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On conditions of potentialness for geometrically exact shearable beams

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In classical mechanics, there are different, yet interrelated, formulations which allow studying mechanical systems from various perspectives and deepening. However, it is desirable that the equations describing the dynamics of a system be the same, whichever path is followed to derive them. In this paper, the focus is on the equations of motion of beams in the geometrically exact framework, for which we also require that they have the additional property of the 'potentialness' (or potentiality), that is they can be derived from a properly defined potential. Extending results from the literature that focus on unshearable beams, this contribution is devoted to the shearable case. For a planar beam whose stress-free configuration is curvilinear, four different sets of balance equations are considered, where two different choices of the components of the internal resultant force and two different curvatures as bending strain measures are taken into account. The conditions to be imposed on constitutive assumptions, relating generalized static entities to strains, are assessed through Helmholtz conditions. Some constitutive assumptions for nonlinear shearable beams available in the literature are also discussed in light of theoretical results obtained in this contribution.

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1. Introduction

Newtonian and Lagrangian mechanics are among the major approaches used in classical mechanics to derive equations of motion. While Newtonian mechanics is based on vector quantities, Lagrangian mechanics is a formulation that uses scalar functionals (known as Lagrangians, commonly defined as the difference between kinetic energy and internal energy) and describes motion by means of stationarity of those functionals in a configurational space. Basically, equations of motion in Newtonian form state that the total resultant acting on a constrained system equals the rate of the total linear momentum [1]. It is noteworthy that in such an approach, some difficulties may arise, for instance when constraining forces are applied, or in dealing with boundary conditions. These are circumvented in Lagrangian mechanics, in which generalized coordinates are introduced [2]. However, a Newtonian potential system is a particular case of a Lagrangian system [3] and, indeed, insight into the physics underlying Lagrange mechanics can be profitably given by showing the direct relationship between Lagrangian and Newtonian mechanics, the latter being, in fact, the most intuitive vector formulation used in classical mechanics [4]. However, if, on the one hand, Newton mechanics is so basic, on the other, the existence of a Lagrangian is considered, at least in some circles, a fundamental prerequisite to the point that a certain set of equations not derivable from a Lagrangian would not be acceptable [5,6]. These considerations apply both to particle systems and to continuous bodies. In the latter, the Lagrangian formulation of the equations of motion calls for the definition of a Lagrangian density [7], which is, in general, a function of n generalized coordinates and their spatial gradients, n generalized velocities, and time. The Lagrange equations, thus, become n simultaneous partial differential equations dependent on the generalized coordinates and time [8].

Although it will be not used in the present contribution, for the sake of completeness, it must be emphasized that a further generalization of the Lagrangian mechanics is represented by the Hamiltonian mechanics. With its peculiar structure, it makes it possible to solve mechanical problems that cannot be solved by other approaches [3]. For instance, a relevant problem is represented by a Hamiltonian system subjected to nonlinear constraints [9].

The different but interrelated formulations allow a given system to be studied in sufficient depth and can be selected according to the needs of the investigation [7].

The present contribution deals with balance equations of geometrically exact shearable beams and the conditions that constitutive assumptions must satisfy in order that Newton equations can be derived from the stationarity of a Lagrangian. We will call such equations either variational or potential. Both adjectives need to be motivated.

The choice of the first adjective simply relies on the fact that the extrema of a functional, as the Lagrangian, are obtained by finding functions for which the variational derivative, or functional derivative, is equal to zero, which leads to writing the Euler–Lagrange equations.

The use of the second adjective emphasizes that we limit our attention to generalized forces that are expected to be *stricto sensu* derivable from a potential.

However, a methodological clarification is mandatory. In the general case, equations of motion contain some terms that are derivable from a potential and some others not. This does not imply that Lagrangian machinery can be used in the former case, while not in the latter. In fact, in the case of forces that are not derivable from a potential, it is possible to use Euler–Lagrange equations containing external terms, as indeed originally conceived by Lagrange and Hamilton, or alternatively define a suitable, generalized Lagrangian. In this case, it simply no longer represents the difference between kinetic energy and internal energy, because otherwise it could not include forces that cannot be derived from a potential (a similar argument holds for a generalized Hamiltonian) [7].

It must be further emphasized that geometrically exact beam theory and variational or potential beam theory are not synonyms. If we think to the ‘geometrical exactness’ and to the ‘potentialness’ as two properties of a model, we can have models with only one of them or both. In fact, writing balance equations in a geometrically exact framework and using any constitutive hypothesis does not guarantee that the equations are also potential. *Vice versa*, writing a Lagrangian or an energy functional with any constitutive hypothesis does not guarantee that the resulting variational equations are also geometrically exact.

At first glance, it might seem sufficient to let a deformation energy function depend on the deformation variables used, write the Lagrangian, derive its Euler–Lagrange equations, and simply compare them with the equilibrium equations to be tested. In reality, this is not a good methodological approach, because it does not reveal what general properties the constitutive hypotheses must have and requires case-by-case verification.

What interests us, instead, is answering the question of how to formulate the one-dimensional constitutive hypotheses that relate kinematic and static variables so that the equations of motion are geometrically exact and variational, or, looking at it from another perspective, we seek the relations that the one-dimensional constitutive hypotheses must satisfy so that the stationarity of a Lagrangian functional leads to geometrically exact equations of motion.

The problem of knowing when nonlinear equations can be deduced from the stationarity of a functional and how to find it is analyzed in [10], with some examples given in [11].

Planar beams having a generic initial configuration, under three internal kinematical constraints affecting the plane cross sections, which remain (i) plane, (ii) rigid, and (iii) perpendicular to the beam axis during the deformation process, are considered in [12]. Since no *a priori* restrictions are placed on the magnitude of displacements and rotations, nor are they approximated, the model is a geometrically exact, unshearable planar beam, which is inherently nonlinear. It is addressed as the Euler–Bernoulli beam, in analogy with the linear problem of the classical engineering theory of beams. Indeed, the constraints (i)–(iii) are among the main ingredients of the linear Euler–Bernoulli beam, where small displacements and rotations are considered and the material obeys Hooke's law. In [12], the balance equations are derived by means of the Newton approach before introducing the constitutive assumptions between static and kinematic variables. Then, the equations are particularized to the two cases of the bending moment as a function of the derivative of the cross-sectional rotation with respect to the reference length or the arc length, addressed as the mechanical curvature or the geometrical curvature, respectively. By means of the generalized Helmholtz conditions, the two systems of differential equations have been checked to see if they are variational or not. A major result reported in [12] is that the equations adopting as constitutive assumption a relation between the bending moment and the mechanical curvature can be derived from a Lagrangian, while those using the geometrical curvature cannot.

In the present contribution, we remove constraint (iii), therefore, shearable beams are investigated and call the resulting model, which is a geometrically exact, shearable planar beam, a Timoshenko beam, in analogy to its linear counterpart. Four different sets of balance equations are considered, depending on the mechanical and geometrical curvature, as in [12], and two different choices of the components of the resultant of the internal force.

We must emphasize that, from an intuitive point of view, one would expect to find results that perfectly overlap with those already found in [12]. Indeed, although reasonable, such an expectation is revealed to be wrong.

For the sake of comparison, the organization of the paper follows, in the first part, as much as possible that used in [12]. Therefore, the exact kinematics of the beam is introduced in §2, the Newton balance is reported in §3, the constitutive assumptions are detailed in §4, the Euler–Lagrange equations are given in §5 and Helmholtz conditions are summarized in §6. Then, the conditions for the existence of a Lagrangian in all the cases of interest are shown in §§7 and 8, where indeed the main results of this work are reported. Some extra conditions quite relevant for constitutive assumptions, but not required in mathematical proofs, appear in §9. Details about the relations between the internal energy density and generalized static entities are reported in §10. Some constitutive assumptions available in the literature are examined in §11. Conclusions are illustrated in §12. Finally, in order to make the reading of the paper as smooth as possible, without giving up details that we feel are important to report, many formulas used throughout the developments are collected in the electronic supplementary material [13] as appendices A–E.

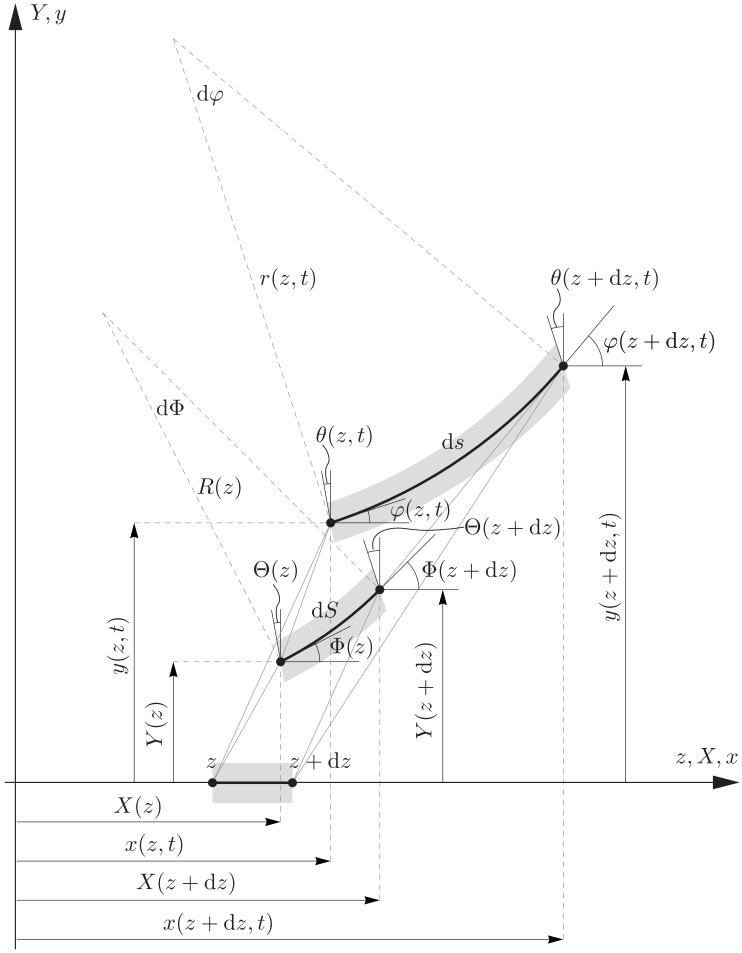

Figure 1. The beam element in its three relevant configurations (reference, initial stress-free and deformed).

2. The exact kinematics and the strain measures

For the scope of the present work, as in director theories of rods [14], the beam is defined by a one-dimensional line, with a cross section attached to each of its points, undergoing a deformation process from an initial to a current configuration, with a transformation map assumed to be biunivocal and continuous. The beam is characterized by a curved shape in both configurations, with the restriction that the axis is a plane curve which remains in the initial plane during the whole deformation process. The cross sections are rigid; thus they do not experience any change in shape during the deformation of the beam, and cross-sectional rotation takes place in the same plane of the axis. Therefore, the plane of the axis coincides with the plane of the motion. For convenience, each cross-sectional centre is equipped with four unit vectors in the plane of motion, namely $\boldsymbol{n}$ tangent to the beam axis, $\boldsymbol{t}$ orthogonal to $\boldsymbol{n}$, $\boldsymbol{\nu}$ normal to the cross section and $\boldsymbol{\tau}$ orthogonal to $\boldsymbol{\nu}$.

The initial and current configurations of the beam are described by means of a generic straight beam element assumed as the reference, having no physical meaning, and any configuration of the beam axis is expressed in a parametric form in a suitable rectangular coordinate system, as shown in figure 1.

We assume that displacements (i.e. changes in placement from the initial to current configuration) are positive if concordant with the reference axes and rotations are positive if oriented counterclockwise.

Let (O, X, Y) and (o, x, y) be two coordinate systems and assume that O is coincident with o $(O \equiv o)$, X and Y are collinear to x and y, respectively. Introducing a space variable z and time t, any point of the beam axis in the initial configuration is marked by the ordered couple $(X(z), Y(z))$ and any point of the beam axis in the current configuration by the ordered couple $(x(z,t), y(z,t))$. The variable z, chosen without loss of generality collinear to X, x axes, is responsible for remapping the actual configurations, both initial and current, to the reference one. We assume that $X(z)$, $x(z,t)$, $Y(z)$ and $y(z,t)$, which we call the abscissas and the ordinates of points in the initial and current configuration, are continuous function of their arguments z and t and are differentiable at least up to the order suitable for the needed developments. Hereafter, for the sake of brevity, once the functional dependence has been defined, it will no longer be written, unless this would lead to a loss of clarity.

It must be emphasized that the exact kinematics of the beam we consider here mostly recall that adopted in [12], where the axial and flexural strains were defined with reference to Euler–Bernoulli beams, whose motion is characterized by the fact that $\boldsymbol{\nu} \equiv \boldsymbol{n}$ and $\boldsymbol{\tau} \equiv \boldsymbol{t}$, due to the unshearability internal constraint. For convenience, the definitions adopted in [12] are briefly recalled here or updated to the current case whenever necessary.

Taken an infinitesimal beam element, let us denote its lengths in the initial and current configurations as $\mathrm{d}S = S'(z)\,\mathrm{d}z$ and $\mathrm{d}s = s'(z,t)\,\mathrm{d}z$, where $\mathrm{d}z$ is the reference length and prime means derivative with respect to z, and let $\Phi = \Phi(z)$ and $\varphi = \varphi(z,t)$ be the angles between $\mathbf{n}$ and the axis x. Therefore, it can be easily recognized from figure 1 that

$$s' = \sqrt{x'^2 + y'^2} \tag{2.1}$$

and

$$\varphi = \arctan\left(\frac{y'}{x'}\right). \tag{2.2}$$

Since in shearable beams $\boldsymbol{\nu} \not\equiv \boldsymbol{n}$ and $\boldsymbol{\tau} \not\equiv \boldsymbol{t}$, in general (i.e. plane cross sections do not stay orthogonal to the current axes during the deformation process), the cross-sectional rotation $\theta = \theta(z,t)$ differs from rotation φ of the shearing angle $\gamma = \gamma(z,t)$. Here and henceforth, it is assumed that

$$\theta = \varphi + \gamma. \tag{2.3}$$

For the sake of formal analogy with equation (2.3), the initial cross-sectional rotation, expressed by $\Theta = \Theta(z)$, is assumed to be the sum of Φ and $\Gamma = \Gamma(z)$, the initial shearing angle that could be interpreted as a shearing prestrain.

The engineering extensional strain $\varepsilon(z,t)$ is defined as the ratio between the relative elongation $\mathrm{d}s - \mathrm{d}S$ and the initial length $\mathrm{d}S$ of the line element as

$$\varepsilon = \frac{\mathrm{d}s}{\mathrm{d}S} - 1 = \frac{s'}{S'} - 1 = \frac{\sqrt{x'^2 + y'^2}}{S'} - 1, \tag{2.4}$$

and changes of curvature $\Delta k_m = \Delta k_m(z,t)$ and $\Delta k_g = \Delta k_g(z,t)$, here and henceforth addressed as *mechanical* and *geometrical*, respectively, are introduced as

$$\Delta k_m = k_m - K_m = \theta' - \Theta' \tag{2.5}$$

and

$$\Delta k_g = k_g - K_g = \frac{\theta'}{s'} - \frac{\Theta'}{S'}, \tag{2.6}$$

where the various curvatures in initial and current configurations are implicitly defined.

From a nomenclature standpoint, the concepts of changes of curvature for both Δk_m and Δk_g must not be confused with the change of geometrical curvature of the beam axis related to Φ and φ as reciprocal or $R = R(z)$ and $r = r(z,t)$, which are the radii of the osculans circles at the initial and current element.

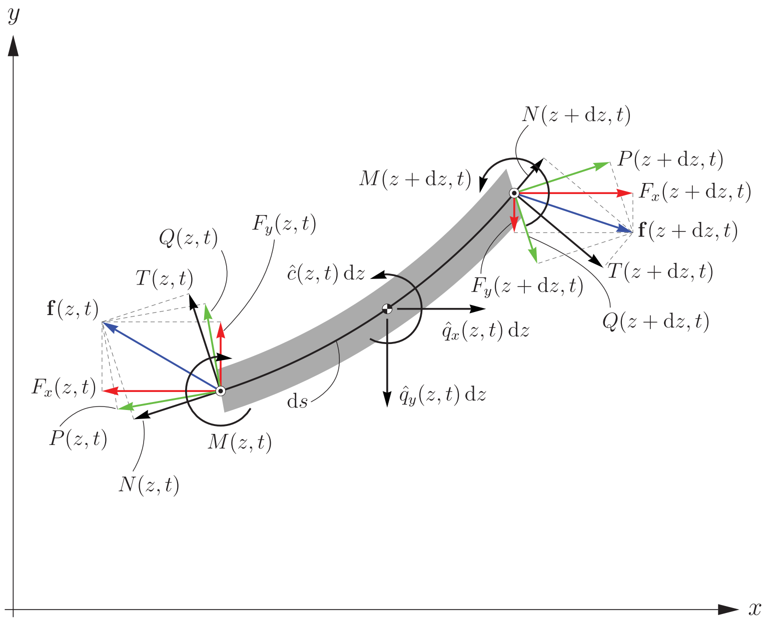

Figure 2. Internal and external forces and moments in the deformed configuration.

The shearing strain is defined as

$$\Delta\gamma = \gamma - \Gamma = \theta - \arctan\left(\frac{y'}{x'}\right) - \Gamma. \tag{2.7}$$

Definitions and equations above allow deriving a number of identities, which reveal to be useful to demonstrate various equalities throughout the paper. For the sake of readability of the work, such identities are not reported here, but in the electronic supplementary materials [13] (appendix A).

3. The Newtonian balance of forces and moments

Let $\boldsymbol{f} = \boldsymbol{f}(z,t)$ and $\boldsymbol{m} = \boldsymbol{m}(z,t)$ be the internal force vector and the internal moment vector. Recalling that the beam element is assumed to deform in the x, y-plane, the out-of-plane component of $\boldsymbol{f}$ and in-plane components of $\boldsymbol{m}$ must vanish.

In the following, $F_x = F_x(z,t)$ and $F_y = F_y(z,t)$ are the components of $\boldsymbol{f}$ along the global reference axes x and y, under the assumption that F_x is positive if oriented as x, F_y is positive if opposite to y; $N = N(z,t)$ and $T = T(z,t)$ are the components of $\boldsymbol{f}$ along the directions $\boldsymbol{n}$ and $\boldsymbol{t}$; $P = P(z,t)$ and $Q = Q(z,t)$ are the components of $\boldsymbol{f}$ along the directions $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$. As shown in figure 2, these are related, being components of the same vector in three different bases. Therefore,

$$F_x = N\,\cos\varphi + T\,\sin\varphi \tag{3.1}$$

and

$$F_y = N\,\sin\varphi - T\,\cos\varphi, \tag{3.2}$$

or similarly

$$F_x = P\,\cos\theta + Q\,\sin\theta \tag{3.3}$$

and

$$F_y = P\,\sin\theta - Q\,\cos\theta. \tag{3.4}$$

From these, by exploiting equations (2.3) and (2.7), it is, therefore, immediate obtaining

$$N = P\cos(\Delta\gamma + \Gamma) + Q\sin(\Delta\gamma + \Gamma) \tag{3.5}$$

and

$$T = Q\cos(\Delta\gamma + \Gamma) - P\sin(\Delta\gamma + \Gamma). \tag{3.6}$$

Since the orthogonal unit vector to the x, y-plane is orthogonal also to $\boldsymbol{n}$ and $\boldsymbol{t}$ and to $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, the only non-vanishing component on $\boldsymbol{m}$ is the same in the three bases. Such a component is the bending moment here denoted as $M = M(z, t)$. It is assumed to be positive if oriented counterclockwise.

To write the balance in the current configuration, external loads and inertial forces and moments must be defined. For the sake of the present work, by virtue of d'Alembert's approach, $\hat{q}_x = \hat{q}_x(z, t)$ and $\hat{q}_y = \hat{q}_y(z, t)$ group together external and inertial terms along x and y and $\hat{c} = \hat{c}(z, t)$ contains external and inertial moments. As for the signs, $\hat{q}_x$, $\hat{q}_y$ and $\hat{c}$ follow the same convention adopted for F_x, F_y and M.

With these ingredients, the balance equations in the directions of the coordinate system take the form

$$F'_x + \hat{q}_x = 0, \tag{3.7}$$

$$F'_y + \hat{q}_y = 0 \tag{3.8}$$

and

$$M' + F_y\, x' - F_x\, y' + \hat{c} = 0. \tag{3.9}$$

The derivation of equations (3.7)–(3.9) not reported here for the sake of conciseness is detailed in the electronic supplementary material [13] (appendix B).

By means of equations (3.1) and (3.2), it is possible to recast equations (3.7)–(3.9) as

$$\left(NS'\frac{\partial\varepsilon}{\partial x'} + T(1+\varepsilon)S'\frac{\partial\Delta\gamma}{\partial x'}\right)' + \hat{q}_x = 0, \tag{3.10}$$

$$\left(N\,S'\frac{\partial\varepsilon}{\partial y'} + T(1+\varepsilon)S'\frac{\partial\Delta\gamma}{\partial y'}\right)' + \hat{q}_y = 0 \tag{3.11}$$

and

$$M' - T(1+\varepsilon)S' + \hat{c} = 0, \tag{3.12}$$

where equations (A.13) and (A.14) in [13] (appendix A) are taken into account. Similarly, equations (3.7)–(3.9) are rewritten in the form

$$(P\,\cos\theta + Q\,\sin\theta)' + \hat{q}_x = 0, \tag{3.13}$$

$$(P\,\sin\theta - Q\,\cos\theta)' + \hat{q}_y = 0 \tag{3.14}$$

and

$$M' - (Py' + Qx')\cos\theta + (Px' - Qy')\sin\theta + \hat{c} = 0, \tag{3.15}$$

using equations (3.3) and (3.4). Furthermore, taking the shearing angle γ from equation (2.3), equation (3.15) reads as

$$M' + \Big(P\sin(\Delta\gamma + \Gamma) - Q\cos(\Delta\gamma + \Gamma)\Big)(1+\varepsilon)S' + \hat{c} = 0, \tag{3.16}$$

which transforms into equation (3.12) if equation (3.6) is taken into account.

It must be emphasized that, in beam theories, it is customary to assign names as *axial* or *normal* force to a component of the resultant force on the cross section ($\boldsymbol{f}$ in figure 2) and *transverse* or *shear* force to the other component. Actually, both N and P can be assumed as axial force, and both T and Q could be called shear force. Indeed, N is the axial force and Q is the shear force according to Engesser, while P is retained as the axial force and Q as the shear force by Haringx (using symbols in figure 2) with arguments for and against each of the two hypotheses. Reissner [15], supporting the approach of Haringx, used P and Q, while Ziegler [16] adopted the approach of Engesser but introduced a change, assuming that the shear force must be T, instead of Q. The different choices are named after various authors. In this regard, fig. 1 in [17] and fig. 1 in [18] are

Table 1. Terms used as axial force and shear force in four beam theories.

theory	axial force	shear force	mutually orthogonal
Engesser	N, tangent to the beam axis	Q, tangent to the cross section	no
Haringx	P, normal to the cross section	Q, tangent to the cross section	yes
Reissner	P, normal to the cross section	Q, tangent to the cross section	yes
Ziegler	N, tangent to the beam axis	T, normal to the beam axis	yes

instructive. For convenience, we summarize these concepts in table 1. Based on these arguments, we find it appropriate to forgo assigning in the following any names to the components of the vector $\boldsymbol{f}$, though we call ε and $\Delta\gamma$ as axial and shearing strain, respectively.

4. The constitutive relations

The arguments shown in §3 allow us to recognize that equations (3.10)–(3.12) and (3.13), (3.14) and (3.16), derived using Newton's approach, are equivalent systems.

However, at this stage, we ignore the form that the constitutive assumptions for the internal forces and the internal moment must take for the balance equations to be variational, in the sense that they can be derived from the stationarity of a suitable Lagrangian. In order not to put any constraint on the possible choices, in the following, we assume that N, T, P, Q and M are all dependent on axial, shearing and flexural strains, namely ε, $\Delta\gamma$ and Δk_m or Δk_g.

Additionally, since without further information it can be possible to argue that, at least in principle, the constitutive relations of N and T and those of P and Q are related to each other in an arbitrary manner, we do not make any further *a priori* assumption on them.

Therefore, we must examine four systems of equations, each characterized by a specific set of constitutive assumptions.

Assuming a local, elastic behaviour, they are abstractly described by

$$N = \overline{N}(\varepsilon, \Delta\gamma, \Delta k_m), \quad T = \overline{T}(\varepsilon, \Delta\gamma, \Delta k_m) \quad \text{and} \quad M = \overline{M}(\varepsilon, \Delta\gamma, \Delta k_m), \tag{4.1}$$

or alternatively

$$N = \widehat{N}(\varepsilon, \Delta\gamma, \Delta k_g), \quad T = \widehat{T}(\varepsilon, \Delta\gamma, \Delta k_g) \quad \text{and} \quad M = \widehat{M}(\varepsilon, \Delta\gamma, \Delta k_g), \tag{4.2}$$

to be inserted in equations (3.10)–(3.12), or

$$P = \overline{P}(\varepsilon, \Delta\gamma, \Delta k_m), \quad Q = \overline{Q}(\varepsilon, \Delta\gamma, \Delta k_m) \quad \text{and} \quad M = \overline{M}(\varepsilon, \Delta\gamma, \Delta k_m), \tag{4.3}$$

or alternatively

$$P = \widehat{P}(\varepsilon, \Delta\gamma, \Delta k_g), \quad Q = \widehat{Q}(\varepsilon, \Delta\gamma, \Delta k_g) \quad \text{and} \quad M = \widehat{M}(\varepsilon, \Delta\gamma, \Delta k_g), \tag{4.4}$$

to be adopted for equations (3.13), (3.14) and (3.16).

It must be noted that, as in [12], constitutive relationships do not directly depend on z and t, that is the attention is restricted to beams whose properties are homogeneous and independent of time. This will play an important role in §9.

An important comment is in order. Inserting the constitutive relations, both in terms of Δk_m or in terms of Δk_g, into equations (3.5) and (3.6), we get

$$N(\varepsilon, \Delta\gamma, \Delta k_{m,g}) = P(\varepsilon, \Delta\gamma, \Delta k_{m,g}) \cos(\Delta\gamma + \Gamma) + Q(\varepsilon, \Delta\gamma, \Delta k_{m,g}) \sin(\Delta\gamma + \Gamma) \tag{4.5}$$

and

$$T(\varepsilon, \Delta\gamma, \Delta k_{m,g}) = Q(\varepsilon, \Delta\gamma, \Delta k_{m,g}) \cos(\Delta\gamma + \Gamma) - P(\varepsilon, \Delta\gamma, \Delta k_{m,g}) \sin(\Delta\gamma + \Gamma). \tag{4.6}$$

We note that if the prestrain Γ is not constant along the beam, i.e. $\Gamma = \Gamma(z)$, then it is not possible that both $\overline{N}$ and $\overline{T}$, from the one side, an $\overline{P}$ and $\overline{Q}$, from the other side, are independent of z for any arbitrary strain. In other words, in the presence of non-constant shear prestrain, the homogeneity of the beam can only be assessed in terms of $\overline{N}$ and $\overline{T}$ or, alternatively, in terms of $\overline{P}$ and $\overline{Q}$. The same conclusion, of course, holds for $\widehat{N}$, $\widehat{T}$, $\widehat{P}$ and $\widehat{Q}$. Thus, the homogeneity hypothesis mentioned above must be associated with the choice of N and T or, alternatively, to P and Q. If not specified differently, in the following, we assume that, when Γ is not constant, the homogeneity holds in terms of N and T.

For the relevance that it will have in the following developments and to further stress the fact that, in the present work, the dependence of kinematic and static entities on z and t is always implicit, we conveniently report here a recap of functional dependencies. To this end, from §2, we recognize

$$\left.\begin{aligned} \varepsilon &= \varepsilon(x', y', S'), \\ \Delta\gamma &= \Delta\gamma(x', y', \theta, \Gamma), \\ \Delta k_m &= \Delta k_m(\theta', \Theta') \\ \text{and} \qquad \Delta k_g &= \Delta k_g(x', y', \theta', S', \Theta'). \end{aligned}\right\} \tag{4.7}$$

However, the variables S' and Θ' only describe the passage from reference to initial configuration, while Γ is the given, and fixed in time, initial shear prestrain. Thus, for the functions of interest, the only relevant dependencies are those on x', y', θ and θ', which are, in fact, the only ones that evolve in time. Therefore, coherently, the mathematical structure of F_x, F_y, N, T, P, Q, M, must be in general depend on the same set of variables, i.e. $\{x', y', \theta, \theta'\}$.

5. Euler–Lagrange equations

As stated above, at this stage, we do not know how to formulate the constitutive assumptions for the internal forces and the internal moment so that the balance equations are equivalent to the stationarity of an appropriate Lagrangian. If it indeed exists, its stationarity is obtained from the Euler–Lagrange equations.

In the case of a beam with properties independent of space and time and having length L, it is assumed that there exists a Lagrangian density $\mathcal{L}$, which depends on the kinematics unknowns and their derivatives as

$$\mathcal{L} = \mathcal{L}(x, y, \theta, x', y', \theta'), \tag{5.1}$$

the Lagrangian functional takes the form

$$\mathbb{L} = \int_0^L \mathcal{L}\,\mathrm{d}z. \tag{5.2}$$

The equations of motion correspond to the stationarity of equation (5.2) obtained through the Euler–Lagrange equations, in the present case taking the form

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial \mathcal{L}}{\partial x'} - \frac{\partial \mathcal{L}}{\partial x} = 0, \tag{5.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0 \tag{5.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial \mathcal{L}}{\partial \theta'} - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \tag{5.5}$$

together with appropriate boundary conditions that, however, are not used in this work and thus are not reported for the sake of conciseness.

Typically, the Lagrangian of a motion is written as the difference of the kinetic and potential energy, although other definitions can be found in the literature in which they are added together, depending on the sign chosen for the potential energy, as pointed out in [19].

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For the problem we are dealing with, the Lagrangian density can be put in the form

$$\mathcal{L} = \mathcal{U} - \mathcal{W}, \tag{5.6}$$

where $\mathcal{U} = \mathcal{U}(x', y', \theta, \theta')$ is the strain energy density and $\mathcal{W} = \mathcal{W}(x, y, \theta)$ is the work of external and inertial forces and moments (we recall that the d'Alembert approach is used) expressed as

$$\mathcal{W} = \hat{q}_x x + \hat{q}_y y + \hat{c}\theta. \tag{5.7}$$

Note that $\mathcal{W}$ is not essential for the developments described in the next sections of this work. On the contrary, $\mathcal{U}$ is of paramount importance and, if it exists, it must depend on the choice of constitutive assumptions, that is on one of the sets of equations (4.1)–(4.4). This also explains why $\mathcal{U}$ is expected to depend only on x', y', θ and θ'.

6. Conditions for the existence of a Lagrangian

The inverse problem of the calculus of variations [20] is to find what conditions must be met for the balance equations to be equal to Euler–Lagrange equations. In such a case, we will say that those equations are potential or variational.

For a system of equations containing second order derivatives of the unknown functions, the necessary and sufficient conditions for the existence of $\mathcal{L}$ were obtained by Helmholtz [21]. After him, they are known as Helmholtz conditions, which in the case we are dealing with can be expressed as [22, pp. 8, 9]

$$\frac{\partial P_i}{\partial p_k''} = \frac{\partial P_k}{\partial p_i''}, \tag{6.1}$$

$$\frac{\partial P_i}{\partial p_k'} + \frac{\partial P_k}{\partial p_i'} = \left(\frac{\partial P_i}{\partial p_k''} + \frac{\partial P_k}{\partial p_i''}\right)' \tag{6.2}$$

and

$$\frac{\partial P_i}{\partial p_k} - \frac{\partial P_k}{\partial p_i} = \frac{1}{2}\left(\frac{\partial P_i}{\partial p_k'} - \frac{\partial P_k}{\partial p_i'}\right)'. \tag{6.3}$$

By setting

$$P_1 := F_x' + \hat{q}_x, \tag{6.4}$$

$$P_2 := F_y' + \hat{q}_y, \tag{6.5}$$

$$P_3 := M' + F_y\, x' - F_x\, y' + \hat{c}, \tag{6.6}$$

and

$$p_1 := x(z, t), \tag{6.7}$$

$$p_2 := y(z, t), \tag{6.8}$$

$$p_3 := \theta(z, t), \tag{6.9}$$

equations (6.1)–(6.3) lead to a set of 27 equations, not necessarily linearly independent.

Before specifying if forces F_x and F_y in equations (6.4)–(6.6) are expressed in terms of N and T, by means of equations (3.1) and (3.2), or in terms of P and Q, using equations (3.3) and (3.4) instead, we manipulate equations (6.1)–(6.3), to simplify as much as possible the forthcoming computations.

For this scope, we introduce the relations

$$\frac{\partial}{\partial \xi}\frac{\mathrm{d}f}{\mathrm{d}z}=\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial f}{\partial \xi}, \tag{6.10}$$

$$\frac{\partial}{\partial \xi'}\frac{\mathrm{d}f}{\mathrm{d}z}=\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial f}{\partial \xi'}+\frac{\partial f}{\partial \xi} \tag{6.11}$$

and

$$\frac{\partial}{\partial \xi''}\frac{\mathrm{d}f}{\mathrm{d}z}=\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial f}{\partial \xi''}+\frac{\partial f}{\partial \xi'}, \tag{6.12}$$

whose proofs are reported in the electronic supplementary material [13] (appendix C). It must be noted that equation (6.10) allows interchanging the partial and total differentiation (cf. [23, pp. 19, 20], or [24, p. 231]) and the derivatives $\partial/\partial\xi$ and $\mathrm{d}/\mathrm{d}z$ commute. On the contrary, the derivatives $\partial/\partial\xi'$ and $\partial/\partial\xi''$ do not commute with $\mathrm{d}/\mathrm{d}z$, as shown by equations (6.11) and (6.12), respectively. Here, ξ and f are two dummy functions and may be used to represent x, y or θ and F_x, F_y or M, respectively.

It is possible to show that operating on equation (6.1) yields (Proposition 1 in [13])

$$\frac{\partial F_x}{\partial y'}=\frac{\partial F_y}{\partial x'}, \tag{6.13}$$

$$\frac{\partial F_x}{\partial \theta'}=\frac{\partial M}{\partial x'} \tag{6.14}$$

and

$$\frac{\partial F_y}{\partial \theta'}=\frac{\partial M}{\partial y'}, \tag{6.15}$$

operating on equation (6.2) yields (Proposition 2 in [13])

$$\frac{\partial M}{\partial \theta}=\frac{\partial M}{\partial x'}y'-\frac{\partial M}{\partial y'}x', \tag{6.16}$$

$$F_y=\frac{\partial F_x}{\partial x'}y'-\frac{\partial F_x}{\partial y'}x'-\frac{\partial F_x}{\partial \theta} \tag{6.17}$$

and

$$F_x=-\frac{\partial F_y}{\partial x'}y'+\frac{\partial F_y}{\partial y'}x'+\frac{\partial F_y}{\partial \theta}, \tag{6.18}$$

and equation (6.3) is automatically satisfied (Proposition 3 in [13]). Therefore, at this stage, we observe that, for the problem under investigation, Helmholtz conditions reduce to this set of six simpler equations.

7. Balance equations in terms of *N* and *T*

Expressing the components of the internal force $\boldsymbol{f}$ as N and T, the balance is given by equations (3.10)–(3.12). The two cases of constitutive assumptions as in equations (4.1) or (4.2) are investigated. In the following, we will show that in both cases equations could be potential.

(a) The case of mechanical curvature

We recall that using equations (4.1) as constitutive assumptions, the components of $\boldsymbol{f}$ are indicated as $\overline{N}$ and $\overline{T}$ and the bending moment is $\overline{M}$. Notably, this leads to considering the mechanical curvature Δk_m as bending strain. It is possible to show (see [13], proposition 4) that in order for

Helmholtz conditions to be satisfied the following relations:

$$\frac{\partial}{\partial \Delta\gamma}\left(S'\overline{N}\right) = \frac{\partial}{\partial \varepsilon}\left(S'(1+\varepsilon)\overline{T}\right), \tag{7.1}$$

$$\frac{\partial}{\partial \Delta k_m}\left(S'\overline{N}\right) = \frac{\partial \overline{M}}{\partial \varepsilon} \tag{7.2}$$

and

$$\frac{\partial}{\partial \Delta k_m}\left(S'(1+\varepsilon)\,\overline{T}\right) = \frac{\partial \overline{M}}{\partial \Delta\gamma}, \tag{7.3}$$

must be true.

An immediate corollary is that a fully decoupled constitutive behaviour, i.e. $\overline{N} = \overline{N}(\varepsilon)$, $\overline{T} = \overline{T}(\Delta\gamma)$ and $\overline{M} = \overline{M}(\Delta k_m)$, does not satisfy equations (7.1)–(7.3). In particular, while equations (7.2) and (7.3) turn out to be two identities, i.e. $0 = 0$, equation (7.1) would be fulfilled only for $T = 0$ for any $\Delta\gamma$, which is a clear inconsistency. Therefore, fully decoupled constitutive assumptions do not lead to potentiality. This is very surprising and unexpected. However, if one assumes $\overline{N} = f_N(\varepsilon)/S'$, $\overline{T} = f_T(\Delta\gamma)/(S'(1+\varepsilon))$ and $\overline{M} = \overline{M}(\Delta k_m)$, then equations (7.1)–(7.3) are satisfied. Note that for the case of small displacements, where $|\varepsilon| \ll 1$ and $S' = 1$, the latter assumption is equivalent to having decoupled constitutive behaviour. In a more speculative way, we observe that if $\overline{M} = \overline{M}(\Delta k_m)$, then $S'\overline{N}$ and $S'(1+\varepsilon)\overline{T}$ cannot depend on Δk_m. When $\overline{T} = f_T(\Delta\gamma)/((1+\varepsilon)S')$, then $S'\overline{N}$ and $\overline{M}$ cannot depend on $\Delta\gamma$. When $\overline{N} = f_N(\varepsilon)/S'$, then $S'(1+\varepsilon)\overline{T}$ and $\overline{M}$ cannot depend on ε.

A special case of partially decoupled constitutive assumptions implying potentiality is given by $\overline{N} = \overline{N}(\varepsilon, \Delta\gamma)$, $\overline{T} = \overline{T}(\varepsilon, \Delta\gamma)$ and $\overline{M} = \overline{M}(\Delta k_m)$. In fact, in such a case equations (7.2) and (7.3) become $0 = 0$ and equation (7.1) remains formally unchanged.

In §11, some cases of constitutive assumptions for $\overline{N}$, $\overline{T}$ and $\overline{M}$ taken from the literature are considered. Both the cases leading to balance equations which are potential or not potential are discussed.

(b) The case of geometrical curvature

We recall that using equations (4.2) as constitutive assumptions, the components of $\boldsymbol{f}$ are indicated as $\widehat{N}$ and $\widehat{T}$ and the bending moment is $\widehat{M}$. Notably, this leads to considering the geometrical curvature Δk_g as bending strain. It is possible to show (see [13], Proposition 5) that in order for Helmholtz conditions to be satisfied the following relations:

$$\frac{\partial}{\partial \Delta\gamma}\left(S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right)\right) = \frac{\partial}{\partial \varepsilon}\left(S'(1+\varepsilon)\,\widehat{T}\right), \tag{7.4}$$

$$\frac{\partial}{\partial \Delta k_g}\left(S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right)\right) = \frac{\partial}{\partial \varepsilon}\left(S'(1+\varepsilon)\,\widehat{M}\right) \tag{7.5}$$

and

$$\frac{\partial}{\partial \Delta k_g}\left(S'(1+\varepsilon)\,\widehat{T}\right) = \frac{\partial}{\partial \Delta\gamma}\left(S'(1+\varepsilon)\,\widehat{M}\right), \tag{7.6}$$

must be true.

An immediate corollary is that a fully decoupled constitutive behaviour, i.e. $\widehat{N} = \widehat{N}(\varepsilon)$, $\widehat{T} = \widehat{T}(\Delta\gamma)$ and $\widehat{M} = \widehat{M}(\Delta k_g)$ does not satisfy equations (7.4)–(7.6), and so does not lead to potentiality, mainly because of the term $\partial\widehat{M}/\partial\Delta k_g$ (cf. equation (E.78) in [13]). In particular, while equation (7.6) becomes a $0 = 0$ identity, equation (7.4) would be fulfilled only if $T = 0$ holds for any $\Delta\gamma$ and equation (7.5) if the derivative of $\widehat{M}$ with respect to Δk_g is null (i.e. $\widehat{M}$ must be constant for any Δk_g). Therefore, fully decoupled constitutive assumptions do not lead to potential equations, similarly to what has already been observed in the case of mechanical curvature in §7(a).

Unlike from the case of mechanical curvature, the partially decoupled constitutive assumptions $\widehat{N} = \widehat{N}(\varepsilon, \Delta\gamma)$, $\widehat{T} = \widehat{T}(\varepsilon, \Delta\gamma)$ and $\widehat{M} = \widehat{M}(\Delta k_g)$ do not imply potentiality. In fact, in such a case equation (7.4) remains formally unchanged and equation (7.6) becomes an identity $(0 = 0)$, but equation (7.5) would be satisfied only for $\widehat{M}$ constant for any Δk_g.

In §11, some cases of constitutive assumptions for $\widehat{N}$, $\widehat{T}$ and $\widehat{M}$ taken from the literature are considered. Both the cases leading to balance equations which are potential or not potential are discussed.

Remark 7.1. It has been shown that the derivatives of the bending moment, whether it is expressed as $\overline{M}$ or $\widehat{M}$, satisfy equation (6.16). Noting that such an equation does not contain components of the internal force $\boldsymbol{f}$, it is satisfied whatever such components are, namely F_x and F_y, or N and T, or P and Q.

8. Balance equations in terms of P and Q

Expressing the components of the internal force $\boldsymbol{f}$ as P and Q, the balance is given by equations (3.13)–(3.15). The two cases of constitutive assumptions as in equations (4.3) or equations (4.4) are investigated. In the following, we will show that in both cases equations could be potential.

(a) The case of mechanical curvature

We recall that using equations (4.3) as constitutive assumptions, the components of $\boldsymbol{f}$ are indicated as $\overline{P}$ and $\overline{Q}$ and the bending moment is $\overline{M}$. Notably, this leads to considering the mechanical curvature Δk_m as bending strain. On introducing four functions $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\varepsilon, \Delta\gamma, \Delta k_m)$, $\overline{\mathcal{B}} = \overline{\mathcal{B}}(\varepsilon, \Delta\gamma, \Delta k_m)$, $\overline{\mathcal{C}} = \overline{\mathcal{C}}(\varepsilon, \Delta\gamma, \Delta k_m)$, $\overline{\mathcal{D}} = \overline{\mathcal{D}}(\varepsilon, \Delta\gamma, \Delta k_m)$, defined as

$$\overline{\mathcal{A}} = \frac{\partial \overline{P}}{\partial \varepsilon} + \frac{\partial \overline{Q}}{\partial \Delta\gamma}\frac{1}{1+\varepsilon}, \tag{8.1}$$

$$\overline{\mathcal{B}} = \frac{\partial \overline{P}}{\partial \Delta\gamma}\frac{1}{1+\varepsilon} - \frac{\partial \overline{Q}}{\partial \varepsilon}, \tag{8.2}$$

$$\overline{\mathcal{C}} = \frac{\partial \overline{M}}{\partial \varepsilon}\frac{\cos\Delta\gamma}{S'} - \frac{\partial \overline{M}}{\partial \Delta\gamma}\frac{\sin\Delta\gamma}{S'(1+\varepsilon)} \tag{8.3}$$

and

$$\overline{\mathcal{D}} = \frac{\partial \overline{M}}{\partial \varepsilon}\frac{\sin\Delta\gamma}{S'} + \frac{\partial \overline{M}}{\partial \Delta\gamma}\frac{\cos\Delta\gamma}{S'(1+\varepsilon)}, \tag{8.4}$$

it is possible to show (see [13], propositions 6 and 7) that in order for Helmholtz conditions to be satisfied the following relations:

$$\left(\overline{\mathcal{A}}\sin\Delta\gamma + \overline{\mathcal{B}}\cos\Delta\gamma\right)\cos\Gamma + \left(\overline{\mathcal{A}}\cos\Delta\gamma - \overline{\mathcal{B}}\sin\Delta\gamma\right)\sin\Gamma = 0, \tag{8.5}$$

$$\frac{\partial \overline{P}}{\partial \Delta k_m} = \overline{\mathcal{C}}\cos\Gamma - \overline{\mathcal{D}}\sin\Gamma \tag{8.6}$$

and

$$\frac{\partial \overline{Q}}{\partial \Delta k_m} = \overline{\mathcal{C}}\sin\Gamma + \overline{\mathcal{D}}\cos\Gamma, \tag{8.7}$$

must be true. It is possible to recognize from equations (8.5)–(8.7) that the derivatives of $\overline{P}$, $\overline{Q}$ and $\overline{M}$ are mutually related. Indeed, equation (8.5) involves the first derivatives of both $\overline{P}$ and $\overline{Q}$ with respect to both ε and $\Delta\gamma$, while equations (8.6) and (8.7) relate the first derivatives of both $\overline{P}$ and $\overline{Q}$ with respect to Δk_m with the first derivatives of $\overline{M}$ with respect to both ε and $\Delta\gamma$.

Similar comments written for the case analyzed in §7(a) apply here. In fact, equations (8.5)–(8.7) are related to equations (7.1)–(7.3). To show this fact, it is sufficient to solve equations (3.5) and (3.6) with respect to $\overline{P}$ and $\overline{Q}$ and insert the results into equations (8.5)–(8.7). Solving equation (8.5) with respect to $\partial\overline{N}/\partial\Delta\gamma$ gives directly equation (7.1), while equations (8.6) and (8.7) solved together with respect to $\partial\overline{N}/\partial\Delta k_m$ and $\partial\overline{T}/\partial\Delta k_m$ allow for obtaining equations (7.2) and (7.3).

In §11, some cases of constitutive assumptions for $\overline{P}$, $\overline{Q}$ and $\overline{M}$ taken from the literature are considered.

(b) The case of geometrical curvature

We recall that using equations (4.4) as constitutive assumptions, the components of f are indicated as $\widehat{P}$ and $\widehat{Q}$ and the bending moment is $\widehat{M}$. Notably, this leads to considering the geometrical curvature Δk_g as bending strain. On introducing four functions $\widehat{\mathcal{A}}=\widehat{\mathcal{A}}(\varepsilon,\Delta\gamma,\Delta k_g)$, $\widehat{\mathcal{B}}=\widehat{\mathcal{B}}(\varepsilon,\Delta\gamma,\Delta k_g)$, $\widehat{\mathcal{C}}=\widehat{\mathcal{C}}(\varepsilon,\Delta\gamma,\Delta k_g)$, $\widehat{\mathcal{D}}=\widehat{\mathcal{D}}(\varepsilon,\Delta\gamma,\Delta k_g)$, defined as

$$\widehat{\mathcal{A}}=\frac{\partial\widehat{P}}{\partial\varepsilon}-\frac{1}{1+\varepsilon}\left(\Delta k_g+\frac{\Theta'}{S'}\right)\frac{\partial\widehat{P}}{\partial\Delta k_g}+\frac{1}{1+\varepsilon}\frac{\partial\widehat{Q}}{\partial\Delta\gamma}, \tag{8.8}$$

$$\widehat{\mathcal{B}}=-\frac{\partial\widehat{Q}}{\partial\varepsilon}+\frac{1}{1+\varepsilon}\left(\Delta k_g+\frac{\Theta'}{S'}\right)\frac{\partial\widehat{Q}}{\partial\Delta k_g}+\frac{1}{1+\varepsilon}\frac{\partial\widehat{P}}{\partial\Delta\gamma}, \tag{8.9}$$

$$\widehat{\mathcal{C}}=\left((1+\varepsilon)\frac{\partial\widehat{M}}{\partial\varepsilon}-\left(\Delta k_g+\frac{\Theta'}{S'}\right)\frac{\partial\widehat{M}}{\partial\Delta k_g}\right)\cos\Delta\gamma-\frac{\partial\widehat{M}}{\partial\Delta\gamma}\sin\Delta\gamma \tag{8.10}$$

and $$\widehat{\mathcal{D}}=\left((1+\varepsilon)\frac{\partial\widehat{M}}{\partial\varepsilon}-\left(\Delta k_g+\frac{\Theta'}{S'}\right)\frac{\partial\widehat{M}}{\partial\Delta k_g}\right)\sin\Delta\gamma+\frac{\partial\widehat{M}}{\partial\Delta\gamma}\cos\Delta\gamma, \tag{8.11}$$

it is possible to show (see [13], propositions 8 and 9) that in order for Helmholtz conditions to be satisfied the following relations:

$$\left(\widehat{\mathcal{A}}\sin\Delta\gamma+\widehat{\mathcal{B}}\cos\Delta\gamma\right)\cos\Gamma+\left(\widehat{\mathcal{A}}\cos\Delta\gamma-\widehat{\mathcal{B}}\sin\Delta\gamma\right)\sin\Gamma=0, \tag{8.12}$$

$$\frac{\partial\widehat{P}}{\partial\Delta k_g}=\widehat{\mathcal{C}}\cos\Gamma-\widehat{\mathcal{D}}\sin\Gamma \tag{8.13}$$

and $$\frac{\partial\widehat{Q}}{\partial\Delta k_g}=\widehat{\mathcal{C}}\sin\Gamma+\widehat{\mathcal{D}}\cos\Gamma, \tag{8.14}$$

must be true. As for the mechanical curvature, also in the geometrical case, the derivatives of $\widehat{P}$, $\widehat{Q}$ and $\widehat{M}$ are mutually related, as it can be seen from equations (8.12)–(8.14) and comments to the case analyzed in §7(b) apply. In fact equations (8.12)–(8.14) can be transformed into equations (7.4)–(7.6). To show this fact, it is sufficient to solve equations (3.5) and (3.6) with respect to $\widehat{P}$ and $\widehat{Q}$ and insert the results into equations (8.12)–(8.14). Solving equation (8.12) with respect to $\partial\widehat{N}/\partial\Delta\gamma$ gives directly equation (E.77) in [13], i.e. a simplified version of equation (7.4), while equations (8.13) and (8.14) solved together with respect to $\partial\widehat{N}/\partial\Delta k_g$ and $\partial\widehat{T}/\partial\Delta k_g$ allow for obtaining equations (E.78) and (E.79), simplified versions of equations (7.5) and (7.6), respectively.

9. Extra conditions

All the constitutive relations introduced in abstract form in §4 do not explicitly depend on z. Since they are related in specified ways by the conditions for the existence of a Lagrangian, these conditions have to not be explicitly independent of z, too.

We start with inspecting equations (7.1)–(7.3) and (8.5)–(8.7), which now we know are related as both consider the mechanical curvature as a bending strain. Since S' appears in equations (7.2), (7.3), (8.3), and (8.4), S' must be independent of z, i.e.

$$S(z)=A_0+A_1 z, \tag{9.1}$$

with A_0 and A_1 ($A_1>0$) constants to be chosen. Such a condition has been exploited in [12], as necessary to satisfy Helmholtz conditions in the case of the Euler–Bernoulli beam with mechanical curvature.

It is worth noting that while A_0 can be freely chosen, A_1 depends on constitutive assumptions. Notice that, since it is not necessary for the development in this work, no explicit definition has been given for S' so far. However, from figure 1 and in analogy with equation (2.1), it can be recognized that

$$S' = \sqrt{X'^2 + Y'^2}, \tag{9.2}$$

which is identical to the expression introduced in [12]. It states that if Y is assigned, in order for equation (9.1) to be satisfied, X must be chosen in the form

$$X(z) = A_2 \mp \int_0^z \sqrt{A_1^2 - Y'(\zeta)^2}\mathrm{d}\zeta, \tag{9.3}$$

where ζ is a dummy variable and A_2 is a further constant to be chosen. It is worth remarking that this is not a mechanical constraint, since the choice of the reference system is free and is not related to the mechanical behaviour. Equation (9.1) affirms that the reference system must be chosen conscientiously.

In addition, in order for equations (8.5)–(8.7) to be independent of z, the shear prestrain Γ must be constant and its value related to the constitutive assumptions chosen.

Let us now consider equations (7.4)–(7.6) and (8.12)–(8.14) that retain the geometrical curvature as the bending strain. From equations (7.4)–(7.6) (or, more directly from their simplified versions, equations (E.77)–(E.79) in [13]) we see that the initial geometrical curvature $K_g = \Theta'/S'$ must be constant. Therefore, with a proper parametrization for S we can force, at least in principle, the ratio Θ'/S' to be independent of z, that is

$$\frac{\Theta'}{S'} = \frac{1}{B_1}, \tag{9.4}$$

B_1 being a constant dependent on constitutive assumptions. It immediately gives

$$S(z) = B_0 + B_1\,\Theta(z) = B_0 + B_1(\Phi(z) + \Gamma(z)), \tag{9.5}$$

where B_0 is a free constant and Φ is defined as

$$\Phi = \arctan\left(\frac{Y'}{X'}\right), \tag{9.6}$$

in analogy with equation (2.2).

Using equation (9.2) and assuming that Y is assigned, X must satisfy

$$\left(X'^2 + Y'^2\right)^{3/2} = B_1\left(\left(X'^2 + Y'^2\right)\Gamma' + X'Y'' - X''Y'\right). \tag{9.7}$$

Unfortunately, equation (9.7) seems not to admit a closed form solution for a general $\Gamma(z)$. However, by virtue of equations (8.12)–(8.14), it can be recognized that, as in the case of mechanical curvature, Γ must be constant and its value related to the constitutive assumptions chosen. Therefore, in the following it is set $\Gamma = \Gamma_0$, Γ_0 a constant, and Γ' vanishes in equation (9.7), which can then be solved with respect to X, getting two solutions in the form

$$X = B_3 \pm \sqrt{B_1^2 - (Y(z) - B_2)^2}. \tag{9.8}$$

Thus, in the case of geometrical curvature and constant shearing prestrain, S can be finally written in the form

$$S(z) = B_0 + B_1\left(\Gamma_0 \pm \arctan\left(\frac{\sqrt{B_1^2 - (Y(z) - B_2)^2}}{Y(z) - B_2}\right)\right). \tag{9.9}$$

It must be noted that equation (9.8) can be rewritten in the form

$$(X - B_3)^2 + (Y - B_2)^2 = B_1^2, \tag{9.10}$$

which is the equation of a circle, thus also providing a meaning to the constants B_1 (radius), B_2 and B_3 (coordinates of the centre).

This fact clearly highlights that, in the case Δk_g is assumed as the bending strain, the initial shape of the beam must be straight or a circular arch, in order to assure the potentiality. Such a condition is indeed peculiar to the shearable beams under consideration, without an equivalent in the case of unshearable (Euler–Bernoulli) beams. In fact, in the latter, constitutive assumptions embedding geometrical curvature are always not variational, as shown in [12].

10. The potential function, or the internal energy

Here, the potential functions are obtained in both the cases of mechanical and geometrical curvature.

(a) Mechanical curvature

It can be recognized that equations (7.1)–(7.3) are the closure conditions of a differential form. Since the domain, in this work the space of deformations, is simply connected, this is equivalent to its exactness, i.e. to the existence of a potential function $\overline{\mathcal{U}}=\overline{\mathcal{U}}(\varepsilon,\Delta\gamma,\Delta k_m)$ such that

$$\frac{\partial\overline{\mathcal{U}}}{\partial\varepsilon}=S'\,\overline{N},\tag{10.1}$$

$$\frac{\partial\overline{\mathcal{U}}}{\partial\Delta\gamma}=S'(1+\varepsilon)\,\overline{T},\tag{10.2}$$

and

$$\frac{\partial\overline{\mathcal{U}}}{\partial\Delta k_m}=\overline{M}.\tag{10.3}$$

It is an easy task to verify that the $\overline{\mathcal{U}}$ in the previous equations is exactly $\mathcal{U}$ appearing in equation (5.6).

It is possible to write the potential function by integrating equations (10.1)–(10.3). Since $\overline{\mathcal{U}}$ is expected to be independent of the order in which equations (10.1)–(10.3) are integrated (one can indeed attack the three equations ordered in six different ways), in the remainder we will follow the order in which they are written. In addition, for formal reasons starred functions $\varepsilon^\star$, $\Delta\gamma^\star$, and $\Delta k_m^\star$, playing the role of dummy variables, will be introduced.

By integrating equation (10.1) with respect to ε, we attain

$$\overline{\mathcal{U}}=\int_0^{\varepsilon}S'\overline{N}\,\mathrm{d}\varepsilon^\star+\bar{u}_1,\tag{10.4}$$

where $\bar{u}_1=\bar{u}_1(\Delta\gamma,\Delta k_m)$ is determined by inserting equations (10.4) into (10.2) and integrating with respect to $\Delta\gamma$. We get

$$\bar{u}_1=\int_0^{\Delta\gamma}S'(1+\varepsilon)\overline{T}\,\mathrm{d}\Delta\gamma^\star-\int_0^{\Delta\gamma}\int_0^{\varepsilon}\frac{\partial(S'\overline{N})}{\partial\Delta\gamma^\star}\,\mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta\gamma^\star+\bar{u}_2,\tag{10.5}$$

with the function $\bar{u}_2=\bar{u}_2(\Delta k_m)$ determined by inserting equations (10.5) into (10.4) and the result in equation (10.3) and integrating with respect to Δk_m. Then, it is attained

$$\begin{aligned}\bar{u}_2=&\int_0^{\Delta k_m}\overline{M}\,\mathrm{d}\Delta k_m^\star-\int_0^{\Delta k_m}\int_0^{\Delta\gamma}\frac{\partial(S'(1+\varepsilon)\,\overline{T})}{\partial\Delta k_m^\star}\,\mathrm{d}\Delta\gamma^\star\,\mathrm{d}\Delta k_m^\star\\&-\int_0^{\Delta k_m}\int_0^{\varepsilon}\frac{\partial(S'\overline{N})}{\partial\Delta k_m^\star}\,\mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta k_m^\star+\int_0^{\Delta k_m}\int_0^{\Delta\gamma}\int_0^{\varepsilon}\frac{\partial^2(S'\overline{N})}{\partial\Delta k_m^\star\partial\Delta\gamma^\star}\,\mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta\gamma^\star\,\mathrm{d}\Delta k_m^\star+\bar{u}_3,\end{aligned}\tag{10.6}$$

where $\bar{u}_3$ is a constant that does not play a role, since it disappears when writing equations (5.3)–(5.5).

By means of equations (7.1)–(7.3) and (10.1)–(10.3), it can be easily verified that the second mixed derivatives of equation (10.4), taking into account equations (10.5) and (10.6), satisfy Schwarz's theorem.

We observe that the potential function $\overline{\mathcal{U}}$ written so far is purely abstract since, at this stage, equations (4.1) are not explicit. For properly chosen constitutive assumptions, $\overline{\mathcal{U}}$ is a strain energy density.

In addition, since equations (7.1)–(7.3) and (8.5)–(8.7) are related, $\overline{\mathcal{U}}$ could also be rewritten to contain $\overline{P}$ and $\overline{Q}$. It would be straightforwardly obtained by inserting equations (3.5) and (3.6) into equations (10.1) and (10.2). However, this is omitted here for the sake of conciseness.

(b) Geometrical curvature

Following the same steps as in the previous case, a potential function in terms of $\widehat{N}$, $\widehat{T}$ and $\widehat{M}$ can be obtained. In fact, it can be recognized that equations (7.4)–(7.6) (cf. their simplified versions, equations (E.77)–(E.79) in [13]) are the closure conditions of a differential form, which is also exact, the domain being simply connected. Therefore, there exists a potential function $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(\varepsilon, \Delta\gamma, \Delta k_g)$ such that

$$\frac{\partial \widehat{\mathcal{U}}}{\partial \varepsilon} = S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right), \tag{10.7}$$

$$\frac{\partial \widehat{\mathcal{U}}}{\partial \Delta\gamma} = S'(1+\varepsilon)\widehat{T} \tag{10.8}$$

and

$$\frac{\partial \widehat{\mathcal{U}}}{\partial \Delta k_g} = S'(1+\varepsilon)\widehat{M}. \tag{10.9}$$

By integrating equation (10.7) with respect to ε, we attain

$$\widehat{\mathcal{U}} = \int_0^{\varepsilon} S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right) \mathrm{d}\varepsilon^\star + \widehat{u}_1, \tag{10.10}$$

with $\widehat{u}_1 = \widehat{u}_1(\Delta\gamma, \Delta k_g)$ determined by inserting equation (10.10) into equation (10.8) and integrating with respect to $\Delta\gamma$ as

$$\widehat{u}_1 = \int_0^{\Delta\gamma} S'(1+\varepsilon)\widehat{T}\,\mathrm{d}\Delta\gamma^\star - \int_0^{\Delta\gamma}\int_0^{\varepsilon} \frac{\partial}{\partial \Delta\gamma^\star}\left(S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right)\right) \mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta\gamma^\star + \widehat{u}_2. \tag{10.11}$$

Then, by inserting equation (10.11) into equation (10.10), the result into equation (10.9) and integrating with respect to Δk_g, the function $\widehat{u}_2 = \widehat{u}_2(\Delta k_g)$ is obtained as

$$\begin{aligned}\widehat{u}_2 = &\int_0^{\Delta k_g} S'(1+\varepsilon)\widehat{M}\,\mathrm{d}\Delta k_g^\star - \int_0^{\Delta k_g}\int_0^{\Delta\gamma} \frac{\partial(S'(1+\varepsilon)\widehat{T})}{\partial \Delta k_g^\star}\,\mathrm{d}\Delta\gamma^\star\,\mathrm{d}\Delta k_g^\star \\ &- \int_0^{\Delta k_g}\int_0^{\varepsilon} \frac{\partial}{\partial \Delta k_g^\star}\left(S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right)\right) \mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta k_g^\star \\ &+ \int_0^{\Delta k_g}\int_0^{\Delta\gamma}\int_0^{\varepsilon} \frac{\partial^2}{\partial \Delta k_g^\star \partial \Delta\gamma^\star}\left(S'\left(\widehat{N} + \left(\Delta k_g + \frac{\Theta'}{S'}\right)\widehat{M}\right)\right) \mathrm{d}\varepsilon^\star\,\mathrm{d}\Delta\gamma^\star\,\mathrm{d}\Delta k_g^\star + \widehat{u}_3,\end{aligned} \tag{10.12}$$

where $\widehat{u}_3$ is a constant playing no role.

Finally, as stated in the case of mechanical curvature, knowing that equations (7.4)–(7.6) and (8.12)–(8.14) are related, it might make sense to rewrite $\widehat{\mathcal{U}}$ to contain $\widehat{P}$ and $\widehat{Q}$, although it is omitted here.

11. Analysis of some constitutive assumptions in the literature

Some constitutive relations available in the literature are reconsidered in light of the results obtained in the present work. It may happen that equations of motion that satisfy the Principle of Virtual Work, and, therefore, are mechanically well-posed, are indeed not potential [25]. Therefore, it is instructive to analyze different possibilities. We consider here the constitutive relations used in [26–29].

It is noteworthy that notations and symbols may differ from paper to paper or the same symbols are used differently. For instance, symbols such as ε and N have different meanings here and in [26]. Indeed, ε is the engineering strain here and Euler–Lagrange strain there; N is the force tangent to the beam axis here, while ε is the axial force divided by the square of the stretch there.

Therefore, for the sake of comparison, nomenclature may need to be adjusted.

We consider [26] first. Since signs depend on axis orientation, we must observe that abscissas have the same orientation here and there, but ordinates here are positive if upward oriented, while there is the opposite. Rotations and internal forces and moments are oriented in the same manner here and there. Paying attention to the meaning of symbols, we recognize that x, $u_{,x}$, $v_{,x}$, γ, κ, $\hat{\kappa}$ and F_a in [26] here stand for z, $x'-1$, $-y'$, $\Delta\gamma+\Gamma$, $\Delta k_m+\Theta'$, $\Delta k_g+\Theta'/S'$ and N, respectively. In addition, since external loads and inertial terms do not play a role in the theoretical developments of the present work, we use the symbols $\hat{q}_x$, $\hat{q}_y$ and $\hat{c}$ without any further specification.

These details are sufficient to rewrite the balance equations derived in [26] (from the principle of virtual work) recognizing that they coincide with equations (3.10)–(3.12).

The constitutive laws are derived in [26] by comparing the internal virtual works written for a one-dimensional beam element and a three-dimensional body. After internal forces and bending moment are written as generalized stress resultants of the second Piola–Kirchhoff stress tensor components over the beam cross section, constitutive laws are specified for a linear hyperelastic Green material as (using current symbols)

$$\begin{aligned}\overline{N} &= K_S\,\lambda\sin^2(\Delta\gamma)+\frac{K_0}{2}\lambda(\lambda^2-1)+\frac{K_1}{2}(3\lambda^2-1)\cos(\Delta\gamma)\,\Delta k_m\\ &\quad+\frac{K_2}{2}\lambda(\cos(2\Delta\gamma)+2)\Delta k_m{}^2+\frac{K_3}{2}\cos(\Delta\gamma)\,\Delta k_m{}^3,\end{aligned}\tag{11.1}$$

$$\begin{aligned}\overline{T} &= K_S\,\lambda\cos(\Delta\gamma)\sin(\Delta\gamma)-\frac{K_1}{2}(\lambda^2-1)\sin(\Delta\gamma)\,\Delta k_m\\ &\quad-K_2\lambda\cos(\Delta\gamma)\sin(\Delta\gamma)\Delta k_m{}^2-\frac{K_3}{2}\sin(\Delta\gamma)\Delta k_m{}^3\end{aligned}\tag{11.2}$$

and

$$\begin{aligned}\overline{M} &= \frac{K_1}{2}\lambda(\lambda^2-1)\cos(\Delta\gamma)+\frac{K_2}{2}(\lambda^2(\cos(2\Delta\gamma)+2)-1)\Delta k_m\\ &\quad+\frac{3}{2}K_3\,\lambda\,\cos(\Delta\gamma)\Delta k_m{}^2+\frac{K_4}{2}\Delta k_m{}^3,\end{aligned}\tag{11.3}$$

or, alternatively, as

$$\begin{aligned}\widehat{N} &= K_S\,\lambda\sin^2(\Delta\gamma)+\frac{K_0}{2}\lambda(\lambda^2-1)+\frac{K_1}{2}\lambda(3\lambda^2-1)\cos(\Delta\gamma)\Delta k_g\\ &\quad+\frac{K_2}{2}\lambda^3(\cos(2\Delta\gamma)+2)\Delta k_g{}^2+\frac{K_3}{2}\lambda^3\cos(\Delta\gamma)\Delta k_g{}^3,\end{aligned}\tag{11.4}$$

$$\begin{aligned}\widehat{T} &= K_S\,\lambda\cos(\Delta\gamma)\sin(\Delta\gamma)-\frac{K_1}{2}\lambda(\lambda^2-1)\sin(\Delta\gamma)\Delta k_g\\ &\quad-\frac{K_2}{2}\lambda^3\sin(2\Delta\gamma)\,\Delta k_g{}^2-\frac{K_3}{2}\lambda^3\sin(\Delta\gamma)\,\Delta k_g{}^3\end{aligned}\tag{11.5}$$

and

$$\begin{aligned}\widehat{M} &= \frac{K_1}{2}\lambda(\lambda^2-1)\cos(\Delta\gamma)+\frac{K_2}{2}\lambda(\lambda^2(\cos(2\Delta\gamma)+2)-1)\Delta k_g\\ &\quad+\frac{3}{2}K_3\lambda^3\cos(\Delta\gamma)\Delta k_g{}^2+\frac{K_4}{2}\lambda^3\Delta k_g{}^3,\end{aligned}\tag{11.6}$$

where $\lambda=S'(1+\varepsilon)$, K_S and K_i, $i=\{0,\ldots,4\}$ are cross-sectional stiffnesses (see [26]) and $\Theta=0$ and $\Gamma=0$ hold since [26] refers to straight beams without shear prestrain. It should be noted that λ has the physical meaning of the stretch, and, therefore, it could be used as strain in constitutive assumptions, although here we exclusively use ε as the axial strain.

The constitutive relations in [26] are written in a coordinate system that does not coincide with the central reference of the inertia of the beam, chosen instead in [30]. The corresponding equations can be directly obtained from equations (11.1)–(11.3), respectively, from equations (11.4)–(11.6), by setting $K_1=K_3=0$. By direct check, it can be verified that equations (11.1)–(11.3) satisfy equations (7.1)–(7.3), and equations (11.4)–(11.6) satisfy equations (7.4)–(7.6). Therefore, in both cases, balance equations are potential.

The energies are obtained using equations (10.4)–(10.6) as

$$\overline{\mathcal{U}} = \frac{K_S}{2}\lambda^2 \sin^2(\Delta\gamma) + \frac{K_0}{8}(\lambda^2 - S'^2)(\lambda^2 + S'^2 - 2) + \frac{K_1}{2}\lambda(\lambda^2 - 1)\cos(\Delta\gamma)\,\Delta k_m \\ + \frac{K_2}{4}(\lambda^2(\cos(2\Delta\gamma) + 2) - 1)\Delta {k_m}^2 + \frac{K_3}{2}\lambda\cos(\Delta\gamma)\Delta {k_m}^3 + \frac{K_4}{8}\Delta {k_m}^4, \tag{11.7}$$

or using equations (10.10)–(10.12) as

$$\widehat{\mathcal{U}} = \frac{K_S}{2}\lambda^2 \sin^2(\Delta\gamma) + \frac{K_0}{8}(\lambda^2 - S'^2)(\lambda^2 + S'^2 - 2) + \frac{K_1}{2}\lambda^2(\lambda^2 - 1)\cos(\Delta\gamma)\,\Delta k_g \\ + \frac{K_2}{4}\lambda^2(\lambda^2(\cos(2\Delta\gamma) + 2) - 1)\Delta {k_g}^2 + \frac{K_3}{2}\lambda^4\cos(\Delta\gamma)\Delta {k_g}^3 + \frac{K_4}{8}\lambda^4\Delta {k_g}^4. \tag{11.8}$$

It is interesting to observe that equations (11.7) and (11.8) show differences only in terms containing the curvatures. In fact, the ratio between terms of the same stiffness in equations (11.7) and (11.8) is $(\Delta k_m/(\lambda\Delta k_g))^p$, i.e. $(\Delta k_m/(S'(1+\varepsilon)\Delta k_g))^p$, with p being the power of curvature appearing in that term. This is in agreement with the fact that strains ε, Δk_m and Δk_g are related by

$$\Delta k_g = \frac{\Delta k_m - \varepsilon\Theta'}{S'(1+\varepsilon)}, \tag{11.9}$$

as can be seen by combining equations (2.4)–(2.6), similar to what is observed in [12].

Constitutive relationships which are linear in ε, $\Delta\gamma$ and Δk_m,

$$\overline{N} = K_0\varepsilon, \quad \overline{T} = K_S\Delta\gamma, \quad \overline{M} = K_2\Delta k_m, \tag{11.10}$$

are adopted in [31]. This kind of constitutive law is often used in simple nonlinear planar rod models with nonlinearity embedded in the kinematics only [32]. They are the first-order Taylor–Maclaurin approximation of equations (11.1)–(11.3) and result *not* in satisfying equations (7.1)–(7.3). It is interesting to observe that in the case of the linear Timoshenko beams, that is, strains in equations (11.10) are small, being small displacement and rotations, it is possible to write potential equations, but of course they will not be geometrically exact. Similarly, the constitutive assumptions of the linear Timoshenko beams, where the static entities are linearly dependent on the strains, independently of the magnitude of displacements and rotations, cannot be used in a geometrically exact setting, if it is also desired that equations of motion can be derived from a Lagrangian. In fact, quite surprisingly and interestingly, neither the approximations of equations (11.1)–(11.3) nor those of equations (11.4)–(11.6) constructed by means of Taylor expansion to the first up to the fourth order in ε, $\Delta\gamma$ and Δk_m, or Δk_g, taking the same order for all the strains, are potential. For instance, the first- and second-order approximations of equations (11.1)–(11.3) do not satisfy equation (7.1), and third- and fourth-order approximations do not satisfy equation (7.1), nor equation (E.79). Although approximations above the fourth order were not checked and other approximations should be checked in a systematic way, it is notable that constitutive assumptions which lead to potential equations of motion lose the potentiality in the Taylor expansions analyzed.

A special case of approximation is considered in [33,34], where the fact that, based on numerical experiences, $\Delta\gamma$ (current nomenclature) is small enough is exploited. Considering a homogeneous straight beam with the cross section held constant along the beam axis, the constitutive assumptions in [33,34] are therefore equations (11.1)–(11.3) with $K_1 = K_3 = 0$ (as those in [30]) expanded in Taylor series at the first order in $\Delta\gamma$. The resulting constitutive assumptions do not pass the potentiality check, since in this specific case only equation (7.2) is satisfied.

Let us now consider terms and equations reported in [27], which refers to straight beams without shear prestrain ($\Theta = 0$ and $\Gamma = 0$). By the meaning of the symbols, we recognize that the axial strain used there is equivalent to the engineering strain used here. Thus, the balance equations there, using symbols introduced in the present work, are recast as

$$(K_0\varepsilon\cos\varphi + K_S\Delta\gamma\sin\varphi)' + \hat{q}_x = 0, \tag{11.11}$$

$$(K_0\varepsilon\sin\varphi - K_S\Delta\gamma\cos\varphi)' + \hat{q}_y = 0 \tag{11.12}$$

and

$$(K_2\Delta k_g)' - K_S\Delta\gamma S'(1+\varepsilon) + \hat{c} = 0. \tag{11.13}$$

These equations, taking into account equations (3.1), (3.2), (3.10)–(3.12) and (4.2), allow us to recognize that in [27], as well as in [35], it is assumed (still using the present symbols) that

$$\widehat{N} = K_0\varepsilon, \quad \widehat{T} = K_S\Delta\gamma, \quad \widehat{M} = K_2\Delta k_g, \tag{11.14}$$

indeed corresponding to the first-order approximation of equations (11.4)–(11.6) and equivalent versions of equations (11.10) when Δk_g is used as bending strain.

As stated above, such an approximation does not pass the potentiality check, since in this specific case only equation (7.5) is satisfied. Therefore, we must conclude that equations (11.11)–(11.13) are not variational, as already reported in [25,36].

Now, we examine [28], where the constitutive assumptions relate strains to the bending moment M and the forces orthogonal and tangent to the cross sections in the current configuration, which are, respectively, P and Q in the present contribution. In [28], a continuum-mechanics interpretation of the Reissner theory [15] is presented, and generalized static entities are resultants of components of the second Piola–Kirchhoff stress tensor. Such results can be directly compared to those in [26], which we have already checked to pass the potentiality test. It must be noted that the orientations of axes, displacements and rotations, forces, and moments are the same in the two papers, although names and symbols differ. Using current symbols, it is possible to recognize that z, Λ_{x0}, χ, φ', N and Q in [28] correspond here to $-y$, $S'(1+\varepsilon)$, $\Delta\gamma + \Gamma$, $\Delta k_m + \Theta'$, $\overline{P}$ and $\overline{Q}$. Due to the different choices of axes, the components of the second Piola–Kirchhoff tensor S_{xx} and S_{xz} in [28] correspond to S_{11} and S_{12} in [26]. For comparison, we take the St.Venant–Kirchhoff material law, which is considered in [28] (along with one for hyperelastic isotropic materials) since it corresponds to the choice in [26] provided that the quantities $\lambda + 2\mu$ and μ in the former are replaced by the elasticities $\mathcal{E}$ and $\mathcal{G}$ introduced in the latter, λ and μ being the two Lamè constants. Then, it is straightforwardly recognized that $\overline{P}$ and $\overline{Q}$ (current symbols) in [28] correspond to equation (3.5) and (3.6), with N and T given by equations (11.1) and (11.2). By virtue of correspondences among symbols, it is also easy to see that M in [28] is the same as M in [26]. All these, together with arguments developed above, allow us to conclude that the constitutive assumptions in [28] pass the potentiality check. It must be emphasized that there are no equivalents to $\widehat{P}$, $\widehat{Q}$ and $\widehat{M}$ in [28]. This fact is indeed in agreement with the assumptions and results in [15], to which also reduces the three-dimensional formulation proposed in [37], when applied to the plane problem.

Finally, we analyze [29], where both the Reissner model [15] and the Ziegler model [16] are considered. Using current symbols, it is possible to recognize that ξ, x_s, z_s, λ_s, χ, φ', N, Q, $\widetilde{N}$, $\widetilde{Q}$ and Q^* in [29] have here the meaning of z, x, $-y$, $S'(1+\varepsilon)$, $\Delta\gamma + \Gamma$, $\Delta k_m + \Theta'$, $\overline{P}$, $\overline{Q}$, $\overline{N}$, $S'(1+\varepsilon)\overline{T}$ and $\overline{T}$. With this information, it can be recognized that the equilibrium equations in [29] correspond to equations (3.13)–(3.16) for the case of the Reissner model, and to equations (3.10)–(3.12) for the case of the Ziegler model, provided that different orientation of axes is taken into account. The constitutive assumptions, using the present symbols and setting $\Theta = 0$ and $\Gamma = 0$ (the considered beams are straight without shear prestrain), are written as

$$\overline{P} = K_0(S'(1+\varepsilon)\cos(\Delta\gamma) - 1), \quad \overline{Q} = K_S S'(1+\varepsilon)\sin(\Delta\gamma), \quad \overline{M} = K_2\Delta k_m, \tag{11.15}$$

for the Reissner model and as

$$\overline{N} = K_0(S'(1+\varepsilon)-1), \quad \overline{T} = K_S\frac{\Delta\gamma}{S'(1+\varepsilon)}, \quad \overline{M} = K_2\Delta k_m, \tag{11.16}$$

for the Ziegler model. It can be easily seen that equations (11.15) satisfy equations (8.12)–(8.14) (by means of equations (8.8)–(8.11)), and equations (11.16) satisfy equations (7.1)–(7.3), and, therefore, both systems of constitutive assumptions are potential.

Interestingly, using $S'(1+\varepsilon)\cos(\Delta\gamma)-1$ and $S'(1+\varepsilon)\sin(\Delta\gamma)$ (respectively, indicated as ε and γ in [29]) as the deformation variables in the Reissner model, the constitutive assumptions are written in a diagonal form. In addition, they are also linear in the strain measures adopted in [29] and the corresponding elastic energy density is quadratic (see [29, p. 24]). However, as shown by equations (11.15) they are neither diagonal in the sense of equations (11.10), nor linear with respect to the strains that we use in the present work.

Remark 11.1. It has already been commented at the end of §§7(a) and (b) that, employing ε, $\Delta\gamma$ and Δk_m or Δk_g as strains, fully decoupled constitutive assumptions cannot be used in geometrically exact balance equations without renouncing to potentiality or, conversely, using diagonal assumptions in a Lagrangian functional would not lead to equations (3.10)–(3.12) that are reproduced below for the reader's convenience. To further enlighten this fact, we add here a straightforward example assuming equations (11.10) as constitutive assumptions. In such a case, the strain energy density, as given by equations (10.4)–(10.6), takes the (indeed quite standard) form

$$\overline{U} = \frac{1}{2}\left(K_0S'\varepsilon^2 + K_SS'(1+\varepsilon)\Delta\gamma^2 + K_2\Delta k_m^2\right), \tag{11.17}$$

which, inserted in equations (5.3)–(5.5), allows

$$\left(\overline{N}S'\frac{\partial\varepsilon}{\partial x'} + \overline{T}S'(1+\varepsilon)\frac{\partial\Delta\gamma}{\partial x'} + \frac{1}{2}\overline{T}\Delta\gamma S'\frac{\partial\varepsilon}{\partial x'}\right)' + \hat{q}_x = 0, \tag{11.18}$$

to be compared with

$$\left(NS'\frac{\partial\varepsilon}{\partial x'} + TS'(1+\varepsilon)\frac{\partial\Delta\gamma}{\partial x'}\right)' + \hat{q}_x = 0, \tag{3.10}$$

$$\left(\overline{N}\,S'\frac{\partial\varepsilon}{\partial y'} + \overline{T}S'(1+\varepsilon)\frac{\partial\Delta\gamma}{\partial y'} + \frac{1}{2}\overline{T}\Delta\gamma S'\frac{\partial\varepsilon}{\partial y'}\right)' + \hat{q}_y = 0, \tag{11.19}$$

to be compared with

$$\left(N\,S'\frac{\partial\varepsilon}{\partial y'} + TS'(1+\varepsilon)\frac{\partial\Delta\gamma}{\partial y'}\right)' + \hat{q}_y = 0, \tag{3.11}$$

and

$$\overline{M}' - S'(1+\varepsilon)\,\overline{T} + \hat{c} = 0, \tag{11.20}$$

to be compared with

$$M' - S'(1+\varepsilon)\,T + \hat{c} = 0. \tag{3.12}$$

It is immediately recognized that while equation (11.20) is equivalent to equation (3.12), the same is not true for equations (11.18) and (11.19) when compared to equations (3.10) and (3.11) due to the additional terms appearing in the former.

12. Concluding remarks

In this paper, equations of motion of a geometrically exact shearable beam are considered. The initial shape of the beam axis is generic, while the cross section is constant along the length of the beam. The only restriction on the material is that it is homogeneous. From a model standpoint, since the time variable is of minor concern for the theoretical aspects the paper is focused on, inertial terms are gathered with external load by virtue of the d'Alembert approach. Balance is written by using both Newton and Lagrangian formulations of mechanics. Four different sets of balance equations are taken into account. They rely on two different choices of mutually orthogonal components of the resultant of the internal force (with a component tangent to the deformed axis or normal to the rotated cross section) and two curvatures (the mechanical and geometrical ones). After testing Newton equations by means of Helmholtz conditions, the relations that the constitutive assumptions of generalized static entities must satisfy for the potentialness of equations are obtained.

The main results are:

- all the four sets of equations can be written to be potential;
- chosen a curvature, whatever mechanical or geometrical, the two sets of equations which can be written as a function of the internal force components are equivalent to each other;
- relations between strain energy densities and generalized static entities are expressed;
- constitutive assumptions, which lead to variational equations when used, if approximated through certain Taylor expansions lose such a property;
- totally decoupled constitutive assumptions does not pass the potentiality check, and thus the Lagrangian approach does not lead to the same geometrically exact equations of motions written through the Newton approach;
- by considering the geometrical curvature as bending strain, the beam could be variational only if it is initially circular or straight; and
- the parametrization with respect to the reference configuration cannot be chosen freely to have potentiality.

Other ancillary results are also described in the paper.

We emphasize that the first result was unexpected since, in the case of unshearable beams, the equations written using mechanical curvature are variational, while that is not true for those written taking into account the geometrical curvature. Understanding how potentiality is lost by passing from shearable beams, investigated in this paper, to unshearable beam, investigated in [12] and being always not potential in the case of geometrical curvature, is one of the challenges for future works.

In our opinion, the findings of this work are particularly relevant from a theoretical point of view and can also have interesting consequences from a practical standpoint. Nevertheless, we want to stress that we are not claiming that variational, geometrically exact equations are the only possible system to be used in studying statics or dynamics of beams. Indeed, various approximations can be profitably used to illuminate some aspects relevant from a modelling point of view, leaving in the background aspects that can be of minor concern for the case study on which the focus is centred.

Data accessibility. The data are provided in the electronic supplementary material [13].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. E.B.: conceptualization, formal analysis, investigation, methodology, writing—original draft, writing—review and editing; S.L.: conceptualization, formal analysis, investigation, methodology, writing—original draft, writing—review and editing.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein. The authors contributed in equal parts to the paper.

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