal{W}} \geq C. \tag{4.5}$$

In fact, if $\|u\|_{\mathcal{W}} \geq 1$, then (4.5) holds. If $0 < \|u\|_{\mathcal{W}} < 1$, then we can argue as in the proof of $(MP2)$ to see that

$$0 = \langle L'(u), u\rangle \geq \begin{cases} c_1 \|u\|^{p_2}_{\mathcal{W}} - c_2 \|u\|^{p_2^*}_{\mathcal{W}} & \text{if } p_2 < N, \\ c_1' \|u\|^{p_2}_{\mathcal{W}} - c_2' \|u\|^{q}_{\mathcal{W}} & \text{if } p_2 = N, \end{cases}$$

for some $c_1, c_2, c_1', c_2' > 0$, where $q > N$ in the case $p_2 = N$. Since $p_2 < p_2^*$ when $p_2 < N$, $p_2 < q$ when $p_2 = N$, and $\|u\|_{\mathcal{W}} > 0$, we deduce

$$\|u\|_{\mathcal{W}} \geq \begin{cases} (c_1/c_2)^{\frac{1}{p_2^*-p_2}} & \text{if } p_2 < N, \\ (c_1'/c_2')^{\frac{1}{q-p_2}} & \text{if } p_2 = N. \end{cases}$$

Therefore, (4.5) is valid.

Taking Lemma 4.1 into account, we can define the mountain pass level

$$c_{\mathrm{MP}} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} L(\gamma(t)), \tag{4.6}$$

and the set of paths

$$\Gamma := \{\gamma \in C([0,1], \mathcal{W}) : \gamma(0) = 0, L(\gamma(1)) < 0\}. \tag{4.7}$$

Motivated by [27], we produce a Palais-Smale sequence of L at the level c_{MP} that satisfies asymptotically the Pohozaev identity.

Proposition 4.1 *There exists a Pohozaev-Palais-Smale sequence $(u_n) \subset \mathcal{W}$ for L at the level c_{MP}, that is,*

$$L(u_n) \to c_{\mathrm{MP}},\ L'(u_n) \to 0 \text{ in } \mathcal{W}',\ P(u_n) \to 0. \tag{4.8}$$

Proof It suffices to argue as in [7, Proposition 3.1]. For reader's convenience, we provide the details. Let us introduce the map $\Phi : \mathbb{R} \times \mathcal{W} \to \mathcal{W}$ by setting

$$\Phi(\theta, u)(x) := u(e^{-\theta}x),$$

for $\theta \in \mathbb{R}$, $u \in \mathcal{W}$, and $x \in \mathbb{R}^N$. Here $\mathbb{R} \times \mathcal{W}$ is equipped with the norm

$$\|(\theta, u)\|_{\mathbb{R}\times\mathcal{W}} := |\theta| + \|u\|_{\mathcal{W}}.$$

For every $\theta \in \mathbb{R}$ and $u \in \mathcal{W}$, the functional $L \circ \Phi$ is given by

$$L(\Phi(\theta, u)) = \sum_{i=1}^{2} \frac{e^{(N-p_i)\theta}}{p_i} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - e^{N\theta} \int_{\mathbb{R}^N} G(u)\, dx.$$

Clearly, $L \circ \Phi \in C^1(\mathcal{W}, \mathbb{R})$. Reasoning as in the proof of Lemma 4.1, we can easily check that $L \circ \Phi$ has a mountain pass geometry, and so it is well-defined the mountain pass level of $L \circ \Phi$:

$$\widetilde{c}_{\mathrm{MP}} := \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \max_{t\in[0,1]} (L \circ \Phi)(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma} := \{\widetilde{\gamma} \in C([0,1], \mathbb{R} \times \mathcal{W}) : \widetilde{\gamma}(0) = (0,0), (L \circ \Phi)(\widetilde{\gamma}(1)) < 0\}.$$

It is readily verified that $\widetilde{c}_{\mathrm{MP}} = c_{\mathrm{MP}}$. Invoking the general minimax principle [46, Theorem 2.8], we can select a sequence $((\theta_n, v_n)) \subset \mathbb{R} \times \mathcal{W}$ such that, as $n \to +\infty$,

(i) $(L \circ \Phi)(\theta_n, v_n) \to c_{\mathrm{MP}}$,
(ii) $(L \circ \Phi)'(\theta_n, v_n) \to 0$ in $(\mathbb{R} \times \mathcal{W})'$,
(iii) $\theta_n \to 0$.

Indeed, due to (4.6) and (4.7), we can find $(\gamma_n) \subset \Gamma$ such that $\max_{t\in[0,1]} L(\gamma_n(t)) \le c_{\mathrm{MP}} + \frac{1}{n^2}$. Put $\tilde{\gamma}_n(t) := (0, \gamma_n(t)) \in \widetilde{\Gamma}$. Thus,

$$\max_{t\in[0,1]} (L \circ \Phi)(\tilde{\gamma}_n(t)) = \max_{t\in[0,1]} L(\gamma_n(t)) \le c_{\mathrm{MP}} + \frac{1}{n^2}.$$

According to [46, Theorem 2.8], there exists $((\theta_n, v_n)) \subset \mathbb{R} \times \mathcal{W}$ such that (i) and (ii) are true, and

$$\mathrm{dist}_{\mathbb{R}\times\mathcal{W}}((\theta_n, v_n), \{0\} \times \gamma_n([0,1])) \to 0,$$

which implies (iii). Here, we have used the notation

$$\mathrm{dist}_{\mathbb{R}\times\mathcal{W}}((\theta,u),A) := \inf_{(\tau,v)\in A}(|\theta-\tau|+\|u-v\|_{\mathcal{W}}) \quad \text{for all } A\subset\mathbb{R}\times\mathcal{W}.$$

For all $(h,w)\in\mathbb{R}\times\mathcal{W}$, it holds

$$\langle (L\circ\Phi)'(\theta_n,v_n),(h,w)\rangle = \langle L'(\Phi(\theta_n,v_n)),\Phi(\theta_n,w)\rangle + P(\Phi(\theta_n,v_n))h. \tag{4.9}$$

Put $u_n := \Phi(\theta_n,v_n)$. Thanks to (i), we deduce that $L(u_n)\to c_{\mathrm{MP}}$. Choosing $h=1$ and $w=0$ in (4.9), and using (ii), we obtain

$$P(u_n)\to 0.$$

Finally, for every fixed $\varphi\in\mathcal{W}$, taking $w(x)=\varphi(e^{\theta_n}x)$ and $h=0$ in (4.9), it follows from (ii) and (iii) that

$$\langle L'(u_n),\varphi\rangle = o_n(1)\|\varphi(e^{\theta_n}\cdot)\|_{\mathcal{W}} = o_n(1)\|\varphi\|_{\mathcal{W}},$$

and so $L'(u_n)\to 0$ in $\mathcal{W}'$. Therefore, the sequence (u_n) fulfills (4.8). □

Next we prove the boundedness of Pohozaev-Palais-Smale sequences of L.

Lemma 4.2 *Every sequence $(u_n)\subset\mathcal{W}$ satisfying (4.8) is bounded in $\mathcal{W}$.*

Proof From (4.8), we know

$$c_{\mathrm{MP}}+o_n(1) = L(u_n)-\frac{1}{N}P(u_n) = \sum_{i=1}^{2}\frac{1}{N}\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)},$$

which implies that $(\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i=1,2$. In particular, when $p_2<N$, $(\|u_n\|_{L^{p_i^*}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i=1,2$. It remains to verify that $(\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)}+\|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$. Arguing indirectly, suppose that

$$\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)}+\|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}\to+\infty.$$

Define

$$t_n := \left(\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)}+\|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}\right)^{-\frac{1}{N}}\to 0,$$

and

$$v_n(x) := u_n(x/t_n).$$

For all $i=1,2$, we see

$$\|v_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} = t_n^N\|u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\le 1,$$
$$\|\nabla v_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} = t_n^{N-p_i}\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}.$$

We claim

$$\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n|^{p_2}\,dx\to 0 \quad \text{as } n\to+\infty. \tag{4.10}$$

To this end, we show that $\tilde{v}_n := v_n(\cdot+y_n)\rightharpoonup 0$ in $\mathcal{W}$ for every sequence $(y_n)\subset\mathbb{R}^N$. Since $(\tilde{v}_n)$ is bounded in $\mathcal{W}$, up to a subsequence, we may assume that $\tilde{v}_n\rightharpoonup\tilde{v}$ in $\mathcal{W}$, $\tilde{v}_n\to\tilde{v}$ in $L^r_{loc}(\mathbb{R}^N)$

for all $r \in [1, p_2^*)$, and $\tilde{v}_n \to \tilde{v}$ a.e. in $\mathbb{R}^N$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $\varphi_n(x) := \varphi(t_n x - y_n)$. Then it holds

$$\begin{aligned} t_n^N \langle L'(u_n), \varphi_n \rangle &= t_n^N \left[\sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla u_n|^{p_i-2} \nabla u_n \nabla \varphi_n \, dx - \int_{\mathbb{R}^N} g(u_n)\varphi_n \, dx \right] \\ &= \sum_{i=1}^2 \left[t_n^{p_i} \int_{\mathbb{R}^N} |\nabla \tilde{v}_n|^{p_i-2} \nabla \tilde{v}_n \nabla \varphi \, dx \right] - \int_{\mathbb{R}^N} g(\tilde{v}_n)\varphi \, dx. \end{aligned} \tag{4.11}$$

Let us observe that $t_n^N \langle L'(u_n), \varphi_n \rangle = o_n(1)$ owing to

$$\begin{aligned} \left| t_n^N \langle L'(u_n), \varphi_n \rangle \right| &\le t_n^N \|L'(u_n)\|_{\mathcal{W}'} \|\varphi_n\|_{\mathcal{W}} \\ &= t_n^N \|L'(u_n)\|_{\mathcal{W}'} \left\{ \sum_{i=1}^2 \left[t_n^{-\frac{N-p_i}{p_i}} \|\nabla \varphi\|_{L^{p_i}(\mathbb{R}^N)} + t_n^{-\frac{N}{p_i}} \|\varphi\|_{L^{p_i}(\mathbb{R}^N)} \right] \right\} \\ &= \|L'(u_n)\|_{\mathcal{W}'} \left\{ \sum_{i=1}^2 \left[t_n^{1+\frac{(p_i-1)N}{p_i}} \|\nabla \varphi\|_{L^{p_i}(\mathbb{R}^N)} + t_n^{\frac{(p_i-1)N}{p_i}} \|\varphi\|_{L^{p_i}(\mathbb{R}^N)} \right] \right\} \\ &\to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Exploiting this fact, $(\tilde{v}_n)$ is bounded in $\mathcal{W}$, $\tilde{v}_n \to \tilde{v}$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, p_2^*)$, $\tilde{v}_n \to \tilde{v}$ a.e. in $\mathbb{R}^N$, $t_n \to 0$, and the compactness lemma of Strauss [12, Theorem A.I], it follows from (4.11) that

$$\int_{\mathbb{R}^N} g(\tilde{v})\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Therefore, $g(\tilde{v}) \equiv 0$. Since $\tilde{v} \in \mathcal{W}$ and $t = 0$ is an isolated solution of $g(t) = 0$ (by $(g1)$), we have that $\tilde{v} \equiv 0$. Hence, $\tilde{v}_n \rightharpoonup 0$ in $\mathcal{W}$ and $\tilde{v}_n \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, p_2^*)$. As a result, (4.10) is true. Then, by Lemma 2.2, we infer that $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (p_2, p_2^*)$. Now, using $\langle L'(u_n), t_n^N u_n \rangle = o_n(1)$ (note that $t_n^N u_n \to 0$ in $\mathcal{W}$ because $\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} \to +\infty$ and $(\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i = 1, 2$), $t_n \to 0$, and the boundedness of $(\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)})$ in $\mathbb{R}$ for all $i = 1, 2$, we see

$$\begin{aligned} \int_{\mathbb{R}^N} g(v_n)v_n \, dx &= t_n^N \int_{\mathbb{R}^N} g(u_n)u_n \, dx \\ &= t_n^N \left[\sum_{i=1}^2 \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} \right] - \langle L'(u_n), t_n^N u_n \rangle = o_n(1). \end{aligned} \tag{4.12}$$

Let us recall that $g = g_1 - g_2$. In light of (3.1) and (3.2), we can apply Lemma 2.4 with $\Psi(t) = g_1(t)t$ (see Remark 2.2) to discover

$$\int_{\mathbb{R}^N} g_1(v_n)v_n \, dx \to 0 \quad \text{as } n \to +\infty. \tag{4.13}$$

Combining (3.3) with (4.12) and (4.13), we obtain

$$\begin{aligned} 2\nu(\|v_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|v_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}) &\le \int_{\mathbb{R}^N} g_2(v_n)v_n \, dx \\ &= \int_{\mathbb{R}^N} g_1(v_n)v_n \, dx + o_n(1) = o_n(1), \end{aligned}$$

and thus $\|v_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|v_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} \to 0$, which is a contradiction due to

$$\|v_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|v_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} = t_n^N \left[\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}\right] = 1 \quad \text{for all } n \in \mathbb{N}.$$

Hence, $(\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$, and so $(\|u_n\|_{L^{p_i}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i = 1, 2$. Therefore, (u_n) is bounded in $\mathcal{W}$. □

Remark 4.3 When $p_2 < N$, we can show that $(\|u_n\|_{L^{p_i}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i = 1, 2$ in a more direct way. Indeed, because $P(u_n) = o_n(1)$, we have

$$\sum_{i=1}^{2} \left(\frac{N - p_i}{p_i}\right) \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + N \int_{\mathbb{R}^N} G_2(u_n)\, dx = N \int_{\mathbb{R}^N} G_1(u_n)\, dx + o_n(1).$$

Using (3.4) with $\varepsilon = \frac{1}{2}$, and the boundedness of $(\|u_n\|_{L^{p_2^*}(\mathbb{R}^N)})$, we see

$$\begin{aligned}\int_{\mathbb{R}^N} G_2(u_n)\, dx &= \int_{\mathbb{R}^N} G_1(u_n)\, dx - \sum_{i=1}^{2} \left(\frac{N - p_i}{N p_i}\right) \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + o_n(1) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} G_2(u_n)\, dx + C'_{\frac{1}{2}} \|u_n\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} + o_n(1) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} G_2(u_n)\, dx + C'' + o_n(1),\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} G_2(u_n)\, dx \leq C''' + o_n(1).$$

Since (3.3) guarantees that

$$G_2(t) \geq 2\nu \left(\frac{|t|^{p_1}}{p_1} + \frac{|t|^{p_2}}{p_2}\right) \quad \text{for all } t \in \mathbb{R},$$

we deduce that $(\|u_n\|_{L^{p_i}(\mathbb{R}^N)})$ is bounded in $\mathbb{R}$ for all $i = 1, 2$. Consequently, (u_n) is bounded in $\mathcal{W}$.

The result below will be crucial to ensure the almost everywhere convergence of the gradients of Pohozaev-Palais-Smale sequences.

Lemma 4.3 *Let $(u_n) \subset \mathcal{W}$ be a bounded sequence such that $L'(u_n) \to 0$ in $\mathcal{W}'$ as $n \to +\infty$. Up to a subsequence, we assume that for some $u \in \mathcal{W}$, as $n \to +\infty$,*

$$u_n \rightharpoonup u \text{ in } \mathcal{W}, \;\; u_n \to u \text{ in } L^s_{loc}(\mathbb{R}^N) \;\text{ for all } s \in [1, p_2^*), \;\; u_n \to u \text{ a.e. in } \mathbb{R}^N. \tag{4.14}$$

Then, up to a subsequence, as $n \to +\infty$,

$$\begin{aligned}&\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \\ &|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u \quad \text{in } \left(L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N)\right)^N \text{ for all } i = 1, 2.\end{aligned}$$

Proof We follow [7, formula (58) in Proposition 4.1]. Pick $\eta > 0$ and consider the truncation function $T_\eta : \mathbb{R} \to \mathbb{R}$ at height η defined as

$$T_\eta(t) := \begin{cases} t & \text{if } |t| \le \eta, \\ \eta \frac{t}{|t|} & \text{if } |t| \ge \eta. \end{cases}$$

Take $R > 0$ and select $\psi_R \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \psi_R \le 1$ in $\mathbb{R}^N$, $\psi_R = 1$ in $B_R(0)$, and $\psi_R = 0$ in $B_{2R}^c(0)$. We can write

$$\begin{aligned} &\int_{\mathbb{R}^N} \psi_R \left[|\nabla u_n|^{p_1-2}\nabla u_n - |\nabla u|^{p_1-2}\nabla u\right] \nabla T_\eta(u_n - u)\, dx \\ &\quad + \int_{\mathbb{R}^N} \psi_R \left[|\nabla u_n|^{p_2-2}\nabla u_n - |\nabla u|^{p_2-2}\nabla u\right] \nabla T_\eta(u_n - u)\, dx \\ &= -\int_{\mathbb{R}^N} T_\eta(u_n - u) \left[|\nabla u_n|^{p_1-2}\nabla u_n + |\nabla u_n|^{p_2-2}\nabla u_n\right] \nabla \psi_R\, dx \\ &\quad - \int_{\mathbb{R}^N} \psi_R \left[|\nabla u|^{p_1-2}\nabla u + |\nabla u|^{p_2-2}\nabla u\right] \nabla T_\eta(u_n - u)\, dx + \langle L'(u_n), \psi_R T_\eta(u_n - u)\rangle \\ &\quad + \int_{\mathbb{R}^N} g(u_n)\psi_R T_\eta(u_n - u)\, dx =: X^1_{n,\eta,R} + X^2_{n,\eta,R} + X^3_{n,\eta,R} + X^4_{n,\eta,R}. \end{aligned} \tag{4.15}$$

Employing (4.15), we obtain that $T_\eta(u_n - u) \rightharpoonup 0$ in $\mathcal{W}$ and $T_\eta(u_n - u) \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for all $s \in [1, p_2^*)$. On the other hand, $\langle L'(u_n), \psi_R T_\eta(u_n - u)\rangle \to 0$ due to $L'(u_n) \to 0$ in $\mathcal{W}'$ and the boundedness of $(\psi_R T_\eta(u_n - u))$ in $\mathcal{W}$. Hence,

$$X^j_{n,\eta,R} \to 0 \quad \text{as } n \to +\infty \quad \text{for all } j = 1, 2, 3. \tag{4.16}$$

Utilizing $|T_\eta(t)| \le \eta$ for all $t \in \mathbb{R}$, $0 \le \psi_R \le 1$, $\text{supp}(\psi_R) \subset B_{2R}(0)$, the growth assumptions on g, the Hölder inequality, and the boundedness of (u_n) in $\mathcal{W}$, we get

$$|X^4_{n,\eta,R}| \le C_R \eta \quad \text{for all } n \in \mathbb{N}, \tag{4.17}$$

where $C_R > 0$ is a constant that depends only on R. Let us observe that to arrive at (4.17), we exploit $|g(t)| \le C(|t|^{p_1-1} + |t|^{p_2^*-1})$ for all $t \in \mathbb{R}$ when $p_2 < N$, whereas if $p_2 = N$ then we invoke the Trudinger-Moser inequality. Since the verification in the case $p_2 < N$ is straightforward, we provide the details in the case $p_2 = N$. Because (u_n) is bounded in $\mathcal{W}$, there exists $C_0 > 0$ such that $\|u_n\|_{\mathcal{W}} \le C_0$ for all $n \in \mathbb{N}$. Fix $q > N$ and $\alpha > 0$ such that $\alpha q' C_0^{\frac{N}{N-1}} < \alpha_N$, where $q' := \frac{q}{q-1}$. In view of $(g1)$ and $(g2)'$, there exists $C_1 > 0$ such that

$$|g(t)| \le C_1 \left(|t|^{p_1-1} + \Psi_N\left(\alpha|t|^{\frac{N}{N-1}}\right)\right) \quad \text{for all } t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} |X^4_{n,\eta,R}| &\le C_1\eta \int_{\mathbb{R}^N} |u_n|^{p_1-1}\psi_R\, dx + C_1\eta \int_{\mathbb{R}^N} \Psi_N\left(\alpha|u_n|^{\frac{N}{N-1}}\right)\psi_R\, dx \\ &=: \eta C_1 A_{n,R} + \eta C_1 B_{n,R}. \end{aligned}$$

Thanks to the Hölder inequality, $0 \le \psi_R \le 1$, $\text{supp}(\psi_R) \subset B_{2R}(0)$, and $\|u_n\|_{\mathcal{W}} \le C_0$ for all $n \in \mathbb{N}$, we see that $A_{n,R} \le C_2 C'_R$ for all $n \in \mathbb{N}$. Concerning $B_{n,R}$, using the Hölder inequality, (2.4), $0 \leq \psi_R \leq 1$, $\mathrm{supp}(\psi_R) \subset B_{2R}(0)$, (2.3), and $\|u_n\|_{\mathcal{W}} \leq C_0$ for all $n \in \mathbb{N}$, we learn

$$B_{n,R} \leq \left(\int_{\mathbb{R}^N} \Phi_N \left(\alpha q' C_0^{\frac{N}{N-1}} \left(\frac{|u_n|}{C_0} \right)^{\frac{N}{N-1}} \right) dx \right)^{\frac{1}{\sigma_2}} \|\psi_R\|_{L^q(\mathbb{R}^N)} \leq C_3 C_R'' \quad \text{for all } n \in \mathbb{N}.$$

Thus we can deduce that (4.17) is true even if $p_2 = N$.

Let us now note that the integrands on the left-hand side in (4.15) are nonnegative by virtue of $\psi_R \geq 0$, (3.25), and the definition of T_η. Then, combining (4.15), (4.16), (4.17), and recalling that $\psi_R = 1$ in $B_R(0)$, we have

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{B_R(0)} \left[(|\nabla u_n|^{p_1-2}\nabla u_n - |\nabla u|^{p_1-2}\nabla u) + (|\nabla u_n|^{p_2-2}\nabla u_n - |\nabla u|^{p_2-2}\nabla u) \right] \nabla T_\eta(u_n - u)\, dx \\ &\leq C_R \eta. \end{aligned} \tag{4.18}$$

Define

$$\begin{aligned} e_n(x) := &(|\nabla u_n(x)|^{p_1-2}\nabla u_n(x) - |\nabla u(x)|^{p_1-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) \\ &+ (|\nabla u_n(x)|^{p_2-2}\nabla u_n(x) - |\nabla u(x)|^{p_2-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)). \end{aligned}$$

In light of (3.25), we know that $e_n \geq 0$ in $\mathbb{R}^N$. Moreover, (e_n) is bounded in $L^1(\mathbb{R}^N)$ because (∇u_n) is bounded in $(L^{p_i}(\mathbb{R}^N))^N$ and $(|\nabla u_n|^{p_i-2}\nabla u_n)$ is bounded in $(L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N$ for all $i = 1, 2$. Take $\theta \in (0, 1)$ and split $B_R(0)$ by considering the sets

$$\mathcal{X}_R^\eta := \{x \in B_R(0) : |u_n(x) - u(x)| \leq \eta\}$$

and

$$\mathcal{Y}_R^\eta := \{x \in B_R(0) : |u_n(x) - u(x)| \geq \eta\}.$$

From the Hölder inequality, it follows that

$$\begin{aligned} \int_{B_R(0)} e_n^\theta\, dx &= \int_{\mathcal{X}_R^\eta} e_n^\theta\, dx + \int_{\mathcal{Y}_R^\eta} e_n^\theta\, dx \\ &\leq \left(\int_{\mathcal{X}_R^\eta} e_n\, dx \right)^\theta |\mathcal{X}_R^\eta|^{1-\theta} + \left(\int_{\mathcal{Y}_R^\eta} e_n\, dx \right)^\theta |\mathcal{Y}_R^\eta|^{1-\theta}. \end{aligned} \tag{4.19}$$

We stress that, for $\eta > 0$ fixed, $|\mathcal{Y}_R^\eta| \to 0$ as $n \to +\infty$. Employing this, the boundedness of (e_n) in $L^1(\mathbb{R}^N)$, (4.18), and (4.19), we obtain

$$\limsup_{n\to+\infty} \int_{B_R(0)} e_n^\theta\, dx \leq (C_R\eta)^\theta |B_R(0)|^{1-\theta}.$$

Letting $\eta \to 0^+$, we deduce that $e_n^\theta \to 0$ in $L^1(B_R(0))$. Hence, up to a subsequence, $e_n \to 0$ a.e. in $B_R(0)$. Since $R > 0$ is arbitrary, up to a subsequence, $e_n \to 0$ a.e. in $\mathbb{R}^N$. This fact and (3.25) ensure that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Given that $(|\nabla u_n|^{p_i-2}\nabla u_n)$ is bounded in $(L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N$ for all $i = 1, 2$, we can infer that, up to a subsequence, $|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u$ in $(L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N$ for all $i = 1, 2$. □

We now focus on the convergence of Pohozaev-Palais-Smale sequences of L. More precisely, we establish the next result.

Proposition 4.2 *Let $(u_n) \subset \mathcal{W}$ be a sequence such that*

(i) $(L(u_n))$ is bounded,
(ii) $L'(u_n) \to 0$ in $\mathcal{W}'$ and $P(u_n) \to 0$ as $n \to +\infty$.

Then,

(1) either up to a subsequence, $u_n \to 0$ in $\mathcal{W}$ as $n \to +\infty$,
(2) or we can find $u \in \mathcal{W}\backslash\{0\}$ such that $L'(u) = 0$ and $(x_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot - x_n) \rightharpoonup u$ in $\mathcal{W}$ as $n \to +\infty$.

Proof We start by observing that Lemma 4.2 implies that (u_n) is bounded in $\mathcal{W}$. Let us suppose that (1) does not hold. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} \|u_n\|_{W^{1,p_2}(\mathbb{R}^N)} > 0. \tag{4.20}$$

We claim that, for every $r \in (p_2, p_2^*)$,

$$\liminf_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} > 0. \tag{4.21}$$

We first examine the case $p_2 < N$. Suppose by contradiction that, for some $r \in (p_2, p_2^*)$,

$$\liminf_{n\to+\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} = 0.$$

On account of

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx = \frac{N-p_1}{Np_1}\|\nabla u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{N-p_2}{Np_2}\|\nabla u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} G_2(u_n)\,dx - \frac{1}{N}P(u_n),$$

and using (3.3), $P(u_n) \to 0$, and (4.20), we see

$$\begin{aligned} &\liminf_{n\to+\infty}\int_{\mathbb{R}^N} G_1(u_n)\,dx \\ &\geq \liminf_{n\to+\infty}\Big[\frac{N-p_1}{Np_1}\|\nabla u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{N-p_2}{Np_2}\|\nabla u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)} \\ &\quad + 2\nu\left(\frac{1}{p_1}\|u_n\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{1}{p_2}\|u_n\|^{p_2}_{L^{p_2}(\mathbb{R}^N)}\right) - \frac{1}{N}P(u_n)\Big] > 0. \end{aligned} \tag{4.22}$$

On the other hand, thanks to (3.1), (3.2), and Lemma 2.1 with $t = p_2$ and $s = r$, we can argue as in the proof of Lemma 2.4 with $\Psi(t) = G_1(t)$ to get

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N} G_1(u_n)\,dx = 0,$$

which contradicts (4.22). Hence, (4.21) is valid. Now we deal with the case $p_2 = N$. Since (u_n) is bounded in $W^{1,N}(\mathbb{R}^N)$, there exists $M > 0$ such that $\|\nabla u_n\|_{L^N(\mathbb{R}^N)} \leq M$ for all $n \in \mathbb{N}$. Pick $r \in (N, +\infty)$ and $\alpha > 0$ such that $\alpha M^{\frac{N}{N-1}} < \alpha_N$. From (3.1) and (3.2), we know that fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$g_1(t)t \leq \varepsilon(|t|^{p_1} + \Phi_N(\alpha|t|^{\frac{N}{N-1}})) + C_\varepsilon|t|^r \quad \text{for all } t \in \mathbb{R}. \tag{4.23}$$

Exploiting $\langle L'(u_n), u_n\rangle = o_n(1)$, (3.3), and (4.23), we obtain

$$o_n(1) = \langle L'(u_n), u_n\rangle = \sum_{i=1}^{2}\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} g(u_n)u_n\,dx$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + \int_{\mathbb{R}^N} g_2(u_n)u_n\,dx - \int_{\mathbb{R}^N} g_1(u_n)u_n\,dx \\
&\geq \sum_{i=1}^{2} \left[\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + 2\nu\|u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i}\right] - \varepsilon\,\|u_n\|_{L^{p_1}(\mathbb{R}^N)}^{p_1} \\
&\quad - \varepsilon \int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{\frac{N}{N-1}})\,dx - C_\varepsilon\|u_n\|_{L^r(\mathbb{R}^N)}^r \\
&\geq \sum_{i=1}^{2} \left[\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + 2\nu\|u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i}\right] - \varepsilon\,\|u_n\|_{L^{p_1}(\mathbb{R}^N)}^{p_1} \\
&\quad - C'\,\varepsilon\,\|u_n\|_{L^N(\mathbb{R}^N)}^N - C_\varepsilon\|u_n\|_{L^r(\mathbb{R}^N)}^r,
\end{aligned}
$$

where we have used (2.3) to infer

$$
\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{\frac{N}{N-1}})\,dx = \int_{\mathbb{R}^N} \Phi_N\left(\alpha M^{\frac{N}{N-1}}\left(\frac{|u_n|}{M}\right)^{\frac{N}{N-1}}\right)dx \leq C\frac{\|u\|_{L^N(\mathbb{R}^N)}^N}{M^N} = C'\|u_n\|_{L^N(\mathbb{R}^N)}^N.
$$

Therefore, choosing $\varepsilon \in (0, \min\{2\nu, 2\nu/C'\})$, we can find $c_1, c_2 > 0$ such that

$$
o_n(1) + C_\varepsilon\|u_n\|_{L^r(\mathbb{R}^N)}^r \geq \sum_{i=1}^{2}\left[\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + c_i\|u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i}\right].
$$

By Lemma 2.1 and the boundedness of (u_n) in $W^{1,N}(\mathbb{R}^N)$, we see

$$
\|u_n\|_{L^r(\mathbb{R}^N)}^r \leq C\left(\sup_{x_0\in\mathbb{R}^N}\|u_n\|_{L^r(B_1(x_0))}^r\right)^{1-\frac{N}{r}} \|u_n\|_{W^{1,N}(\mathbb{R}^N)}^N \leq C''\left(\sup_{x_0\in\mathbb{R}^N}\|u_n\|_{L^r(B_1(x_0))}^r\right)^{1-\frac{N}{r}},
$$

and so

$$
o_n(1) + C'''\left(\sup_{x_0\in\mathbb{R}^N}\|u_n\|_{L^r(B_1(x_0))}^r\right)^{1-\frac{N}{r}} \geq \sum_{i=1}^{2}\left[\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} + c_i\|u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i}\right].
$$

Combining this fact with (4.20), we deduce that, for all $r \in (N, +\infty)$,

$$
\liminf_{n\to+\infty}\sup_{x_0\in\mathbb{R}^N}\|u_n\|_{L^r(B_1(x_0))} > 0,
$$

that is, (4.21) holds even in the case $p_1 < p_2 = N$. Accordingly, up to a translation, we may assume that, for some $r \in (p_2, p_2^*)$,

$$
\liminf_{n\to+\infty}\|u_n\|_{L^r(B_1(0))} > 0.
$$

As (u_n) is bounded in $\mathcal{W}$, up to a subsequence, we may suppose that $u_n \rightharpoonup u$ in $\mathcal{W}$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p_2^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$, for some $u \in \mathcal{W}\setminus\{0\}$. Using $L'(u_n) \to 0$ in $\mathcal{W}'$ and Lemma 4.3, we see

$$
\begin{aligned}
&\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \\
&|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2.
\end{aligned}
$$

Utilizing $L'(u_n) \to 0$ in $\mathcal{W}'$, the above weak convergence for $(|\nabla u_n|^{p_i-2}\nabla u_n)$ with $i = 1, 2$, and the compactness lemma due to Strauss [12, Theorem A.I], it is straightforward to verify that $\langle L'(u), \varphi\rangle = 0$ for every $\varphi \in C_c^\infty(\mathbb{R}^N)$. Because $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{W}$, we conclude that u is a weak solution to (1.1). □

In the next result we prove the existence of an optimal path in the spirit of [29, Lemma 2.1].

Lemma 4.4 *Let $w \in \mathcal{W} \setminus \{0\}$ be a weak solution to (1.1). Then there exists $\gamma \in \Gamma$ such that $w \in \gamma([0, 1])$ and*

$$\max_{t\in[0,1]} L(\gamma(t)) = L(w).$$

Proof Put $w_t(x) := w(\frac{x}{t})$ for $x \in \mathbb{R}^N$ and $t > 0$. First we assume that $p_2 < N$. Define $\tilde{\gamma} : [0, +\infty) \to \mathcal{W}$ by setting

$$\tilde{\gamma}(t)(x) := \begin{cases} w_t(x) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Clearly, $\tilde{\gamma} \in C([0, +\infty), \mathcal{W})$. Since $P(w) = 0$, for all $t > 0$ we have

$$L(\tilde{\gamma}(t)) = L(\tilde{\gamma}(t)) - \frac{t^N}{N} P(w) = \sum_{i=1}^{2} \frac{1}{p_i} \|\nabla w\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} \left[t^{N-p_i} - t^N \left(\frac{N - p_i}{N} \right) \right].$$

Differentiating with respect to t, we find

$$\frac{d}{dt} L(\tilde{\gamma}(t)) = 0 \text{ for } t = 1, \quad \frac{d}{dt} L(\tilde{\gamma}(t)) > 0 \text{ for all } t \in (0, 1), \quad \frac{d}{dt} L(\tilde{\gamma}(t)) < 0 \text{ for all } t \in (1, +\infty),$$

which implies

$$\max_{t\ge0} L(\tilde{\gamma}(t)) = L(\tilde{\gamma}(1)) = L(w).$$

Because $L(\tilde{\gamma}(t)) \to -\infty$ as $t \to +\infty$, we can infer that $L(\tilde{\gamma}(T)) < 0$ for some $T > 1$. Letting $\gamma(t)(x) := \tilde{\gamma}(tT)(x)$ for $t \in [0, 1]$ and $x \in \mathbb{R}^N$, we reach the assertion.

Now we assume that $p_2 = N$. In this situation, the construction of the required path is more complicated with respect to the previous case. Our purpose is to select $t_0 \in (0, 1)$, $t_1 \in (1, +\infty)$ and $\theta_1 \in (1, +\infty)$ so that the curve γ, constituted of the three pieces defined below, gives the desired path:

$$[0, 1] \to \mathcal{W}; \ \theta \mapsto \theta w_{t_0}, \tag{4.24}$$

$$[t_0, t_1] \to \mathcal{W}; \ t \mapsto w_t, \tag{4.25}$$

$$[1, \theta_1] \to \mathcal{W}; \ \theta \mapsto \theta w_{t_1}. \tag{4.26}$$

Let us observe that in this context the Pohozaev identity is

$$\left(\frac{N - p_1}{p_1} \right) \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - N \int_{\mathbb{R}^N} G(w)\, dx = 0. \tag{4.27}$$

As $L'(w) = 0$ and $w \in \mathcal{W}\setminus\{0\}$, we obtain

$$\int_{\mathbb{R}^N} g(w)w\, dx = \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \|\nabla w\|^{N}_{L^{N}(\mathbb{R}^N)} > 0.$$

Then we can choose $\theta_1 \in (1, +\infty)$ such that

$$\int_{\mathbb{R}^N} g(\theta w)w \, dx > 0 \quad \text{for all } \theta \in [1, \theta_1]. \tag{4.28}$$

Define

$$\varphi(t) := \begin{cases} \frac{g(t)t}{|t|^{p_1}} & \text{for } t \neq 0, \\ \lim_{t\to 0} \frac{g(t)t}{|t|^{p_1}} & \text{for } t = 0. \end{cases}$$

In view of $(g1)$, we learn that $\varphi \in C(\mathbb{R})$. With this notation, (4.28) becomes

$$\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1} \, dx > 0 \quad \text{for all } \theta \in [1, \theta_1]. \tag{4.29}$$

Let us now observe that

$$\begin{aligned} \frac{d}{d\theta} L(\theta w_t) &= \langle L'(\theta w_t), w_t\rangle \\ &= \theta^{p_1-1}\|\nabla w_t\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \theta^{N-1}\|\nabla w_t\|^N_{L^N(\mathbb{R}^N)} - \theta^{p_1-1}\int_{\mathbb{R}^N} \varphi(\theta w_t)|w_t|^{p_1}\, dx \\ &= \theta^{p_1-1}t^{N-p_1}\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \theta^{N-1}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} - \theta^{p_1-1}t^N\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1}\, dx \\ &= \theta^{N-1}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \theta^{p_1-1}t^{N-p_1}\left[\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - t^{p_1}\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1}\, dx\right]. \end{aligned} \tag{4.30}$$

Take $t_0 \in (0, 1)$ small enough such that

$$\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - t_0^{p_1}\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1}\, dx > 0 \quad \text{for all } \theta \in [0, 1]. \tag{4.31}$$

By virtue of (4.29), we can select $t_1 \in (1, +\infty)$ such that

$$\begin{aligned} &\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - t_1^{p_1}\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1}\, dx \\ &\leq -\frac{1}{\theta_1^{p_1}-1}\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - \frac{p_1}{N}\left(\frac{\theta_1^N}{\theta_1^{p_1}-1}\right)\|\nabla w\|^N_{L^N(\mathbb{R}^N)} \quad \text{for all } \theta \in [1, \theta_1]. \end{aligned} \tag{4.32}$$

From (4.31), we deduce that $L(\theta w_{t_0})$ increases along (4.24) and achieves its maximum at $\theta = 1$. By (4.27), we know

$$L(w_t) = \frac{1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \left[\frac{t^{N-p_1}}{p_1} - t^N\left(\frac{1}{p_1} - \frac{1}{N}\right)\right]\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)}. \tag{4.33}$$

Thus, using (4.30), (4.32), (4.33), $p_1 < N$, $t_1, \theta_1 \in (1, +\infty)$, we see

$$\begin{aligned} L(\theta_1 w_{t_1}) &= L(w_{t_1}) + \int_1^{\theta_1} \frac{d}{d\theta} L(\theta w_{t_1})\, d\theta \\ &= \frac{1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \left[\frac{t_1^{N-p_1}}{p_1} - t_1^N\left(\frac{1}{p_1} - \frac{1}{N}\right)\right]\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} \\ &\quad + \int_1^{\theta_1}\left\{\theta^{N-1}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \theta^{p_1-1}t_1^{N-p_1}\left[\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - t_1^{p_1}\int_{\mathbb{R}^N} \varphi(\theta w)|w|^{p_1}\, dx\right]\right\} d\theta \\ &\leq \frac{1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \left[\frac{t_1^{N-p_1}}{p_1} - t_1^N\left(\frac{1}{p_1} - \frac{1}{N}\right)\right]\|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{\theta_1^N - 1}{N}\|\nabla w\|^N_{L^N(\mathbb{R}^N)} \end{aligned}$$

$$- \int_1^{\theta_1} \theta^{p_1-1} t_1^{N-p_1} \left[\frac{1}{\theta_1^{p_1}-1} \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{p_1}{N} \left(\frac{\theta_1^N}{\theta_1^{p_1}-1} \right) \|\nabla w\|^N_{L^N(\mathbb{R}^N)} \right] d\theta$$

$$= \frac{1}{N} \|\nabla w\|^N_{L^N(\mathbb{R}^N)} + \left[\frac{t_1^{N-p_1}}{p_1} - t_1^N \left(\frac{1}{p_1} - \frac{1}{N} \right) \right] \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{\theta_1^N - 1}{N} \|\nabla w\|^N_{L^N(\mathbb{R}^N)}$$

$$- \frac{t_1^{N-p_1}}{p_1} \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} - \frac{\theta_1^N}{N} t_1^{N-p_1} \|\nabla w\|^N_{L^N(\mathbb{R}^N)}$$

$$= -t_1^N \left(\frac{1}{p_1} - \frac{1}{N} \right) \|\nabla w\|^{p_1}_{L^{p_1}(\mathbb{R}^N)} + \frac{\theta_1^N}{N} (1 - t_1^{N-p_1}) \|\nabla w\|^N_{L^N(\mathbb{R}^N)} < 0.$$

Consequently, $\gamma \in \Gamma$. This completes the proof of the lemma. □

Let us define

$$\mathcal{T} := \{u \in \mathcal{W} \setminus \{0\} : L'(u) = 0\}, \ c_{\mathrm{LE}} := \inf_{u \in \mathcal{T}} L(u),$$

$$\mathcal{P} := \{u \in \mathcal{W} \setminus \{0\} : P(u) = 0\}, \ c_{\mathrm{PO}} := \inf_{u \in \mathcal{P}} L(u).$$

Now we are ready to give our first proof of Theorem 1.1.

First proof of Theorem 1.1 On account of Propositions 4.1 and 4.2, we can find a Pohozaev-Palais-Smale sequence $(u_n) \subset \mathcal{W}$ for L at the level $c_{\mathrm{MP}} > 0$ such that $u_n \rightharpoonup u$ in $\mathcal{W}$, for some $u \in \mathcal{W} \setminus \{0\}$ which satisfies (1.1). Exploiting $P(u) = 0$, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, Fatou's lemma, $L(u_n) \to c_{\mathrm{MP}}$, and $P(u_n) \to 0$, we see

$$L(u) = L(u) - \frac{1}{N} P(u) \le \liminf_{n \to +\infty} \left[L(u_n) - \frac{1}{N} P(u_n) \right] = c_{\mathrm{MP}}. \tag{4.34}$$

Since u is a nontrivial weak solution to (1.1), we derive from the definition of c_{LE} and (4.34) that

$$c_{\mathrm{LE}} \le L(u) \le c_{\mathrm{MP}}. \tag{4.35}$$

Let now $v \in \mathcal{W} \setminus \{0\}$ be any weak solution of (1.1) with $L(v) \le L(u)$. If we lift v to a path as in Lemma 4.4, then it follows from the definition of c_{MP} and (4.35) that $L(v) \ge c_{\mathrm{MP}} \ge L(u)$. As a result, $L(v) = L(u) = c_{\mathrm{MP}} = c_{\mathrm{LE}}$. Finally, we note that $u \in \mathcal{T} \subset \mathcal{P}$, and so $c_{\mathrm{MP}} = L(u) \ge c_{\mathrm{LE}} \ge c_{\mathrm{PO}}$. On the other hand, an inspection of the proof of Lemma 4.4 reveals that for all $w \in \mathcal{P}$ there exists a path $\gamma \in C([0, 1], \mathcal{W})$ such that $\gamma \in \Gamma$ and $\max_{t \in [0,1]} L(\gamma(t)) = L(w)$. Therefore, $c_{\mathrm{MP}} = L(u) = c_{\mathrm{LE}} = c_{\mathrm{PO}}$. □

Next we show the strong convergence of the translated subsequence of Proposition 4.2.

Corollary 4.1 *Under the assumptions of Proposition 4.2, if we assume that*

$$\liminf_{n \to +\infty} \|u_n\|_{\mathcal{W}} > 0,$$

and

$$\limsup_{n \to +\infty} L(u_n) \le c_{\mathrm{LE}},$$

then there exists $u \in \mathcal{W} \setminus \{0\}$ such that $L'(u) = 0$, and a sequence $(x_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot - x_n) \to u$ in $\mathcal{W}$ as $n \to +\infty$.

Proof By Proposition 4.2, up to a subsequence and translations, we may assume that for some $u \in \mathcal{W} \setminus \{0\}$, $u_n \rightharpoonup u$ in $\mathcal{W}$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p_2^*)$, $u_n \to u$ a.e. in $\mathbb{R}^N$, and that

$$\begin{aligned} &\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \\ &|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2. \end{aligned} \tag{4.36}$$

Hence, $u \in \mathcal{W}$ solves (1.1). Thus we have

$$\begin{aligned} c_{\mathrm{LE}} \le L(u) &= L(u) - \frac{1}{N} P(u) \\ &= \frac{1}{N}\left(\sum_{i=1}^{2} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) \\ &\le \liminf_{n\to+\infty} \frac{1}{N}\left(\sum_{i=1}^{2} \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) \\ &\le \limsup_{n\to+\infty} \frac{1}{N}\left(\sum_{i=1}^{2} \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) \\ &= \limsup_{n\to+\infty} \left(L(u_n) - \frac{1}{N} P(u_n)\right) \\ &= \limsup_{n\to+\infty} L(u_n) \le c_{\mathrm{LE}}, \end{aligned}$$

from which $L(u) = c_{\mathrm{LE}}$ and

$$\sum_{i=1}^{2} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} = \lim_{n\to+\infty}\left(\sum_{i=1}^{2} \|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right). \tag{4.37}$$

Recalling that (u_n) is bounded in $\mathcal{W}$ and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ (by (4.36)), we can use the Brezis-Lieb lemma [14, Theorem 1] and (4.37) to arrive at $\nabla u_n \to \nabla u$ in $(L^{p_i}(\mathbb{R}^N))^N$ for all $i = 1, 2$.

Assume that $p_2 < N$. By the Sobolev inequality (2.1), we see that $u_n \to u$ in $L^{p_1^*}(\mathbb{R}^N) \cap L^{p_2^*}(\mathbb{R}^N)$. Using the boundedness of (u_n) in $\mathcal{W}$ and the interpolation of L^p spaces, we obtain that $u_n \to u$ in $L^s(\mathbb{R}^N)$ for all $s \in (p_1, p_2^*]$. When $p_2 = N$, we exploit $\nabla u_n \to \nabla u$ in $L^{p_2}(\mathbb{R}^N)$, the boundedness of (u_n) in $L^{p_2}(\mathbb{R}^N)$, and the Gagliardo-Nirenberg interpolation inequality [2, Theorem 5.8], to deduce that $u_n \to u$ in $L^\tau(\mathbb{R}^N)$ for all $\tau \in (N, +\infty)$. In any case, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (p_2, p_2^*)$. Let us now introduce

$$h(t) := g(t) + \nu(|t|^{p_1-2}t + |t|^{p_2-2}t) \quad \text{for all } t \in \mathbb{R}. \tag{4.38}$$

Then h is an odd continuous function on $\mathbb{R}$ having the following properties:

($h1$) $-\infty < \liminf_{t\to 0^+} \frac{h(t)}{t^{p_1-1}} \le \limsup_{t\to 0^+} \frac{h(t)}{t^{p_1-1}} = -\nu < 0$ if $p_2 < N$, and $\lim_{t\to 0^+} \frac{h(t)}{t^{p_1-1}} = -\nu$ if $p_2 = N$,

($h2$) $\lim_{t\to+\infty} \frac{h(t)}{t^{p_2^*-1}} = 0$ if $p_2 < N$, and $\lim_{t\to+\infty} \frac{h(t)}{e^{\alpha t^{\frac{N}{N-1}}}} = 0$ for all $\alpha > 0$ when $p_2 = N$.

From ($h1$), there exists $t_0 > 0$ such that $(h(t)t)^+ = 0$ for all $|t| \le t_0$. In light of this fact and ($h2$), we see that $(h(t)t)^+$ satisfies (2.14) and (2.15). Thus, applying Lemma 2.6 with

$\Psi(t) = (h(t)t)^+$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (h(u_n)u_n)^+ \, dx = \int_{\mathbb{R}^N} (h(u)u)^+ \, dx. \tag{4.39}$$

Utilizing $\langle L'(u_n), u_n\rangle = o_n(1)$, $\langle L'(u), u\rangle = 0$, (4.39), and Fatou's lemma, we discover

$$\begin{aligned}
&\limsup_{n\to+\infty} \sum_{i=1}^{2} \left(\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu \|u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} \right) \\
&= \limsup_{n\to+\infty} \int_{\mathbb{R}^N} h(u_n)u_n \, dx \\
&= \limsup_{n\to+\infty} \int_{\mathbb{R}^N} [(h(u_n)u_n)^+ - (h(u_n)u_n)^-] \, dx \\
&= \int_{\mathbb{R}^N} (h(u)u)^+ - \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (h(u_n)u_n)^- \, dx \\
&\le \int_{\mathbb{R}^N} [(h(u)u)^+ - (h(u)u)^-] \, dx \\
&= \int_{\mathbb{R}^N} h(u)u \, dx \\
&= \sum_{i=1}^{2} \left(\|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu \|u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} \right).
\end{aligned} \tag{4.40}$$

Combining (4.40) with $\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} \to \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}$ for $i = 1, 2$, we infer that $u_n \to u$ in $L^{p_1}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N)$. Therefore, $u_n \to u$ in $\mathcal{W}$. □

Remark 4.4 With suitable modifications, we can prove the existence of a positive ground state solution to (1.1). In fact, due to $L(|u|) = L(u)$ for all $u \in \mathcal{W}$, we may assume in the proof of Proposition 4.1 that $(\gamma_n) \subset \Gamma$ fulfills $\gamma_n(t)(x) \ge 0$ for all $n \in \mathbb{N}$, $t \in [0, 1]$ and $x \in \mathbb{R}^N$. According to [46, Theorem 2.8], there exists $((\theta_n, v_n)) \subset \mathbb{R} \times \mathcal{W}$ such that $(L \circ \Phi)(\theta_n, v_n) \to c_{\mathrm{MP}}$, $(L \circ \Phi)'(\theta_n, v_n) \to 0$ in $(\mathbb{R} \times \mathcal{W})'$, and

$$\mathrm{dist}_{\mathbb{R}\times\mathcal{W}}((\theta_n, v_n), \{0\} \times \gamma_n([0, 1])) \to 0 \quad \text{as } n \to +\infty.$$

The above relation yields $\|v_n^-\|_{\mathcal{W}} \to 0$ and $\theta_n \to 0$ as $n \to +\infty$. Setting $u_n(x) := v_n(e^{-\theta_n} x)$, we deduce that $(u_n) \subset \mathcal{W}$ is a Pohozaev-Palais-Smale sequence of L at the level c_{MP} such that $\|u_n^-\|_{\mathcal{W}} \to 0$ as $n \to +\infty$. Reasoning as in the proof of Corollary 4.1, we obtain that, up to subsequences and translations, $u_n \to u$ in $\mathcal{W}$ as $n \to +\infty$, for some $u \in \mathcal{W}\setminus\{0\}$ such that $u \ge 0$ in $\mathbb{R}^N$ and u satisfies (1.1). Employing the Harnack inequality [45, Theorem 1.2], we conclude that $u > 0$ in $\mathbb{R}^N$.

Remark 4.5 Define

$$c_{\mathrm{MP,r}} := \inf_{\gamma\in\Gamma_{\mathrm{r}}} \max_{t\in[0,1]} L(\gamma(t)),$$

with

$$\Gamma_{\mathrm{r}} := \{\gamma \in C([0, 1], \mathcal{W}_{\mathrm{r}}) : \gamma(0) = 0, L(\gamma(1)) < 0\}.$$

Let us demonstrate that

$$c_{\mathrm{MP}} = c_{\mathrm{MP,r}}. \tag{4.41}$$

By the definitions of c_{MP} and $c_{\mathrm{MP,r}}$, we know that $c_{\mathrm{MP}} \leq c_{\mathrm{MP,r}}$. To establish the opposite inequality, fix $\gamma \in \Gamma$ and put $\gamma_\varepsilon(t) := \rho_\varepsilon * \gamma(t)$, with $\varepsilon > 0$, where $(\rho_\varepsilon) \subset C_c^\infty(\mathbb{R}^N)$ is a sequence of mollifiers. Then, $\gamma_\varepsilon \in C([0,1], \mathcal{W})$, $\gamma_\varepsilon(0) = 0$, and $\gamma_\varepsilon(t) \in C^\infty(\mathbb{R}^N) \cap \mathcal{W}$ for all $t \in [0,1]$. Moreover, $\gamma(t)$ and $\gamma_\varepsilon(t)$ are uniformly equicontinuous. From

$$\sup_{t\in[0,1]} \|\gamma_\varepsilon(t) - \gamma(t)\|_{\mathcal{W}} \to 0 \quad \text{as } \varepsilon \to 0^+,$$

it follows that

$$\max_{t\in[0,1]} L(\gamma_\varepsilon(t)) \to \max_{t\in[0,1]} L(\gamma(t)) \quad \text{as } \varepsilon \to 0^+.$$

Denote by $\gamma_\varepsilon^*(t)$ the symmetric decreasing rearrangement of $\gamma_\varepsilon(t)$. Using the Polya-Szegö inequality (see [3, Theorem 2.7]), we have

$$\|\nabla \gamma_\varepsilon^*(t)\|_{L^{p_i}(\mathbb{R}^N)} \leq \|\nabla \gamma_\varepsilon(t)\|_{L^{p_i}(\mathbb{R}^N)} \quad \text{for all } i = 1, 2.$$

On the other hand, it holds

$$\int_{\mathbb{R}^N} G(\gamma_\varepsilon^*(t))\, dx = \int_{\mathbb{R}^N} G(\gamma_\varepsilon(t))\, dx.$$

Therefore, $L(\gamma_\varepsilon^*(t)) \leq L(\gamma_\varepsilon(t))$ for all $t \in [0,1]$. Since $\gamma_\varepsilon(t) \in C^\infty(\mathbb{R}^N)$, the convolution $\gamma_\varepsilon(t)$ is co-area regular (see [3, definition 1.2.6]), and so [3, Theorem 1.4] implies that $\gamma_\varepsilon^* \in C([0,1], \mathcal{W}_{\mathrm{r}})$. Hence, $\gamma_\varepsilon^* \in \Gamma_{\mathrm{r}}$ and (4.41) is true. In view of this fact, we can study L on $\mathcal{W}_{\mathrm{r}}$ and, modifying slightly our arguments, we can establish the existence of a radially symmetric ground state solution to (1.1). Note that in this case we can take advantage of the compactness of the embedding in Theorem 2.3 to arrive at $u_n \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (p_2, p_2^*)$.

As byproduct of Corollary 4.1, we obtain that the set of ground state solutions to (1.1) is compact, up to translations.

Proposition 4.3 *The set*

$$\mathcal{S}_{LE} := \left\{ u \in \mathcal{W} : \ L(u) = c_{\mathrm{LE}},\ L'(u) = 0 \right\}$$

is compact in $\mathcal{W}$ endowed with the strong topology up to translations in $\mathbb{R}^N$. Furthermore, there exist two constants $C, c > 0$ independent of $u \in \mathcal{S}_{LE}$ such that

$$|u(x)| \leq Ce^{-c|x|} \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof Let $(u_n) \subset \mathcal{S}_{LE}$. Then $L(u_n) = c_{\mathrm{LE}}$ and $L'(u_n) = 0$ for all $n \in \mathbb{N}$. By Theorem 3.1, $P(u_n) = 0$ for all $n \in \mathbb{N}$. Proceeding as in the proof of Corollary 4.1, we can see that, up to a subsequence and translations, $u_n \to u$ in $\mathcal{W}$ for some $u \in \mathcal{W} \setminus \{0\}$ such that $L(u) = c_{\mathrm{LE}}$ and $L'(u) = 0$. Thus, $\mathcal{S}_{LE}$ is compact up to translations in $\mathbb{R}^N$. It remains to prove the uniform exponential decay estimate. This will be done by following the strategy in Theorem 3.2. Since each u_n solves (1.1), we derive from the proof of Theorem 3.1 that $|u_n|$ fulfills

$$-\Delta_{p_1}|u_n| - \Delta_{p_2}|u_n| + 2\nu(|u_n|^{p_1-1} + |u_n|^{p_2-1}) \leq g_1(u_n) \text{ in } \mathbb{R}^N.$$

Exploiting the growth conditions of g_1 and the boundedness of (u_n) in $\mathcal{W}$, we can adapt the Moser iteration argument performed in Theorem 3.1 to infer that, for some $\Upsilon > 0$, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq \Upsilon$ for all $n \in \mathbb{N}$, that is, $\mathcal{S}_{LE}$ is bounded in $L^\infty(\mathbb{R}^N)$. Because $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq$

Υ and $\|g(u_n)\|_{L^\infty(\mathbb{R}^N)} \le C_\Upsilon$ for all $n \in \mathbb{N}$, it follows from [24, Theorem 1] that $u_n \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ for some $\sigma \in (0, 1)$, and that there exists $C = C(N, p_1, p_2, \Upsilon) > 0$ such that

$$\|\nabla u_n\|_{L^\infty(\mathbb{R}^N)} \le C \quad \text{for all } n \in \mathbb{N}.$$

The above estimate implies that (u_n) is uniformly equicontinuous in $\mathbb{R}^N$, that is, for all $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that, if $x, y \in \mathbb{R}^N$ are such that $|x - y| < \delta$, then $|u_n(x) - u_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$. This fact combined with $u_n \to u$ in $L^{p_1}(\mathbb{R}^N)$ ensures that

$$\lim_{|x|\to+\infty} \sup_{n\in\mathbb{N}} |u_n(x)| = 0.$$

Hence, with the help of (3.1) and $p_1 < p_2$, we can find $R > 0$ such that

$$g_1(u_n) \le \nu(|u_n|^{p_1-1} + |u_n|^{p_2-1}) \text{ in } \overline{B_R(0)}^c.$$

Consequently,

$$-\Delta_{p_1}|u_n| - \Delta_{p_2}|u_n| + \nu(|u_n|^{p_1-1} + |u_n|^{p_2-1}) \le 0 \text{ in } \overline{B_R(0)}^c.$$

Put $\phi(x) := \Upsilon e^{\kappa R} e^{-\kappa|x|}$ where

$$0 < \kappa < \min\left\{\left(\frac{\nu}{(p_1-1)}\right)^{\frac{1}{p_1}}, \left(\frac{\nu}{(p_2-1)}\right)^{\frac{1}{p_2}}\right\}.$$

Clearly, $|u_n(x)| \le \phi(x)$ for all $|x| \le R$ and $n \in \mathbb{N}$. On the other hand, we can see that ϕ satisfies (3.24). Then it suffices to develop the same comparison argument given in Theorem 3.2 to achieve the desired exponential estimate. □

5 Decomposition result of bounded Palais-Smale sequences

This section is devoted to a second proof of Theorem 1.1. Motivated by [30, Theorem 3.1], we prove a new decomposition result for bounded Palais-Smale sequences of L in the (p_1, p_2)-Laplacian setting.

Theorem 5.1 *Let $\beta \in \mathbb{R}$ and $(u_n) \subset \mathcal{W}$ be a bounded Palais-Smale sequence for L at the level β. Then, up to a subsequence of (u_n), there exist $l \in \mathbb{N}$, $(y_n^1), \dots, (y_n^l) \subset \mathbb{R}^N$ and $w_1, \dots, w_l \in \mathcal{W}$ such that the following statements hold:*

(i) $y_n^1 = 0$ *for all* $n \in \mathbb{N}$ *and* $|y_n^j - y_n^{j'}| \to +\infty$ *as* $n \to +\infty$ *for all* $1 \le j < j' \le l$.
(ii) $u_n(\cdot + y_n^k) \rightharpoonup w^k$ *in* $\mathcal{W}$ *with* $L'(w^k) = 0$ *for all* $1 \le k \le l$, *and* $w^k \ne 0$ *if* $2 \le k \le l$.
(iii) $\beta = \lim_{n\to+\infty} L(u_n) = \sum_{k=1}^l L(w^k)$.
(iv) *Let* $v_n^l := u_n - \sum_{k=1}^l w^k(\cdot - y_n^k)$ *for all* $n \in \mathbb{N}$. *Then* $\|v_n^l\|_{\mathcal{W}} \to 0$ *as* $n \to +\infty$.

Proof We divide the proof into three main steps.

Step 1. Let $y_n^1 = 0$ for all $n \in \mathbb{N}$. Since (u_n) is bounded in $\mathcal{W}$, we may assume that, up to a subsequence, $u_n(\cdot + y_n^1) \rightharpoonup w^1$ in $\mathcal{W}$, $u_n(\cdot + y_n^1) \to w^1$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [1, p_2^*)$, and $u_n(\cdot + y_n^1) \to w^1$ a.e. in $\mathbb{R}^N$, for some $w^1 \in \mathcal{W}$. Arguing as in the proof of Lemma 4.3, we obtain

$$\nabla u_n(\cdot + y_n^1) \to \nabla w^1 \quad \text{a.e. in } \mathbb{R}^N,$$
$$|\nabla u_n(\cdot + y_n^1)|^{p_i-2}\nabla u_n(\cdot + y_n^1) \rightharpoonup |\nabla w^1|^{p_i-2}\nabla w^1 \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2.$$

Using these facts, $L'(u_n) \to 0$ in $\mathcal{W}'$, and the compactness lemma of Strauss [12, Theorem A.I], we deduce that $L'(w^1) = 0$. Without loss of generality, we may suppose that $\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(u_n)\,dx$ exists. Set $v_n^1 := u_n - w^1(\cdot - y_n^1) = u_n - w^1$ for every $n \in \mathbb{N}$. Thanks to $(g1)$–$(g2)$, we know that G obeys (2.11). By Lemma 2.5, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(u_n)\,dx = \int_{\mathbb{R}^N} G(w^1)\,dx + \lim_{n\to+\infty} \int_{\mathbb{R}^N} G(v_n^1)\,dx.$$

This combined with the Brezis-Lieb lemma [14, Theorem 1] (applied to (u_n) and (∇u_n)) shows that

$$\beta = \lim_{n\to+\infty} L(u_n) = L(w^1) + \lim_{n\to+\infty} L(v_n^1).$$

Step 2. Assume that $m \geq 1$ and that for each $1 \leq k \leq m$ there are $(y_n^k) \subset \mathbb{R}^N$ and $w^k \in \mathcal{W}$ such that the following statements hold:

$(S1)$ $y_n^1 = 0$ for all $n \in \mathbb{N}$ and $|y_n^j - y_n^{j'}| \to \infty$ as $n \to +\infty$ for all $1 \leq j < j' \leq m$.
$(S2)$ $u_n(\cdot + y_n^k) \rightharpoonup w^k$ in $\mathcal{W}$ with $L'(w^k) = 0$ for all $1 \leq k \leq m$, and $w^k \neq 0$ if $2 \leq k \leq m$.
$(S3)$ Let $v_n^m := u_n - \sum_{k=1}^m w^k(\cdot - y_n^k)$ for all $n \in \mathbb{N}$. We have that (v_n^m) is bounded in $\mathcal{W}$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(v_n^m)\,dx \text{ exists,} \tag{5.1}$$

and

$$\beta = \sum_{k=1}^m L(w^k) + \lim_{n\to+\infty} L(v_n^m). \tag{5.2}$$

Define

$$\sigma^m := \limsup_{n\to+\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n^m|^{p_2}\,dx \right).$$

We distinguish two cases: non vanishing and vanishing.

Non vanishing occurs, that is, $\sigma^m > 0$. Then, up to a subsequence of (u_n), $(S1)$–$(S3)$ hold for $m+1$.

Up to a subsequence, there exists $(y_n^{m+1}) \subset \mathbb{R}^N$ such that

$$\lim_{n\to+\infty} \int_{B_1(y_n^{m+1})} |v_n^m|^{p_2}\,dx > 0.$$

Thus, $|y_n^{m+1} - y_n^k| \to +\infty$ as $n \to +\infty$ for all $1 \leq k \leq m$ (because $v_n^m(\cdot + y_n^k) \to 0$ in $L^{p_2}_{loc}(\mathbb{R}^N)$), and, up to a subsequence, $v_n^m(\cdot + y_n^{m+1}) \rightharpoonup w^{m+1}$ in $\mathcal{W}$ for some $w^{m+1} \in \mathcal{W}\setminus\{0\}$. By the definition of v_n^m, $w^k(\cdot + y_n^k - y_n^{m+1}) \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ and $\nabla w^k(\cdot + y_n^k - y_n^{m+1}) \to 0$ in $(L^r_{loc}(\mathbb{R}^N))^N$ for all $r \in [1, p_2^*)$ and for all $1 \leq k \leq m$, we also have

$$u_n(\cdot + y_n^{m+1}) = v_n^m(\cdot + y_n^{m+1}) + \sum_{k=1}^m w^k(\cdot - y_n^k + y_n^{m+1}) \rightharpoonup w^{m+1} \quad \text{in } \mathcal{W}.$$

Since $(u_n(\cdot + y_n^{m+1})) \subset \mathcal{W}$ is a bounded Palais-Smale sequence of L, we can argue as in the proof of Lemma 4.3 to infer

$$\nabla u_n(\cdot + y_n^{m+1}) \to \nabla w^{m+1} \quad \text{a.e. in } \mathbb{R}^N,$$

$$|\nabla u_n(\cdot + y_n^{m+1})|^{p_i-2}\nabla u_n(\cdot + y_n^{m+1}) \rightharpoonup |\nabla w^{m+1}|^{p_i-2}\nabla w^{m+1} \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2.$$

In particular, $\nabla v_n^m(\cdot + y_n^{m+1}) \to \nabla w^{m+1}$ a.e. in $\mathbb{R}^N$. Reasoning as in Step 1, we can prove that $L'(w^{m+1}) = 0$. Set $v_n^{m+1} := v_n^m - w^{m+1}(\cdot - y_n^{m+1})$ for every $n \in \mathbb{N}$. From (5.1) and Lemma 2.5, we deduce

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(v_n^m(\cdot + y_n^{m+1}))\, dx = \int_{\mathbb{R}^N} G(w^{m+1})\, dx + \lim_{n\to+\infty} \int_{\mathbb{R}^N} G(v_n^{m+1}(\cdot + y_n^{m+1}))\, dx,$$

which combined with (5.2) and the Brezis-Lieb lemma [14, Theorem 1] yields

$$\begin{aligned}\beta &= \sum_{k=1}^{m} L(w^k) + \lim_{n\to+\infty} L(v_n^m)\\ &= \sum_{k=1}^{m} L(w^k) + \lim_{n\to+\infty} L(v_n^m(\cdot + y_n^{m+1}))\\ &= \sum_{k=1}^{m} L(w^k) + \left[L(w^{m+1}) + \lim_{n\to+\infty} L(v_n^{m+1}(\cdot + y_n^{m+1}))\right]\\ &= \sum_{k=1}^{m+1} L(w^k) + \lim_{n\to+\infty} L(v_n^{m+1}).\end{aligned}$$

Hence, up to a subsequence of (u_n), $(S1)$–$(S3)$ hold for $m + 1$.

Vanishing occurs, that is, $\sigma^m = 0$. Then Theorem 5.1 holds with $l = m$.

Since $(S1)$–$(S2)$ and (5.2) hold, it suffices to prove that $\|v_n^m\|_{\mathcal{W}} \to 0$ as $n \to +\infty$. Let us note that if $(v_n) \subset \mathcal{W}$ and $(w_n) \subset \mathcal{W}$ are two bounded sequences such that, as $n \to +\infty$,

$$\begin{aligned}\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N} &\left(|\nabla v_n|^{p_i-2}\nabla v_n - |\nabla w_n|^{p_i-2}\nabla w_n\right)(\nabla v_n - \nabla w_n)\, dx\\ &+ \int_{\mathbb{R}^N} (|v_n|^{p_i-2}v_n - |w_n|^{p_i-2}w_n)(v_n - w_n)\, dx\Big\} \to 0,\end{aligned}$$

then $\|v_n - w_n\|_{\mathcal{W}} \to 0$ as $n \to +\infty$. Indeed, invoking the well-known Simon's inequalities [43, formula (2.2)] :

$$(|\eta_1|^{r-2}\eta_1 - |\eta_2|^{r-2}\eta_2)(\eta_1 - \eta_2) \geq c_1|\eta_1 - \eta_2|^r \quad \text{if } r \geq 2, \tag{5.3}$$

$$(|\eta_1| + |\eta_2|)^{2-r}[(|\eta_1|^{r-2}\eta_1 - |\eta_2|^{r-2}\eta_2)(\eta_1 - \eta_2)] \geq c_2|\eta_1 - \eta_2|^2 \quad \text{if } 1 < r < 2, \tag{5.4}$$

for all $\eta_1, \eta_2 \in \mathbb{R}^N$, where $c_1, c_2 > 0$ are constants depending on r, we can see that, if $p_i \geq 2$ then (5.3) gives

$$\begin{aligned}\int_{\mathbb{R}^N} &\left(|\nabla v_n|^{p_i-2}\nabla v_n - |\nabla w_n|^{p_i-2}\nabla w_n\right)(\nabla v_n - \nabla w_n)\, dx\\ &+ \int_{\mathbb{R}^N} (|v_n|^{p_i-2}v_n - |w_n|^{p_i-2}w_n)(v_n - w_n)\, dx \geq c_1\|v_n - w_n\|^{p_i}_{W^{1,p_i}(\mathbb{R}^N)},\end{aligned}$$

while if $1 < p_i < 2$ then (5.4), the Hölder inequality with exponents $\frac{2}{p_i}$ and $\frac{2}{2-p_i}$, and the boundedness of (v_n) and (w_n) yield

$$c_2^{\frac{p_i}{2}} \|v_n - w_n\|^{p_i}_{W^{1,p_i}(\mathbb{R}^N)}$$

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{R}^N}\left(|\nabla v_n|^{p_i-2}\nabla v_n-|\nabla w_n|^{p_i-2}\nabla w_n\right)(\nabla v_n-\nabla w_n)\,dx\right)^{\frac{p_i}{2}}\left(\int_{\mathbb{R}^N}(|\nabla v_n|+|\nabla w_n|)^{p_i}\,dx\right)^{\frac{2-p_i}{2}}\\
&\quad+\left(\int_{\mathbb{R}^N}\left(|v_n|^{p_i-2}v_n-|w_n|^{p_i-2}w_n\right)(v_n-w_n)\,dx\right)^{\frac{p_i}{2}}\left(\int_{\mathbb{R}^N}(|v_n|+|w_n|)^{p_i}\,dx\right)^{\frac{2-p_i}{2}}\\
&\leq C\Big[\left(\int_{\mathbb{R}^N}\left(|\nabla v_n|^{p_i-2}\nabla v_n-|\nabla w_n|^{p_i-2}\nabla w_n\right)(\nabla v_n-\nabla w_n)\,dx\right)^{\frac{p_i}{2}}\\
&\quad+\left(\int_{\mathbb{R}^N}\left(|v_n|^{p_i-2}v_n-|w_n|^{p_i-2}w_n\right)(v_n-w_n)\,dx\right)^{\frac{p_i}{2}}\Big].
\end{aligned}
$$

Therefore, if we show that, as $n\to+\infty$,

$$
\begin{aligned}
&\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big[|\nabla u_n|^{p_i-2}\nabla u_n-\Big|\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big]\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big[|u_n|^{p_i-2}u_n-\Big|\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big]v_n^m\,dx\Big\}\to 0,
\end{aligned}
\tag{5.5}
$$

then we can conclude that $\|v_n^m\|_{\mathcal{W}}\to 0$ as $n\to+\infty$, as desired. Henceforth, we focus on (5.5). Since (v_n^m) is bounded in $\mathcal{W}$ and $\sigma^m=0$, it follows from Lemma 2.2 that $v_n^m\to 0$ in $L^q(\mathbb{R}^N)$ for all $q\in(p_2,p_2^*)$. Now, recalling the definition of h in (4.38), we observe that

$$
\begin{aligned}
0&\leq\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big[|\nabla u_n|^{p_i-2}\nabla u_n-\Big|\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big]\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big[|u_n|^{p_i-2}u_n-\Big|\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big]v_n^m\,dx\Big\}\\
&=\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}|\nabla u_n|^{p_i-2}\nabla u_n\nabla v_n^m\,dx+\nu\int_{\mathbb{R}^N}|u_n|^{p_i-2}u_nv_n^m\,dx\Big\}\\
&\quad-\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big|\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big|\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)v_n^m\,dx\Big\}\\
&=\int_{\mathbb{R}^N}h(u_n)v_n^m\,dx+\langle L'(u_n),v_n^m\rangle\\
&\quad-\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big|\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big|\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m}w^k\big(\cdot-y_n^k\big)\Big)v_n^m\,dx\Big\}\\
&=\int_{\mathbb{R}^N}h(u_n)v_n^m\,dx+o_n(1)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big|\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\nabla v_n^m\,dx\\
&+\nu\int_{\mathbb{R}^N}\Big|\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)v_n^m\,dx\Big\},
\end{aligned}
$$

where we have used the fact that $\langle L'(u_n), v_n^m\rangle = o_n(1)$ because $L'(u_n)\to 0$ in $\mathcal{W}'$ and (v_n^m) is bounded in $\mathcal{W}$. In view of $\langle L'(w^k), v_n^m(\cdot+y_n^k)\rangle = 0$ for all $k=1,\dots,m$, we see

$$
\begin{aligned}
\int_{\mathbb{R}^N} h(u_n)v_n^m\,dx &= \int_{\mathbb{R}^N}\left[h(u_n)-\sum_{k=1}^{m} h\Big(w^k\Big(\cdot-y_n^k\Big)\Big)\right]v_n^m\,dx+\sum_{k=1}^{m}\int_{\mathbb{R}^N} h(w^k\Big(\cdot-y_n^k\Big))v_n^m\,dx\\
&= \int_{\mathbb{R}^N}\left[h(u_n)-\sum_{k=1}^{m} h\Big(w^k\Big(\cdot-y_n^k\Big)\Big)\right]v_n^m\,dx+\sum_{k=1}^{m}\int_{\mathbb{R}^N} h\Big(w^k\Big)v_n^m\Big(\cdot+y_n^k\Big)\,dx\\
&= \int_{\mathbb{R}^N}\left[h(u_n)-\sum_{k=1}^{m} h\Big(w^k\Big(\cdot-y_n^k\Big)\Big)\right]v_n^m\,dx-\sum_{k=1}^{m}\langle L'(w^k), v_n^m\Big(\cdot+y_n^k\Big)\rangle\\
&\quad+\sum_{k=1}^{m}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[|\nabla w^k|^{p_i-2}\nabla w^k\nabla v_n^m\Big(\cdot+y_n^k\Big)+\nu|w^k|^{p_i-2}w^k v_n^m\Big(\cdot+y_n^k\Big)\right]dx\\
&= \int_{\mathbb{R}^N}\left[h(u_n)-\sum_{k=1}^{m} h\Big(w^k\Big(\cdot-y_n^k\Big)\Big)\right]v_n^m\,dx\\
&\quad+\sum_{k=1}^{m}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[|\nabla w^k\Big(\cdot-y_n^k\Big)|^{p_i-2}\nabla w^k\Big(\cdot-y_n^k\Big)\nabla v_n^m+\nu|w^k\Big(\cdot-y_n^k\Big)|^{p_i-2}w_i\Big(\cdot-y_n^k\Big)v_n^m\right]dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 &\le \sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big[|\nabla u_n|^{p_i-2}\nabla u_n-\Big|\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\Big]\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big[|u_n|^{p_i-2}u_n-\Big|\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\Big]v_n^m\,dx\Big\}\\
&= \int_{\mathbb{R}^N}\left[h(u_n)-\sum_{k=1}^{m} h(w^k\Big(\cdot-y_n^k\Big))\right]v_n^m\,dx+o_n(1)\\
&\quad+\sum_{k=1}^{m}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left[|\nabla w^k\Big(\cdot-y_n^k\Big)|^{p_i-2}\nabla w^k\Big(\cdot-y_n^k\Big)\nabla v_n^m+\nu|w^k\Big(\cdot-y_n^k\Big)|^{p_i-2}w^k\Big(\cdot-y_n^k\Big)v_n^m\right]dx\\
&\quad-\sum_{i=1}^{2}\Big\{\int_{\mathbb{R}^N}\Big|\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\Big|^{p_i-2}\nabla\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)\nabla v_n^m\,dx\\
&\quad+\nu\int_{\mathbb{R}^N}\Big|\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big|^{p_i-2}\Big(\sum_{k=1}^{m} w^k\Big(\cdot-y_n^k\Big)\Big)v_n^m\,dx\Big\}\\
&=: A_n+o_n(1)+\sum_{k=1}^{m} B_{n,k}-C_n.
\end{aligned}
\tag{5.6}
$$

We claim that

$$
\limsup_{n\to+\infty} A_n \le 0, \tag{5.7}
$$

$$\lim_{n\to+\infty}\sum_{k=1}^{m} B_{n,k}=0, \tag{5.8}$$

and

$$\lim_{n\to+\infty} C_n=0. \tag{5.9}$$

Once verified (5.7), (5.8), and (5.9), it follows from (5.6) that (5.5) is true. We start by proving (5.7). For all $n\in\mathbb{N}$ and $M>0$, put $\Omega_{n,M}:=\{x\in\mathbb{R}^N : |v_n^m(x)|\geq M\}$. We begin by showing that

$$\limsup_{M\to+\infty}\left(\sup_{n\in\mathbb{N}}\int_{\Omega_{n,M}}\left|h(u_n)-\sum_{k=1}^{m}h(w^k(\cdot-y_n^k))\right||v_n^m|\,dx\right)=0. \tag{5.10}$$

Pick $s>N$ and set

$$\omega_1:=\begin{cases}\frac{p_2^*}{p_2^*-1} & \text{if } p_2<N,\\ \frac{s}{s-1} & \text{if } p_2=N,\end{cases}$$

and

$$\omega_2:=\begin{cases}p_2^* & \text{if } p_2<N,\\ s & \text{if } p_2=N.\end{cases}$$

Note that $\omega_1\in(1,\frac{p_2}{p_2-1})$ and $\sum_{i=1}^{2}\frac{1}{\omega_i}=1$. Using the Hölder inequality and the boundedness of (v_n^m) in $L^{\omega_2}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}&\int_{\Omega_{n,M}}\left|h(u_n)-\sum_{k=1}^{m}h(w^k(\cdot-y_n^k))\right||v_n^m|\,dx\\&\leq\left(\|h(u_n)\|_{L^{\omega_1}(\Omega_{n,M})}+\sum_{k=1}^{m}\|h(w^k(\cdot-y_n^k))\|_{L^{\omega_1}(\Omega_{n,M})}\right)\|v_n^m\|_{L^{\omega_2}(\mathbb{R}^N)}\\&\leq C_0\left(\|h(u_n)\|_{L^{\omega_1}(\Omega_{n,M})}+\sum_{k=1}^{m}\|h(w^k(\cdot-y_n^k))\|_{L^{\omega_1}(\Omega_{n,M})}\right),\end{aligned} \tag{5.11}$$

for some $C_0>0$ independent of ε, n, and M. Exploiting the boundedness of (v_n^m) in $L^{\omega_2}(\mathbb{R}^N)$ once again, we have

$$C_1\geq\|v_n^m\|_{L^{\omega_2}(\mathbb{R}^N)}^{\omega_2}\geq\|v_n^m\|_{L^{\omega_2}(\Omega_{n,M})}^{\omega_2}\geq M^{\omega_2}|\Omega_{n,M}|\quad\text{for all } n\in\mathbb{N}\text{ and } M>0,$$

and so

$$\sup_{n\in\mathbb{N}}|\Omega_{n,M}|\to 0\quad\text{as } M\to+\infty. \tag{5.12}$$

Now, we observe that, for all $v\in\mathcal{W}$, it holds

$$\|v\|_{L^{\omega_1(p_2-1)}(\Omega_{n,M})}^{\omega_1(p_2-1)}\leq|\Omega_{n,M}|^{1-\frac{\omega_1(p_2-1)}{p_2}}\|v\|_{L^{p_2}(\mathbb{R}^N)}^{\omega_1(p_2-1)}. \tag{5.13}$$

Let us recall that $(h1)$ and $(h2)$ imply that for all $\varepsilon>0$ and $\alpha>0$ there exist $C_\varepsilon, C_{\varepsilon,\alpha}>0$ such that, for all $t\in\mathbb{R}$,

$$|h(t)|\leq\begin{cases}2^{\frac{1-\omega_1}{\omega_1}}C_\varepsilon^{\frac{1}{\omega_1}}|t|^{p_1-1}+2^{\frac{1-\omega_1}{\omega_1}}\varepsilon^{\frac{1}{\omega_1}}|t|^{p_2^*-1} & \text{if } p_2<N,\\ 2^{\frac{1-\omega_1}{\omega_1}}C_{\varepsilon,\alpha}^{\frac{1}{\omega_1}}|t|^{p_1-1}+2^{\frac{1-\omega_1}{\omega_1}}\varepsilon^{\frac{1}{\omega_1}}\Phi_N\left(\alpha|t|^{\frac{N}{N-1}}\right) & \text{if } p_2=N.\end{cases} \tag{5.14}$$

Using (5.14), (4.2), (2.4), and that $|t|^{p_1-1} \le |t|^{p_2-1} M^{p_1-p_2}$ for all $|t| \ge M$, we see that, for all $|t| \ge M$,

$$|h(t)|^{\omega_1} \le \begin{cases} C_\varepsilon |t|^{\omega_1(p_2-1)} M^{\omega_1(p_1-p_2)} + \varepsilon\, |t|^{p_2^*} & \text{if } p_2 < N, \\ C_{\varepsilon,\alpha} |t|^{\omega_1(p_2-1)} M^{\omega_1(p_1-p_2)} + \varepsilon\, \Phi_N \left(\alpha\omega_1 |t|^{\frac{N}{N-1}}\right) & \text{if } p_2 = N, \end{cases} \tag{5.15}$$

where $\alpha > 0$ is such that $\alpha\omega_1 K^{\frac{N}{N-1}} < \alpha_N$, and $K > 0$ is such that $\|u_n\|_{W^{1,p_2}(\mathbb{R}^N)} \le K$ for all $n \in \mathbb{N}$. Then, taking into account (5.13), (5.15), and the boundedness of (u_n) in $\mathcal{W}$, we have, when $p_2 < N$,

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \left(\|h(u_n)\|^{\omega_1}_{L^{\omega_1}(\Omega_{n,M})} + \sum_{k=1}^m \|h(w^k(\cdot - y_n^k))\|^{\omega_1}_{L^{\omega_1}(\Omega_{n,M})} \right) \\
&\quad \le \sup_{n\in\mathbb{N}} \Bigg\{ \Big[C_\varepsilon \|u_n\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon\, \|u_n\|^{p_2^*}_{L^{p_2^*}(\Omega_{n,M})} \Big] \\
&\qquad + \sum_{k=1}^m \Big[C_\varepsilon \|w^k(\cdot - y_n^k)\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon\, \|w^k(\cdot - y_n^k)\|^{p_2^*}_{L^{p_2^*}(\Omega_{n,M})} \Big] \Bigg\} \\
&\quad \le \sup_{n\in\mathbb{N}} \Bigg\{ \Big[C_\varepsilon \|u_n\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon\, \|u_n\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} \Big] \\
&\qquad + \sum_{k=1}^m \Big[C_\varepsilon \|w^k(\cdot - y_n^k)\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon\, \|w^k\|^{p_2^*}_{L^{p_2^*}(\mathbb{R}^N)} \Big] \Bigg\} \\
&\quad \le C_2 \left\{ C_\varepsilon M^{\omega_1(p_1-p_2)} \sup_{n\in\mathbb{N}} |\Omega_{n,M}|^{1-\frac{\omega_1(p_2-1)}{p_2}} + \varepsilon \right\},
\end{aligned}$$

and when $p_2 = N$, utilizing (2.3), we find

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \left(\|h(u_n)\|^{\omega_1}_{L^{\omega_1}(\Omega_{n,M})} + \sum_{k=1}^m \|h(w^k(\cdot - y_n^k))\|^{\omega_1}_{L^{\omega_1}(\Omega_{n,M})} \right) \\
&\quad \le \sup_{n\in\mathbb{N}} \Bigg\{ \Big[C_{\varepsilon,\alpha} \|u_n\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon \left\| \Phi_N \left(\alpha\omega_1 |u_n|^{\frac{N}{N-1}} \right) \right\|_{L^1(\Omega_{n,M})} \Big] \\
&\qquad + \sum_{k=1}^m \Big[C_{\varepsilon,\alpha} \|w^k(\cdot - y_n^k)\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon \left\| \Phi_N \left(\alpha\omega_1 |w^k(\cdot - y_n^k)|^{\frac{N}{N-1}} \right) \right\|_{L^1(\Omega_{n,M})} \Big] \Bigg\} \\
&\quad \le \sup_{n\in\mathbb{N}} \Bigg\{ \Big[C_{\varepsilon,\alpha} \|u_n\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon \left\| \Phi_N \left(\alpha\omega_1 |u_n|^{\frac{N}{N-1}} \right) \right\|_{L^1(\mathbb{R}^N)} \Big] \\
&\qquad + \sum_{k=1}^m \Big[C_{\varepsilon,\alpha} \|w^k(\cdot - y_n^k)\|^{\omega_1(p_2-1)}_{L^{\omega_1(p_2-1)}(\Omega_{n,M})} M^{\omega_1(p_1-p_2)} + \varepsilon \left\| \Phi_N \left(\alpha\omega_1 |w^k|^{\frac{N}{N-1}} \right) \right\|_{L^1(\mathbb{R}^N)} \Big] \Bigg\} \\
&\quad \le C_2 \left\{ C_{\varepsilon,\alpha} M^{\omega_1(p_1-p_2)} \sup_{n\in\mathbb{N}} |\Omega_{n,M}|^{1-\frac{\omega_1(p_2-1)}{p_2}} + \varepsilon \right\},
\end{aligned}$$

for some $C_2 > 0$ independent of ε, n, and M. Exploiting the above estimates, (5.11), and (5.12), we arrive at

$$\limsup_{M\to+\infty} \left(\sup_{n\in\mathbb{N}} \int_{\Omega_{n,M}} \left| h(u_n) - \sum_{k=1}^m h(w^k(\cdot - y_n^k)) \right| |v_n^m|\, dx \right) \le C_3\, \varepsilon\,.$$

Because $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{M\to+\infty}\left(\sup_{n\in\mathbb{N}}\int_{\Omega_{n,M}}\left|h(u_n)-\sum_{k=1}^{m}h(w^k(\cdot-y_n^k))\right||v_n^m|\,dx\right)=0. \tag{5.16}$$

Now, we denote by $\chi_{n,M}$ the characteristic function of the set $\{x\in\mathbb{R}^N : |v_n^m(x)|\leq M\}$. Clearly, for all $j=1,\dots,m$ and $R>0$, it holds

$$\begin{aligned}
&\int_{B_R(y_n^j)}\chi_{n,M}\left|h(u_n)-\sum_{k=1}^{m}h(w^k(\cdot-y_n^k))\right||v_n^m|\,dx\\
&=\int_{B_R(0)}\chi_{n,M}(\cdot+y_n^j)\left|h(u_n(\cdot+y_n^j))-h(w^j)-\sum_{k\neq j}h(w^k(\cdot+y_n^j-y_n^k))\right||v_n^m(\cdot+y_n^j)|\,dx\\
&\leq M\int_{B_R(0)}\left(|h(u_n(\cdot+y_n^j))-h(w^j)|+\sum_{k\neq j}|h(w^k(\cdot+y_n^j-y_n^k))|\right)dx.
\end{aligned}$$

Since $u_n(\cdot+y_n^j)\to w^j$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r\in[1,p_2^*)$, $|y_n^j-y_n^k|\to+\infty$ for each $j\neq k$ (and so $w^k(\cdot+y_n^j-y_n^k)\to 0$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r\in[1,p_2^*)$ and $j\neq k$), and h is a continuous function satisfying $(h1)$ and $(h2)$, it follows from the compactness lemma of Strauss [12, Theorem A.I] that

$$\lim_{n\to+\infty}\int_{B_R(y_n^j)}\chi_{n,M}\left|h(u_n)-\sum_{k=1}^{m}h(w^k(\cdot-y_n^k))\right||v_n^m|\,dx=0 \quad \text{for all } j=1,\dots,m \text{ and } R>0. \tag{5.17}$$

Define

$$V_R:=\mathbb{R}^N\setminus\bigcup_{k=1}^{m}B_R(y_n^k).$$

Because h fulfills $(h1)$ and $(h2)$, we can find $C_3>0$ such that, for all $t\in\mathbb{R}$,

$$|h(t)|\leq\begin{cases}C_3(|t|^{p_1-1}+|t|^{p_2^*-1}) & \text{if } p_2<N,\\ C_3\left(|t|^{p_1-1}+\Phi_N\left(\alpha|t|^{\frac{N}{N-1}}\right)\right) & \text{if } p_2=N.\end{cases} \tag{5.18}$$

Then, due to (5.18), for all $k=1,\dots,m$, we have, when $p_2<N$,

$$\begin{aligned}
&\int_{V_R}\chi_{n,M}|h(w^k(\cdot-y_n^k))v_n^m|\,dx\\
&\leq C_3\int_{V_R}(|w^k(\cdot-y_n^k)|^{p_1-1}+|w^k(\cdot-y_n^k)|^{p_2^*-1})|v_n^m|\,dx\\
&\leq C_3\left[\|w^k(\cdot-y_n^k)\|^{p_1-1}_{L^{p_1}(V_R)}\|v_n^m\|_{L^{p_1}(\mathbb{R}^N)}+\|w^k(\cdot-y_n^k)\|^{p_2^*-1}_{L^{p_2^*}(V_R)}\|v_n^m\|_{L^{p_2^*}(\mathbb{R}^N)}\right]\\
&\leq C_3\left[\|w^k\|^{p_1-1}_{L^{p_1}(B_R^c(0))}\|v_n^m\|_{L^{p_1}(\mathbb{R}^N)}+\|w^k\|^{p_2^*-1}_{L^{p_2^*}(B_R^c(0))}\|v_n^m\|_{L^{p_2^*}(\mathbb{R}^N)}\right]=o_R(1),
\end{aligned} \tag{5.19}$$

and when $p_2 = N$, fixed $s > N$ such that $\alpha \frac{s}{s-1} < \alpha_N$, we obtain

$$\begin{aligned}
&\int_{V_R} \chi_{n,M} |h(w^k(\cdot - y_n^k)) v_n^m| \, dx \\
&\leq C_3 \int_{V_R} \left(|w^k(\cdot - y_n^k)|^{p_1 - 1} + \Phi_N \left(\alpha |w^k(\cdot - y_n^k)|^{\frac{N}{N-1}} \right) \right) |v_n^m| \, dx \\
&\leq C_3 \left[\|w^k(\cdot - y_n^k)\|_{L^{p_1}(V_R)}^{p_1-1} \|v_n^m\|_{L^{p_1}(\mathbb{R}^N)} + \left\| \Phi_N \left(\alpha \frac{s}{s-1} |w^k(\cdot - y_n^k)|^{\frac{N}{N-1}} \right) \right\|_{L^1(V_R)}^{\frac{s-1}{s}} \|v_n^m\|_{L^s(\mathbb{R}^N)} \right] \\
&\leq C_3 \left[\|w^k\|_{L^{p_1}(B_R^c(0))}^{p_1-1} \|v_n^m\|_{L^{p_1}(\mathbb{R}^N)} + \left\| \Phi_N \left(\alpha \frac{s}{s-1} |w^k|^{\frac{N}{N-1}} \right) \right\|_{L^1(B_R^c(0))}^{\frac{s-1}{s}} \|v_n^m\|_{L^s(\mathbb{R}^N)} \right] = o_R(1),
\end{aligned} \tag{5.20}$$

where $o_R(1) \to 0^+$ uniformly in n and M as $R \to +\infty$. In a similar fashion, we can prove

$$\int_{V_R} \chi_{n,M} |h(u_n)| \left(\sum_{k=1}^{m} |w^k(\cdot - y_n^k)| \right) dx = o_R(1). \tag{5.21}$$

Finally, we estimate

$$\int_{V_R} h(u_n) \chi_{n,M} u_n \, dx.$$

Since h is odd and satisfies $(h1)$, there exists $\tau > 0$ such that $h(t)t/|t|^{p_1} \leq 0$ for all $0 < |t| \leq \tau$, and so

$$h(t)t \leq 0 \quad \text{for all } |t| \leq \tau. \tag{5.22}$$

Take $q \in (p_2, p_2^*)$. From $(h2)$, we deduce that fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for all $|t| \geq \tau$,

$$|h(t)t| \leq \begin{cases} C_\varepsilon |t|^q + \varepsilon \, |t|^{p_2^*} & \text{if } p_2 < N, \\ C_\varepsilon |t|^q + \varepsilon \, \Phi_N \left(\alpha |t|^{\frac{N}{N-1}} \right) & \text{if } p_2 = N, \end{cases} \tag{5.23}$$

where $\alpha > 0$ is such that $\alpha K^{\frac{N}{N-1}} < \alpha_N$. Note that C_ε depends on ε and q when $p_2 < N$, while it depends on ε, q, and α when $p_2 = N$. Therefore, thanks to (5.22), (5.23), and $0 \leq \chi_{n,M} \leq 1$, we get, when $p_2 < N$,

$$\begin{aligned}
\int_{V_R} h(u_n) \chi_{n,M} u_n \, dx &= \int_{V_R} h(\chi_{n,M} u_n) \chi_{n,M} u_n \, dx \\
&\leq \int_{V_R \cap \{|\chi_{n,M} u_n| \geq \tau\}} h(\chi_{n,M} u_n) \chi_{n,M} u_n \, dx \\
&\leq \varepsilon \, \|u_n\|_{L^{p_2^*}(\mathbb{R}^N)}^{p_2^*} + C_\varepsilon \|u_n\|_{L^q(V_R)}^q \\
&\leq C_4 \, \varepsilon + C_\varepsilon \|u_n\|_{L^q(V_R)}^q,
\end{aligned}$$

and when $p_2 = N$, using (2.3), we find

$$\begin{aligned}
\int_{V_R} h(u_n) \chi_{n,M} u_n \, dx &= \int_{V_R} h(\chi_{n,M} u_n) \chi_{n,M} u_n \, dx \\
&\leq \int_{V_R \cap \{|\chi_{n,M} u_n| \geq \tau\}} h(\chi_{n,M} u_n) \chi_{n,M} u_n \, dx
\end{aligned}$$

$$\leq \varepsilon \, \|\Phi_N(\alpha|u_n|^{\frac{N}{N-1}})\|_{L^1(\mathbb{R}^N)} + C_\varepsilon \|u_n\|^q_{L^q(V_R)}$$
$$\leq \varepsilon \, C_4 + C_\varepsilon \|u_n\|^q_{L^q(V_R)},$$

for some $C_4 > 0$ independent of ε, n, M, and R. Now, because $u_n = v_n^m + \sum_{k=1}^m w^k(\cdot - y_n^k)$, $v_n^m \to 0$ in $L^q(\mathbb{R}^N)$, and recalling the definition of V_R, we see

$$\limsup_{n\to+\infty} \|u_n\|_{L^q(V_R)} \leq \limsup_{n\to+\infty} \left(\|v_n^m\|_{L^q(\mathbb{R}^N)} + \sum_{k=1}^m \|w^k(\cdot - y_n^k)\|_{L^q(V_R)} \right)$$
$$\leq \limsup_{n\to+\infty} \left(\sum_{k=1}^m \|w^k\|_{L^q(B_R^c(0))} \right) = o_R(1),$$

which yields

$$\limsup_{n\to+\infty} \int_{V_R} h(u_n)\chi_{n,M} u_n \, dx \leq C_4 \, \varepsilon + C_\varepsilon o_R(1).$$

Accordingly,

$$\limsup_{R\to+\infty} \left(\limsup_{n\to+\infty} \int_{V_R} h(u_n)\chi_{n,M} u_n \, dx \right) \leq C_4 \, \varepsilon \, . \tag{5.24}$$

In view of $v_n^m = u_n - \sum_{k=1}^m w^k(\cdot - y_n^k)$, (5.19), (5.20), (5.21), and (5.24), we obtain

$$\limsup_{R\to+\infty} \left[\limsup_{n\to+\infty} \int_{V_R} \left(h(u_n) - \sum_{k=1}^m h(w^k(\cdot - y_n^k)) \right) \chi_{n,M} v_n^m \, dx \right] \leq C_4 \, \varepsilon \, .$$

Since $\varepsilon > 0$ is arbitrary and taking (5.17) into account, we have

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \left(h(u_n) - \sum_{k=1}^m h(w^k(\cdot - y_n^k)) \right) \chi_{n,M} v_n^m \, dx \leq 0. \tag{5.25}$$

Combining (5.25) with (5.16), we conclude that (5.7) holds. Next we deal with (5.8) and (5.9). We only prove (5.9) because the proof of (5.8) follows the same pattern. We first show that, for all $i = 1, 2$,

$$\int_{\mathbb{R}^N} \left| \sum_{k=1}^m \nabla w^k(\cdot - y_n^k) \right|^{p_i - 2} \left(\sum_{k=1}^m \nabla w^k(\cdot - y_n^k) \right) v_n^m \, dx \to 0 \quad \text{as } n \to +\infty. \tag{5.26}$$

Fix $i \in \{1, 2\}$ and take $R > 0$ such that

$$\|\nabla w^k\|_{L^{p_i}(B_R^c(0))} \leq \varepsilon \qquad \text{for all } k = 1, \dots, m. \tag{5.27}$$

Thus,

$$\left| \int_{\mathbb{R}^N} \left| \sum_{k=1}^m \nabla w^k(\cdot - y_n^k) \right|^{p_i - 2} \left(\sum_{k=1}^m \nabla w^k(\cdot - y_n^k) \right) \nabla v_n^m \, dx \right|$$
$$\leq \int_{\mathbb{R}^N} \left| \sum_{k=1}^m \nabla w^k(\cdot - y_n^k) \right|^{p_i - 1} |\nabla v_n^m| \, dx$$

$$\leq \sum_{j=1}^{m} \int_{B_R(y_n^j)} \left| \sum_{k=1}^{m} \nabla w^k(\cdot - y_n^k) \right|^{p_i-1} |\nabla v_n^m| \, dx + \int_{V_R} \left| \sum_{k=1}^{m} \nabla w^k(\cdot - y_n^k) \right|^{p_i-1} |\nabla v_n^m| \, dx$$
$$=: D_n + E_n.$$

Since $|y_n^j - y_n^k| \to +\infty$ as $n \to +\infty$ for all $j \neq k$, we see that $\nabla w^k(\cdot + y_n^j - y_n^k) \to 0$ in $(L^{p_i}_{loc}(\mathbb{R}^N))^N$ as $n \to +\infty$ for all $j \neq k$. We also know that $\nabla v_n^m(\cdot + y_n^k) \to 0$ in $(L^{p_i}_{loc}(\mathbb{R}^N, \mathbb{R}^N))^N$ as $n \to +\infty$ for all $k = 1, \ldots, m$. Then, applying the Hölder and Minkowski inequalities, we get

$$\begin{aligned}
D_n &\leq \sum_{j=1}^{m} \int_{B_R(0)} \left| \sum_{k=1}^{m} \nabla w^k(\cdot + y_n^j - y_n^k) \right|^{p_i-1} |\nabla v_n^m(\cdot + y_n^j)| \, dx \\
&\leq \sum_{j=1}^{m} \left\| \sum_{k=1}^{m} \nabla w^k(\cdot + y_n^j - y_n^k) \right\|_{L^{p_i}(B_R(0))}^{p_i-1} \|\nabla v_n^m(\cdot + y_n^j)\|_{L^{p_i}(B_R(0))} \\
&= \sum_{j=1}^{m} \left\| \nabla w^j + \sum_{k \neq j} \nabla w^k(\cdot + y_n^j - y_n^k) \right\|_{L^{p_i}(B_R(0))}^{p_i-1} \|\nabla v_n^m(\cdot + y_n^j)\|_{L^{p_i}(B_R(0))} \\
&\leq \sum_{j=1}^{m} \left(\|\nabla w^j\|_{L^{p_i}(B_R(0))} + \sum_{k \neq j} \left\| \nabla w^k(\cdot + y_n^j - y_n^k) \right\|_{L^{p_i}(B_R(0))} \right)^{p_i-1} \\
&\quad \|\nabla v_n^m(\cdot + y_n^j)\|_{L^{p_i}(B_R(0))} = o_n(1).
\end{aligned}$$

On the other hand, using the boundedness of (v_n^m) in $\mathcal{W}$, the definition of V_R, and (5.27), we obtain

$$\begin{aligned}
E_n &\leq \left\| \sum_{k=1}^{m} \nabla w^k(\cdot - y_n^k) \right\|_{L^{p_i}(V_R)}^{p_i-1} \|\nabla v_n^m\|_{L^{p_i}(\mathbb{R}^N)} \\
&\leq \left(\sum_{k=1}^{m} \|\nabla w^k(\cdot - y_n^k)\|_{L^{p_i}(V_R)} \right)^{p_i-1} \|\nabla v_n^m\|_{L^{p_i}(\mathbb{R}^N)} \\
&\leq \left(\sum_{k=1}^{m} \|\nabla w^k\|_{L^{p_i}(B_R^c(0))} \right)^{p_i-1} \|\nabla v_n^m\|_{L^{p_i}(\mathbb{R}^N)} \\
&\leq C_5 \left(\sum_{k=1}^{m} \|\nabla w^k\|_{L^{p_i}(B_R^c(0))} \right)^{p_i-1} \leq C_6\, \varepsilon^{p_i-1}.
\end{aligned}$$

Consequently,

$$0 \leq \limsup_{n \to +\infty} (D_n + E_n) \leq C_6\, \varepsilon^{p_i-1}.$$

Because $\varepsilon > 0$ is arbitrary, we infer that $D_n + E_n \to 0$ as $n \to +\infty$, and so (5.26) is true. Analogously, we can verify that, for all $i = 1, 2$,

$$\int_{\mathbb{R}^N} \left| \sum_{k=1}^{m} w^k(\cdot - y_n^k) \right|^{p_i-2} \left(\sum_{k=1}^{m} w^k(\cdot - y_n^k) \right) v_n^m \, dx \to 0 \quad \text{as } n \to +\infty. \tag{5.28}$$

Combining (5.26) and (5.28), we deduce that (5.9) is valid. As a result, $\|v_n^m\|_{\mathcal{W}} \to 0$ as $n \to +\infty$. This completes the proof of the vanishing case.

Step 3. We proceed by iteration as in Step 2. Indeed, if $\sigma^m > 0$, then the Brezis-Lieb lemma [14, Theorem 1] ensures that, for all $i = 1, 2$,

$$0 \leq \|v_n^m\|_{W^{1,p_i}(\mathbb{R}^N)}^{pi} = \|u_n\|_{W^{1,p_i}(\mathbb{R}^N)}^{p_i} - \sum_{k=1}^{m} \|w^k\|_{W^{1,p_i}(\mathbb{R}^N)}^{p_i} + o_n(1),$$

from which

$$\sum_{k=1}^{m}\sum_{i=1}^{2} \|w^k\|_{W^{1,p_i}(\mathbb{R}^N)}^{p_i} \leq \sum_{i=1}^{2} \|u_n\|_{W^{1,p_i}(\mathbb{R}^N)}^{p_i} + o_n(1). \tag{5.29}$$

Since (u_n) is bounded in $\mathcal{W}$, we know that there exists $M > 0$ such that

$$\|u_n\|_{\mathcal{W}} \leq M \quad \text{for all } n \in \mathbb{N}. \tag{5.30}$$

On the other hand, we can prove that there exists $C > 0$ such that

$$\sum_{i=1}^{2} \|w^k\|_{W^{1,p_i}(\mathbb{R}^N)}^{p_i} \geq C \quad \text{for all } k = 1, \dots, m. \tag{5.31}$$

In fact, using Remark 4.2, we can find $C' > 0$ such that

$$\|w^k\|_{\mathcal{W}} \geq C' \quad \text{for all } k = 1, \dots, m. \tag{5.32}$$

Now, let $z \in \mathcal{W} \setminus \{0\}$ be such that $\|z\|_{\mathcal{W}} \geq K$ for some $K > 0$. We aim to confirm that, for some $K' > 0$,

$$\|z\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} + \|z\|_{W^{1,p_2}(\mathbb{R}^N)}^{p_2} \geq K'.$$

For simplicity, we assume that $\|z\|_{W^{1,p_1}(\mathbb{R}^N)} \leq \|z\|_{W^{1,p_2}(\mathbb{R}^N)}$. Let us consider the following cases:

- if $\|z\|_{W^{1,p_1}(\mathbb{R}^N)} \leq \frac{K}{2} \leq \|z\|_{W^{1,p_2}(\mathbb{R}^N)}$, then
$$\|z\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} + \|z\|_{W^{1,p_2}(\mathbb{R}^N)}^{p_2} \geq \|z\|_{W^{1,p_2}(\mathbb{R}^N)}^{p_2} \geq (K/2)^{p_2},$$
- if $\frac{K}{2} \leq \|z\|_{W^{1,p_1}(\mathbb{R}^N)} \leq \|z\|_{W^{1,p_2}(\mathbb{R}^N)}$, then
$$\|z\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} + \|z\|_{W^{1,p_2}(\mathbb{R}^N)}^{p_2} \geq (K/2)^{p_1} + (K/2)^{p_2},$$
- if $\|z\|_{W^{1,p_1}(\mathbb{R}^N)} \leq \|z\|_{W^{1,p_2}(\mathbb{R}^N)} \leq \frac{K}{2}$, then
$$K \leq \|z\|_{\mathcal{W}} = \|z\|_{W^{1,p_1}(\mathbb{R}^N)} + \|z\|_{W^{1,p_2}(\mathbb{R}^N)} \leq \|z\|_{W^{1,p_1}(\mathbb{R}^N)} + \frac{K}{2}$$
from which $\|z\|_{W^{1,p_1}(\mathbb{R}^N)} \geq \frac{K}{2}$. Hence, $\|z\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} + \|z\|_{W^{1,p_2}(\mathbb{R}^N)}^{p_2} \geq \|z\|_{W^{1,p_1}(\mathbb{R}^N)}^{p_1} \geq (\frac{K}{2})^{p_1}$.

Therefore, (5.32) and the above argument show that (5.31) is true. Combining (5.29), (5.30), and (5.31), we see

$$mC = \sum_{k=1}^{m} C \leq \sum_{i=1}^{2} M^{p_i} + 1,$$

and thus the vanishing case must occur for some $m_0 \in \mathbb{N}$ and Theorem 5.1 holds with $l = m_0$. The proof of Theorem 5.1 is now complete. □

Now we are ready to give a second proof of Theorem 1.1.

Second proof of Theorem 1.1 By Proposition 4.1, we know that there exists a Pohozaev-Palais-Smale sequence $(u_n) \subset \mathcal{W}$ of L at the level $c_{\mathrm{MP}} > 0$. By virtue of Lemma 4.2, (u_n) is a bounded Palais-Smale sequence in $\mathcal{W}$. Now we note that if $w \in \mathcal{W}\backslash\{0\}$ is any critical point of L then $L(w) \geq c_{\mathrm{MP}} > 0$. In fact, due to $P(w) = 0$, we can use Lemma 4.4 to construct a path $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} L(\gamma(t)) = L(w)$. Hence, $L(w) \geq c_{\mathrm{MP}}$, as claimed. Applying Theorem 5.1 with $\beta = c_{\mathrm{MP}} > 0$, we see that, up to a subsequence of (u_n), there exist $l \in \mathbb{N}$, $(y_n^1), \ldots, (y_n^l) \subset \mathbb{R}^N$ and $w_1, \ldots, w_l \in \mathcal{W}$ such that properties (i)–(iv) in Theorem 5.1 hold. If $l \geq 3$, or $l = 2$ but $w^1 \neq 0$, then items (ii) and (iii) of Theorem 5.1 yield

$$c_{\mathrm{MP}} \geq \sum_{k=1}^{l} L(w^k) \geq 2c_{\mathrm{MP}} > c_{\mathrm{MP}},$$

that is a contradiction. Thus, $l = 1$, or $l = 2$ with $w^1 = 0$. Utilizing items (i) and (iv) of Theorem 5.1, we deduce that $u_n - w^1 \to 0$ in $\mathcal{W}$, or $u_n - w^2(\cdot - y_n^2) \to 0$ in $\mathcal{W}$ with $|y_n^2| \to +\infty$. Therefore, up to a subsequence and translations, $u_n \to u$ in $\mathcal{W}$ for some $u \in \mathcal{W} \setminus \{0\}$ such that $L(u) = c_{\mathrm{MP}}$ and $L'(u) = 0$. Arguing as in the last part of the first proof of Theorem 1.1, we conclude that u is a ground state solution to (1.1). □

6 Monotonicity trick: third proof of Theorem 1.1 and proof of Theorem 1.2

The third proof of Theorem 1.1 and the proof of Theorem 1.2 will be obtained by employing two abstract results based on the monotonicity trick. First we introduce some notations and definitions.

Let $(X, \|\cdot\|)$ be a real Banach space with dual X', $I \subset (0, +\infty)$ be a nonempty compact interval. Let (L_λ) be a family of C^1 functionals on X with parameter $\lambda \in I$ of the form

$$L_\lambda(u) := A(u) - \lambda B(u) \quad \text{for } \lambda \in I,$$

where $A, B \in C^1(X, \mathbb{R})$ are such that $A(0) = 0 = B(0)$, $B \geq 0$ on X, and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$.

We say that (L_λ) has a uniform mountain pass geometry if, for every $\lambda \in I$, the set

$$\Gamma_\lambda := \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ L_\lambda(\gamma(1)) < 0\}$$

is nonempty and

$$c_{\mathrm{MP},\lambda} := \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} L_\lambda(\gamma(t)) > 0.$$

The next result is an alternative version of [28, Theorem 1.1] (see also [30, Theorem 2.1]).

Theorem 6.1 *[28, Theorem 1.1] If (L_λ) has a uniform mountain pass geometry, then*

(i) for almost every $\lambda \in I$, L_λ admits a bounded Palais-Smale sequence $(u_n^\lambda) \subset X$ at the mountain pass level $c_{\mathrm{MP},\lambda}$, that is,

$$\sup_{n\in\mathbb{N}} \|u_n^\lambda\| < +\infty, \quad L_\lambda(u_n^\lambda) \to c_{\mathrm{MP},\lambda} \quad \textit{and} \quad L'_\lambda(u_n^\lambda) \to 0 \textit{ in } X',$$

(ii) the mapping $\lambda \mapsto c_{\mathrm{MP},\lambda}$ is left continuous.

When A and B are even, we can extend the previous result by considering a suitable geometric condition.

For every $k \in \mathbb{N}$, let $\mathbb{D}_k := \{x \in \mathbb{R}^k : |x| \leq 1\}$ and $\mathbb{S}^{k-1} := \{x \in \mathbb{R}^k : |x| = 1\}$.

A family of even functionals (L_λ) with parameter $\lambda \in I$ is said to have a uniform symmetric mountain pass geometry if, for every $k \in \mathbb{N}$, there exists an odd continuous mapping $\gamma_{0k} : \mathbb{S}^{k-1} \to X \setminus \{0\}$ such that

$$\max_{l \in \mathbb{S}^{k-1}} L_\lambda(\gamma_{0k}(l)) < 0 \quad \text{uniformly in } \lambda \in I,$$

the class of mappings

$$\Gamma_k := \{\gamma \in C(\mathbb{D}_k, X) : \gamma \text{ is odd and } \gamma = \gamma_{0k} \text{ on } \mathbb{S}^{k-1}\}$$

is nonempty, and

$$c_{k,\lambda} := \inf_{\gamma \in \Gamma_k} \max_{l \in \mathbb{D}_k} L_\lambda(\gamma(l)) > 0.$$

It holds the following result.

Theorem 6.2 *[30, Theorem 2.2] Assume in addition that A and B are even. If (L_λ) has a uniform symmetric mountain pass geometry, then*

(i) for almost every $\lambda \in I$, L_λ admits a bounded Palais-Smale sequence $(u^\lambda_{k,n}) \subset X$ at each level $c_{k,\lambda}$ ($k \in \mathbb{N}$), that is,

$$\sup_{n \in \mathbb{N}} \|u^\lambda_{k,n}\| < +\infty, \quad L_\lambda(u^\lambda_{k,n}) \to c_{k,\lambda} \quad \textit{and} \quad L'_\lambda(u^\lambda_{k,n}) \to 0 \textit{ in } X',$$

(ii) for every $k \in \mathbb{N}$, the mapping $\lambda \mapsto c_{k,\lambda}$ is left continuous.

From $(g3)$ we know that $G_1(\xi) - G_2(\xi) > 0$. Then there exists $\lambda_0 \in (0, 1)$ such that $\lambda_0 G_1(\xi) - G_2(\xi) > 0$. For $t \in \mathbb{R}$ and $\lambda \in [\lambda_0, 1]$, define

$$g^\lambda(t) := \lambda g_1(t) - g_2(t) \quad \text{and} \quad G^\lambda(t) := \int_0^t g^\lambda(s)\, ds.$$

Let us introduce a family of even functionals of class C^1 as follows:

$$L_\lambda(u) := \sum_{i=1}^{2} \frac{1}{p_i} \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} G_2(u)\, dx - \lambda \int_{\mathbb{R}^N} G_1(u)\, dx =: A(u) - \lambda B(u),$$

for all $u \in \mathcal{W}$ and $\lambda \in [\lambda_0, 1]$. Clearly, $A, B \in C^1(\mathcal{W}, \mathbb{R})$, A and B are even, $A(0) = 0 = B(0)$, $B \geq 0$, and $A(u) \to +\infty$ as $\|u\|_{\mathcal{W}} \to +\infty$ (due to (3.3)). Moreover,

$$L(u) = L_1(u) \leq L_\lambda(u) \leq L_{\lambda_0}(u) \quad \text{for all } u \in \mathcal{W} \text{ and } \lambda \in [\lambda_0, 1]. \tag{6.1}$$

Next we prove some uniform geometric properties for the functionals L_λ.

Lemma 6.1 *The functional L_λ fulfills the following properties:*

(i) There exist $r_0 > 0$ and $\rho_0 > 0$ (independent of $\lambda \in [\lambda_0, 1]$) such that

$$L_\lambda(u) \geq L(u) > 0 \quad \textit{for all } u \in \mathcal{W} \textit{ such that } 0 < \|u\|_{\mathcal{W}} \leq r_0,$$
$$L_\lambda(u) \geq L(u) \geq \rho_0 \quad \textit{for all } u \in \mathcal{W} \textit{ such that } \|u\|_{\mathcal{W}} = r_0.$$

(ii) *For every $k \in \mathbb{N}$ there exists an odd continuous map $\gamma_{0k} : \mathbb{S}^{k-1} \to \mathcal{W}_{\mathrm{r}}$ independent of $\lambda \in [\lambda_0, 1]$ such that*

$$L(\gamma_{0k}(l)) \leq L_\lambda(\gamma_{0k}(l)) \leq L_{\lambda_0}(\gamma_{0k}(l)) < 0 \quad \textit{for all } l \in \mathbb{S}^{k-1}.$$

Proof The proof of (i) follows from (6.1), arguing as in the proof of Lemma 4.1, and using Remark 4.1. For what concerns (ii), recalling that $G^{\lambda_0}(\xi) > 0$, we can argue as in the proof of [13, Theorem 10] to see that for every $k \in \mathbb{N}$ there exists an odd continuous map $\pi_k : \mathbb{S}^{k-1} \to \mathcal{W}_{\mathrm{r}}$ such that

$$0 \notin \pi_k(\mathbb{S}^{k-1}) \text{ and } \int_{\mathbb{R}^N} G^{\lambda_0}(\pi_k(l))\, dx \geq 1 \quad \text{for all } l \in \mathbb{S}^{k-1}.$$

Define $\gamma_{0k}(l)(x) := \pi_k(l)(x/t) : \mathbb{S}^{k-1} \to \mathcal{W}_{\mathrm{r}}$, with $t \geq 1$ undetermined. Then,

$$\begin{aligned} L_{\lambda_0}(\gamma_{0k}(l)) &= \sum_{i=1}^{2} \frac{t^{N-p_i}}{p_i} \|\nabla \pi_k(l)\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - t^N \int_{\mathbb{R}^N} G^{\lambda_0}(\pi_k(l))\, dx \\ &\leq \sum_{i=1}^{2} \frac{t^{N-p_i}}{p_i} \|\nabla \pi_k(l)\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - t^N \to -\infty \text{ as } t \to +\infty. \end{aligned}$$

Choosing $t = t_k \geq 1$ sufficiently large, we complete the proof of the lemma. □

Set

$$P_\lambda(u) := \sum_{i=1}^{2} \left(\frac{N - p_i}{p_i}\right) \|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} - N \int_{\mathbb{R}^N} G^\lambda(u)\, dx.$$

Note that if $\lambda = 1$ then $P_1 = P$. Proceeding as in the proof of Lemma 4.4, we obtain the result below.

Lemma 6.2 *Assume that $\lambda \in [\lambda_0, 1]$ is fixed and that $w \in \mathcal{W}\backslash\{0\}$ satisfies $P_\lambda(w) = 0$. Then there exists $\gamma \in \Gamma_\lambda$ such that $w \in \gamma([0, 1])$ and* $\max_{t \geq 0} L_\lambda(\gamma(t)) = L_\lambda(w)$.

Lemma 6.3 *Assume that $\lambda \in [\lambda_0, 1]$ is fixed and that $(u_n) \subset \mathcal{W}$ is a bounded Palais-Smale sequence for L_λ at the level $c_{\mathrm{MP},\lambda}$. Then, up to a subsequence, there exists $(y_n) \subset \mathbb{R}^N$ such that the translated sequence $(u_n(\cdot + y_n))$ is a convergent Palais-Smale sequence for L_λ at the level $c_{\mathrm{MP},\lambda}$.*

Proof The proof is similar to the second proof of Theorem 1.1. However, we give the details for completeness. We aim to determine a suitable sequence $(y_n) \subset \mathbb{R}^N$ such that $(u_n(\cdot + y_n))$ is strongly convergent in $\mathcal{W}$. Note that if $w \in \mathcal{W} \setminus \{0\}$ is any critical point of L_λ then $L_\lambda(w) \geq c_{\mathrm{MP},\lambda} > 0$. In fact, $P_\lambda(w) = 0$, and thanks to Lemma 6.2 we can select $\gamma \in \Gamma_\lambda$ such that $\max_{t \in [0,1]} L_\lambda(\gamma(t)) = L_\lambda(w)$, whence, $L_\lambda(w) \geq c_{\mathrm{MP},\lambda}$, as required.

Now we apply Theorem 5.1 with $L = L_\lambda$ and $\beta = c_{\mathrm{MP},\lambda} > 0$. Thus, up to a subsequence of (u_n), we can find $l \in \mathbb{N}$, $(y_n^1), \ldots, (y_n^l) \subset \mathbb{R}^N$ and $w_1, \ldots, w_l \in \mathcal{W}$ such that properties (i)–(iv) in Theorem 5.1 hold. If $l \geq 3$, or $l = 2$ but $w^1 \neq 0$, then it follows from (ii) and (iii) of Theorem 5.1 that

$$c_{\mathrm{MP},\lambda} \geq \sum_{k=1}^{l} L_\lambda(w^k) \geq 2c_{\mathrm{MP},\lambda} > c_{\mathrm{MP},\lambda},$$

that is a contradiction. Therefore, $l = 1$, or $l = 2$ with $w^1 = 0$. Using items (i) and (iv) of Theorem 5.1, we reach the desired conclusion. □

In order to prove Theorems 1.1 and 1.2, we establish the next useful results.

Lemma 6.4 *Assume that* $(\lambda_n) \subset [\lambda_0, 1]$ *and* $(u_n) \subset \mathcal{W}$. *If*

$$\sup_{n\in\mathbb{N}} L_\lambda(u_n) \leq C \quad \text{and} \quad \inf_{n\in\mathbb{N}} P_\lambda(u_n) \geq -C,$$

for some $C > 0$, *then* (u_n) *is bounded in* $\mathcal{W}$.

Proof Since

$$\frac{1}{N}\sum_{i=1}^{2} \|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i} = L_{\lambda_n}(u_n) - \frac{1}{N}P_{\lambda_n}(u_n) \leq 2C \quad \text{for all } n \in \mathbb{N},$$

we infer that $(\|\nabla u_n\|_{L^{p_i}(\mathbb{R}^N)}^{p_i})$ is bounded in $\mathbb{R}$ for all $i = 1, 2$. At this point, we can argue as in the proof of Lemma 4.2 to obtain the assertion. □

Lemma 6.5 *Assume that* $(\lambda_n) \subset [\lambda_0, 1)$, X *is any subspace of* $\mathcal{W}$, *and* $u_n \in X$ *is a critical point of the restricted functional* $L_\lambda|_X$ *for every* $n \in \mathbb{N}$. *If* $\lambda_n \to 1$ *as* $n \to +\infty$, (u_n) *is bounded in* $\mathcal{W}$ *and* $L_{\lambda_n}(u_n) \to c$ *as* $n \to +\infty$ *for some* $c \in \mathbb{R}$, *then* (u_n) *is a bounded Palais-Smale sequence of* $L_\lambda|_X$ *at the level* c.

Proof Due to the boundedness of (u_n) in $\mathcal{W}$, we deduce from (3.1)–(3.2) that $(\int_{\mathbb{R}^N} G_1(u_n)\,dx)$ is bounded in $\mathbb{R}$ and $(g_1(u_n))$ is bounded in X'. Thanks to $\lambda_n \to 1$ and $J_{\lambda_n}(u_n) \to c$ as $n \to +\infty$, we see

$$L(u_n) = L_{\lambda_n}(u_n) + (\lambda_n - 1)\int_{\mathbb{R}^N} G_1(u_n)\,dx = J_{\lambda_n}(u_n) + o_n(1) = c + o_n(1),$$

$$(L|_X)'(u_n) = (L_{\lambda_n}|_X)'(u_n) + (\lambda_n - 1)g_1(u_n) = (\lambda_n - 1)g_1(u_n) = o_n(1) \text{ in } X'.$$

Consequently, (u_n) is a bounded Palais-Smale sequence of $L_\lambda|_X$ at the level c. □

Third proof of Theorem 1.1 Let $X = \mathcal{W}$. In view of Theorem 6.1, there exists a sequence $(\lambda_n) \subset [\lambda_0, 1)$ such that

(i) $\lambda_n \to 1$ as $n \to +\infty$,
(ii) $c_{\mathrm{MP},\lambda_n} \to c_{\mathrm{MP},1} = c_{\mathrm{MP}}$ as $n \to +\infty$,
(iii) L_{λ_n} has a bounded Palais-Smale sequence at the level $c_{\mathrm{MP},\lambda_n}$ for every $n \in \mathbb{N}$.

Utilizing Lemma 6.3, we obtain a critical point u_n of L_{λ_n} with $L_{\lambda_n}(u_n) = c_{\mathrm{MP},\lambda_n}$. Hence $P_{\lambda_n}(u_n) = 0$ for all $n \in \mathbb{N}$, and because $\sup_{n\in\mathbb{N}} L_{\lambda_n}(u_n) = \sup_{n\in\mathbb{N}} c_{\mathrm{MP},\lambda_n} \leq c_{\mathrm{MP},\lambda_0}$, we can employ Lemma 6.4 to infer that (u_n) is bounded in $\mathcal{W}$. From Lemma 6.5, we derive that (u_n) is a bounded Palais-Smale sequence of L at the mountain pass level c_{MP}. Exploiting Lemma 6.5 once again, we find a nontrivial critical point $u \in \mathcal{W}$ of (1.1) with $L(u) = c_{\mathrm{MP}}$. Arguing as in the first proof of Theorem 1.1, we arrive at $L(u) = c_{\mathrm{MP}} = c_{\mathrm{LE}} = c_{\mathrm{PO}}$. □

From now on, we focus on the proof of Theorem 1.2. Let us begin by proving the following compactness result.

Lemma 6.6 *Every bounded Palais-Smale sequence* (u_n) *of the restricted functional* $L|_{\mathcal{W}_\mathrm{r}}$ *has a strongly convergent subsequence in* $\mathcal{W}_\mathrm{r}$.

Proof Since (u_n) is bounded in $\mathcal{W}_r$, according to Theorem 2.3 we may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $\mathcal{W}_r$, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (p_2, p_2^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. Reasoning as in the proof of Lemma 4.3, we discover that, up to a subsequence, as $n \to +\infty$,

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N,$$
$$|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2.$$

From the proof of the vanishing case in Step 2 of Theorem 5.1, we conclud.e that $v_n^1 = u_n - u \to 0$ in $\mathcal{W}_r$. □

For each $\lambda \in [\lambda_0, 1]$, by Lemma 6.6, we know that $L_\lambda|_{\mathcal{W}_r}$ satisfies the bounded Palais-Smale condition, that is, any bounded Palais-Smale sequence for $L_\lambda|_{\mathcal{W}_r}$ converges, up to a subsequence.

For every $k \in \mathbb{N}$, we consider the family of maps

$$\Gamma_k := \{\gamma \in C(\mathbb{D}_k, \mathcal{W}_r) : \gamma \text{ is odd and } \gamma = \gamma_{0k} \text{ on } \mathbb{S}^{k-1}\},$$

where γ_{0k} is defined in Lemma 6.1-(ii). Note that Γ_k is nonempty because it contains the mapping

$$\gamma_k(\sigma) := \begin{cases} |\sigma|\, \gamma_{0k}\left(\frac{\sigma}{|\sigma|}\right) & \text{for } \sigma \in \mathbb{D}_k \setminus \{0\}, \\ 0 & \text{for } \sigma = 0. \end{cases}$$

From Lemma 6.1-(i), we see that, for all $\gamma \in \Gamma_k$,

$$\gamma(\mathbb{D}_k) \cap \{u \in \mathcal{W}_r : \|u\|_{\mathcal{W}} = r_0\} \neq \emptyset.$$

Then the symmetric mountain pass value $c_{k,\lambda}$ of $L_\lambda|_{\mathcal{W}_r}$ given by

$$c_{k,\lambda} := \inf_{\gamma \in \Gamma_k} \max_{\sigma \in \mathbb{D}_k} L_\lambda(\gamma(\sigma))$$

is well-defined and $c_{k,\lambda} \geq c_{k,1} \geq \rho_0 > 0$. Our aim is to prove that $c_{k,1} \to +\infty$ as $k \to +\infty$. We will use a comparison argument as in [25, Sections 2 and 3]. Fix $p_0 \in (p_2 - 1, p_2^* - 1)$ and define the continuous functions $f, \bar{f} : \mathbb{R} \to \mathbb{R}$ by setting

$$f(t) := \begin{cases} (g(t) + \nu(t^{p_1-1} + t^{p_2-1}))^+ & \text{for } t \geq 0, \\ -f(-t) & \text{for } t < 0, \end{cases}$$

$$\bar{f}(t) := \begin{cases} t^{p_0} \sup_{\tau \in (0,t]} \frac{f(\tau)}{\tau^{p_0}} & \text{for } t > 0, \\ 0 & \text{for } t = 0, \\ -\bar{f}(-t) & \text{for } t < 0. \end{cases}$$

Let $F(t) := \int_0^t f(\tau)\, d\tau$ and $\bar{F}(t) := \int_0^t \bar{f}(\tau)\, d\tau$. Inspired by [25, Lemma 2.1 and Corollary 2.2], we establish the next result.

Lemma 6.7 *The following properties hold:*

(i) There exists $\delta_0 > 0$ such that $\bar{f}(t) = 0 = \bar{F}(t)$ for all $t \in [-\delta_0, \delta_0]$.
(ii) We have $\bar{F}(t) \geq G(t) + \nu\left(\frac{|t|^{p_1}}{p_1} + \frac{|t|^{p_2}}{p_2}\right)$ for all $t \in \mathbb{R}$.
(iii) It holds $0 \leq (p_0 + 1)\bar{F}(t) \leq \bar{f}(t)t$ for all $t \in \mathbb{R}$.
(iv) The map $t \mapsto \bar{f}(t) - \nu(|t|^{p_1-1}t + |t|^{p_2-1}t)$ satisfies $(g1)$, $(g2)'$ and $(g3)$.

(v) $\bar{f}(u_n) \to \bar{f}(u)$ *in* $L^{p_N}(\mathbb{R}^N)$ *for every sequence* $(u_n) \subset \mathcal{W}_{\mathrm{r}}$ *such that* $u_n \rightharpoonup u$ *in* $\mathcal{W}_{\mathrm{r}}$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^N$*, where*

$$p_N := \begin{cases} \frac{p_2^*}{p_2^*-1} & \text{if } p_2 < N, \\ \frac{N}{N-1} & \text{if } p_2 = N. \end{cases}$$

Proof The item (i) is evident from the definition of ν. The item (ii) is a consequence of

$$\bar{f}(t) \geq f(t) \geq g(t) + \nu(t^{p_1-1} + t^{p_2-1}) \quad \text{for all } t \geq 0.$$

Concerning (iii), we first observe that the map $t \in (0, +\infty) \mapsto \bar{f}(t)/t^{p_0}$ is nondecreasing. Thus, for all $t > 0$,

$$\begin{aligned} t\bar{f}(t) - (p_0+1)\bar{F}(t) &= \int_0^t [\bar{f}(t) - (p_0+1)\bar{f}(\tau)]\, d\tau \\ &= \int_0^t \left[t^{p_0}\frac{\bar{f}(t)}{t^{p_0}} - (p_0+1)\tau^{p_0}\frac{\bar{f}(\tau)}{\tau^{p_0}} \right] d\tau \\ &\geq \int_0^t \left[t^{p_0}\frac{\bar{f}(t)}{t^{p_0}} - (p_0+1)\tau^{p_0}\frac{\bar{f}(t)}{t^{p_0}} \right] d\tau = 0. \end{aligned}$$

In order to check (iv), we clearly have that $t \mapsto \bar{f}(t) - \nu(|t|^{p_1-1}t + |t|^{p_2-1}t)$ fulfills $(g1)$ and $(g3)$. To verify $(g2)'$, it is enough to demonstrate that $\bar{f}$ satisfies $(g2')$. Suppose $1 < p_1 < p_2 < N$. Note that, for all $t > 0$,

$$\frac{\bar{f}(t)}{t^{p_2^*-1}} = t^{-(p_2^*-p_0)} \sup_{\tau \in (0,t]} \frac{f(\tau)}{\tau^{p_0}} = \sup_{\tau \in (0,t]} \frac{f(\tau)}{\tau^{p_2^*-1}} \frac{\tau^{p_2^*-1}}{t^{p_2^*-p_0}}.$$

Since f obeys $(g2')$, for all $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\left| \frac{f(\tau)}{\tau^{p_2^*-1}} \right| \leq \varepsilon \quad \text{for all } \tau \geq \tau_\varepsilon.$$

Put

$$C_\varepsilon := \sup_{0<\tau\leq\tau_\varepsilon} \left| \frac{f(\tau)}{\tau^{p_2^*-1}} \right|.$$

Then we see

$$\begin{aligned} \frac{\bar{f}(t)}{t^{p_2^*-1}} &\leq \max\left\{ \sup_{\tau\in(0,\tau_\varepsilon]} \left| \frac{f(\tau)}{\tau^{p_2^*-1}} \right| \frac{\tau_\varepsilon^{p_2^*-p_0}}{t^{p_2^*-p_0}}, \sup_{\tau\in[\tau_\varepsilon,t]} \left| \frac{f(\tau)}{\tau^{p_2^*-1}} \right| \right\} \\ &\leq \max\left\{ C_\varepsilon \frac{\tau_\varepsilon^{p_2^*-p_0}}{\tau^{p_2^*-p_0}}, \varepsilon \right\}, \end{aligned}$$

from which

$$\limsup_{t\to+\infty} \frac{\bar{f}(t)}{t^{p_2^*-1}} \leq \varepsilon\,.$$

Because $\varepsilon > 0$ is arbitrary, we get the desired assertion. When $p_2 = N$, it suffices to show that

$$\lim_{t\to+\infty} \frac{\bar{f}(t)}{t^{p_0} e^{\alpha|t|^{\frac{N}{N-1}}}} = 0 \quad \text{for all } \alpha > 0. \tag{6.2}$$

Considering that

$$\frac{\bar{f}(t)}{t^{p_0} e^{\alpha|t|^{\frac{N}{N-1}}}} = \frac{1}{e^{\alpha|t|^{\frac{N}{N-1}}}} \sup_{\tau\in(0,t]} \frac{f(\tau)}{\tau^{p_0}} = \sup_{\tau\in(0,t]} \frac{f(\tau)}{\tau^{p_0} e^{\alpha|\tau|^{\frac{N}{N-1}}}} \frac{e^{\alpha|\tau|^{\frac{N}{N-1}}}}{e^{\alpha|t|^{\frac{N}{N-1}}}},$$

and f satisfies

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p_0} e^{\alpha|t|^{\frac{N}{N-1}}}} = 0,$$

we can argue as in the case $p_2 < N$ to achieve (6.2). Finally, we prove (v). Let $u_n \rightharpoonup u$ in $\mathcal{W}_r$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Invoking Lemma 2.3, we know that if $N \geq 2$, $p \in [1, +\infty)$ and $v \in W^{1,p}_r(\mathbb{R}^N)$, then

$$|v(x)| \leq C(N, p)\, |x|^{-\frac{N-1}{p}} \|v\|_{W^{1,p}(\mathbb{R}^N)} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Hence we can find $R = R_{\delta_0} > 0$ such that $|u_n(x)|, |u(x)| \leq \delta_0$ for all $|x| \geq R$ and $n \in \mathbb{N}$. Therefore, in light of (i), we only need to ascertain that $\bar{f}(u_n) \to \bar{f}(u)$ in $L^{p_N}(B_R(0))$. As (u_n) is bounded in $W^{1,p_2}(\mathbb{R}^N)$, we may assume that there exists $M > 0$ such that $\|u_n\|_{W^{1,p_2}(\mathbb{R}^N)} \leq M$ for all $n \in \mathbb{N}$. Set

$$\mathcal{R}(t) := \begin{cases} |t|^{p_2^*-1} & \text{if } p_2 < N, \\ \Phi_N(\alpha|t|^{\frac{N}{N-1}}) & \text{if } p_2 = N, \end{cases}$$

where $\alpha > 0$ is such that $\alpha\, p_N M^{\frac{N}{N-1}} < \alpha_N$. From (ii), we deduce that for every $\varepsilon > 0$ there exists $t_\varepsilon > \delta_0$ such that $|\bar{f}(t)| \leq \varepsilon\, \mathcal{R}(t)$ for all $|t| \geq t_\varepsilon$. Define

$$\hat{f}(t) := \begin{cases} \bar{f}(t) & \text{for } |t| \leq t_\varepsilon, \\ \bar{f}(t_\varepsilon) & \text{for } t > t_\varepsilon, \\ \bar{f}(-t_\varepsilon) & \text{for } t < -t_\varepsilon. \end{cases}$$

Let us observe that $|\hat{f}(t) - \bar{f}(t)| \leq 2\,\varepsilon\, \mathcal{R}(t)$ for all $t \in \mathbb{R}$. On the other hand, since $\hat{f}$ is bounded and continuous in $\mathbb{R}$, and $u_n \to u$ a.e. in $\mathbb{R}^N$, we know that $\hat{f}(u_n) \to \hat{f}(u)$ in $L^{p_N}(B_R(0))$. Now we note that

$$\begin{aligned} &\|\bar{f}(u_n) - \bar{f}(u)\|_{L^{p_N}(B_R(0))} \\ &\leq \|\bar{f}(u_n) - \hat{f}(u_n)\|_{L^{p_N}(B_R(0))} + \|\hat{f}(u_n) - \hat{f}(u)\|_{L^{p_N}(B_R(0))} + \|\hat{f}(u) - \bar{f}(u)\|_{L^{p_N}(B_R(0))} \\ &\leq 2\,\varepsilon\, \|\mathcal{R}(u_n)\|_{L^{p_N}(\mathbb{R}^N)} + \|\hat{f}(u_n) - \hat{f}(u)\|_{L^{p_N}(B_R(0))} + 2\,\varepsilon\, \|\mathcal{R}(u)\|_{L^{p_N}(\mathbb{R}^N)} \\ &\leq C\,\varepsilon + \|\hat{f}(u_n) - \hat{f}(u)\|_{L^{p_N}(B_R(0))}, \end{aligned}$$

where we have used the fact that the boundedness of (u_n) in $W^{1,p_2}(\mathbb{R}^N)$, our choice of α, (2.4), and (2.3) yield

$$\sup_{n\in\mathbb{N}} \|\mathcal{R}(u_n)\|_{L^{p_N}(\mathbb{R}^N)} = \sup_{n\in\mathbb{N}} \|u_n\|^{p_2^*-1}_{L^{p_2^*}(\mathbb{R}^N)} \leq C' \quad \text{if } p_2 < N,$$

$$\sup_{n\in\mathbb{N}} \|\mathcal{R}(u_n)\|_{L^{p_N}(\mathbb{R}^N)} \leq \sup_{n\in\mathbb{N}} \left(\int_{\mathbb{R}^N} \Phi_N \left(\alpha\, p_N M^{\frac{N}{N-1}} \left(\frac{|u_n|}{M} \right)^{\frac{N}{N-1}} \right) dx \right)^{\frac{1}{p_N}} \leq C'' \quad \text{if } p_2 = N.$$

Thus,

$$\limsup_{n\to+\infty} \|\bar{f}(u_n) - \bar{f}(u)\|_{L^{p_N}(B_R(0))} \leq C\,\varepsilon,$$

and because $\varepsilon > 0$ is arbitrary we get the assertion. □

Lemma 6.8 *The sequence of symmetric mountain pass values $(c_{k,1})$ is such that $c_{k,1} \to +\infty$ as $k \to +\infty$.*

Proof Let us introduce the comparison functional $J : \mathcal{W}_{\mathrm{r}} \to \mathbb{R}$ by setting

$$J(u) := \sum_{i=1}^{2} \frac{1}{p_i}\left(\|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu\|u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) - \int_{\mathbb{R}^N} \bar{F}(u)\,dx.$$

It is not difficult to check that J has a symmetric mountain pass geometry and that it satisfies the Palais-Smale compactness condition. In fact, Lemma 6.7-(iv) ensures that 0 is a strict local minimum point of J. The odd continuous mapping γ_{0k} given by Lemma 6.1 is still valid here since Lemma 6.7-(ii) guarantees that

$$L(u) \geq J(u) \quad \text{for all } u \in \mathcal{W}_{\mathrm{r}}. \tag{6.3}$$

Let us define the symmetric mountain pass values of J as follows:

$$d_k := \inf_{\gamma\in\Gamma_k} \max_{\sigma\in\mathbb{D}_k} J(\gamma(\sigma)) \quad \text{for all } k \in \mathbb{N}.$$

Using Lemma 6.7-(iii) and $p_1 < p_2 < p_0 + 1$, we can verify that every Palais-Smale sequence of J is bounded in $\mathcal{W}_{\mathrm{r}}$. Indeed, if $(u_n) \subset \mathcal{W}_{\mathrm{r}}$ is any Palais-Smale sequence for J, that is, $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ in $\mathcal{W}_{\mathrm{r}}'$, then we see that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
C(1 + \|u_n\|_{\mathcal{W}}) &\geq J(u_n) - \frac{1}{p_0+1}\langle J'(u_n), u_n\rangle \\
&= \sum_{i=1}^{2}\left(\frac{1}{p_i} - \frac{1}{p_0+1}\right)\left[\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu\|u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right] \\
&\quad - \int_{\mathbb{R}^N}\left[\bar{F}(u_n) - \frac{1}{p_0+1}\bar{f}(u_n)u_n\right] dx \\
&\geq \left(\frac{1}{p_2} - \frac{1}{p_0+1}\right)\sum_{i=1}^{2}\left[\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu\|u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right],
\end{aligned}$$

which implies that (u_n) is bounded in $\mathcal{W}_{\mathrm{r}}$. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $\mathcal{W}_{\mathrm{r}}$, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (p_2, p_2^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. Arguing as in the proof of Lemma 4.3, we find that, up to a subsequence,

$$\begin{aligned}
&\nabla u_n \to \nabla u \quad \text{a.e. in } \mathbb{R}^N, \\
&|\nabla u_n|^{p_i-2}\nabla u_n \rightharpoonup |\nabla u|^{p_i-2}\nabla u \quad \text{in } (L^{\frac{p_i}{p_i-1}}(\mathbb{R}^N))^N \text{ for all } i = 1, 2.
\end{aligned}$$

To confirm that J satisfies the Palais-Smale compactness condition, we demonstrate that (u_n) has a strongly convergent subsequence in $\mathcal{W}_{\mathrm{r}}$. Utilizing $J'(u_n) \to 0$ in $\mathcal{W}_{\mathrm{r}}'$ and the above convergences, we deduce that $\langle J'(u), \varphi\rangle = 0$ for all $\varphi \in \mathcal{W}_{\mathrm{r}}$. In particular, $\langle J'(u), u\rangle = 0$, that is,

$$\sum_{i=1}^{2}\left(\|\nabla u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu\|u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) = \int_{\mathbb{R}^N} \bar{f}(u)u\,dx.$$

This fact, combined with

$$\sum_{i=1}^{2}\left(\|\nabla u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \nu\|u_n\|^{p_i}_{L^{p_i}(\mathbb{R}^N)}\right) = \int_{\mathbb{R}^N} \bar{f}(u_n)u_n\,dx + o_n(1),$$

the Brezis-Lieb lemma [14, Theorem 1], and Lemma 6.7-(v), shows that, up to a subsequence, $u_n \to u$ in $\mathcal{W}_{\rm r}$. Reasoning as in the proof of [25, Lemma 3.2], we can see that d_k is a critical value of J for all $k \in \mathbb{N}$ and that $d_k \to +\infty$ as $k \to +\infty$. In view of (6.3), we obtain that $c_{k,1} \geq d_k$ for all $k \in \mathbb{N}$. Consequently, $c_{k,1} \to +\infty$ as $k \to +\infty$. □

Now we are ready to provide the proof of the second main result of this paper.

Proof of Theorem 1.2 Let $X = \mathcal{W}_{\rm r}$. By Theorem 6.2, there exists a sequence $(\lambda_n) \subset [\lambda_0, 1)$ such that

(i) $\lambda_n \to 1$ as $n \to +\infty$,
(ii) $c_{k,\lambda_n} \to c_{k,1}$ as $n \to +\infty$ for every $k \in \mathbb{N}$,
(iii) $L_{\lambda_n}|_{\mathcal{W}_{\rm r}}$ has a bounded Palais-Smale sequence at the level c_{k,λ_n} for every $k, n \in \mathbb{N}$.

Then, for every $k, n \in \mathbb{N}$, the restricted functional $L_{\lambda_n}|_{\mathcal{W}_{\rm r}}$ has a critical point $u_{k,n}$ with $L_{\lambda_n}(u_{k,n}) = c_{k,\lambda_n}$. The Palais principle of symmetric criticality [40] and the Pohozaev identity yield $P_{\lambda_n}(u_{k,n}) = 0$ for all $k, n \in \mathbb{N}$. Since $\sup_{n\in\mathbb{N}} L_{\lambda_n}(u_{k,n}) = \sup_{n\in\mathbb{N}} c_{k,\lambda_n} \leq c_{k,\lambda_0}$, we can apply Lemma 6.4 to infer that (u_n) is bounded in $\mathcal{W}_{\rm r}$. From Lemma 6.5, we derive that $(u_{k,n})$ is a bounded Palais-Smale sequence of $L|_{\mathcal{W}_{\rm r}}$ at the level $c_{k,1}$. This implies that the restricted functional $L|_{\mathcal{W}_{\rm r}}$ has a critical point $v_k \in \mathcal{W}_{\rm r}$ at each level $c_{k,1}$ ($k \in \mathbb{N}$). By Lemma 6.8, we know that $L(v_k) = c_{k,1} \to +\infty$ as $k \to +\infty$. By the Palais principle of symmetric criticality [40], we have that (v_k) is indeed a sequence of nontrivial solutions to (1.1). □

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