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Abstract. We consider the following nonlinear parabolic equation

$$(\mathcal{F}(v))_x + (\mathcal{G}(v))_\tau = (\mathcal{D}(v))_{xx} + \rho(v), \quad v \in [\alpha, \beta]$$

where $\mathcal{F}$, $\mathcal{G}$ are generic C^1-functions in $[\alpha, \beta]$, $\mathcal{D} \in C^1[\alpha, \beta] \cap C^2(\alpha, \beta)$ is positive inside (α, β) (possibily vanishing at the extreme points), and finally ρ is a monostable reaction term.

We investigate the existence and the properties of travelling wave solutions for such an equation and provide their classification between classical and sharp solutions, together with an estimate of the minimal wave speed.

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1 Introduction

The existence and the properties of travelling wave solutions (t.w.s.'s for short), that is, solutions of the type $v(\tau, x) = u(x - c\tau)$, where c represents the wave speed, for reaction-diffusion equations have been deeply investigated and a very wide literature concerns this matter. Since the simplest Fisher-KPP model

$$v_\tau = v_{xx} + \rho(v), \quad v(\tau, x) \in [0, 1]$$

where typically the reaction term ρ may be of monostable or bistable type, more general models have been introduced, both involving convective processes and considering a density-dependent diffusion coefficient, till to obtain reaction-diffusion-convection equations of the type

$$v_\tau + \mathcal{F}(v)_x = (\mathcal{D}(v))_{xx} + \rho(v), \quad v(\tau, x) \in [0, 1] \tag{1}$$

where the convective term $\mathcal{F}$ is a generic C^1-function, whereas the diffusion term $\mathcal{D} \in C^1([0, 1]) \cap C^2((0, 1))$ is generally assumed to be strictly increasing inside $(0, 1)$, with possibly null derivative at the equilibria $0, 1$.

Many papers have been devoted to the study of t.w.s.'s for equation (1). One of the main reasons of the interest concerning t.w.s.'s lies in the fact that the solution of initial-boundary value problem for the parabolic equation converges for large times, in some sense, to a profile of a t.w.s (see [3], [12], [11], [17], [22]).

In the monostable case (that is for $\rho(v) > 0$ in $(0, 1)$) we can summarize the known results saying that there exists a threshold wave speed c^* such that (1) admits travelling fronts if and only if $c \geq c^*$. The t.w.s.'s having speed $c > c^*$ are defined and continuously differentiable on the whole real line. Instead, the special t.w.s. with speed c^* is smooth on the whole real line when the slope of $\mathcal{D}$ does not vanish at the equilibria. Otherwise, if the derivative of $\mathcal{D}$ is zero at the first equilibrium 0 (degenerate case) or at both the equilibria $0, 1$ (doubly degenerate case), then the t.w.s. having speed c^* can reach one/both the equilibria in a finite time, with a non-zero slope (*sharp* solutions) and the dynamic is said to exhibit the phenomenon of *finite speed of propagation* and/or *finite speed of saturation* (see [1], [3], [10], [14], [15], [16], [18], [20], [25], [26]). Moreover, estimates for the threshold speed are available (see [2], [5], [13], [22], [23]) and also results on continuous dependence for the threshold wave speed and the wave profile have been obtained (see [1], [21]).

More recently, in order to model diffusion-concentration phenomena, models with non-monotone diffusion terms have been considered (see [7], [19]).

The present paper deals with the following highly degenerate fully nonlinear reaction-diffusion-convection equation

$$(\mathcal{F}(v))_x + (\mathcal{G}(v))_\tau = (\mathcal{D}(v))_{xx} + \rho(v), \quad v(\tau, x) \in [\alpha, \beta] \tag{2}$$

where we introduce also a non-constant accumulation term $\mathcal{G}$, which is only assumed to be a C^1-function not necessarily increasing. It appears, for instance, in thermal processes when the heat capacity of the medium depends on temperature and in the theory of filtration of a fluid in a porous media (see [4]).

Equation (2) can be rewritten as

$$f(v)v_x + g(v)v_\tau = (D(v)v_x)_x + \rho(v), \quad v(\tau, x) \in [\alpha, \beta] \tag{3}$$

where f, g, D respectively are the derivative of $\mathcal{F}, \mathcal{G}, \mathcal{D}$. All the terms appearing in (3) are generic continuous functions in $[\alpha, \beta]$, whereas D is also assumed to be continuously differentiable in (α, β).

We consider the monostable case, that is the reaction term $\rho \in C[\alpha, \beta]$ satisfies

$$\rho(u) > 0 \quad \text{for } u \in (\alpha, \beta), \quad \rho(\alpha) = \rho(\beta) = 0. \tag{4}$$

The diffusivity D is quite general, since we only assume that

$$D(u) > 0 \quad \text{for every } u \in (\alpha, \beta). \tag{5}$$

No assumptions are made about the sign at the equilibria α and β, where D can be positive or null, covering in this way the degenerate or doubly-degenerate equations. Finally, as for the accumulation term g, we underline that it may have changes of sign, so that equation (3) can present another type of degeneracy, when g vanishes.

We investigate the existence and the properties of t.w.s.'s connecting the equilibria α and β. In particular we focus on conditions for the existence and the non-existence of t.w.s. in terms of the wave speed c and the structure of the set of admissible wave speeds. According to our knowledge, for this general equations the study of t.w.s.'s is new. Throughout the article we will use the symbol $\fint$ to indicate the integral mean value.

Our main result about existence and non-existence of t.w.s.'s is the following.

Theorem 1 *Assume that the function $u \mapsto D(u)\rho(u)$ is differentiable at α. If*

$$\inf_{u\in(\alpha,\beta)} ⨍_\alpha^u (cg(s)-f(s))\mathrm{d}s > 2\sqrt{\sup_{u\in(\alpha,\beta)} ⨍_\alpha^u \frac{D(s)\rho(s)}{s-\alpha}\mathrm{d}s}, \tag{6}$$

then there exists a t.w.s. having speed c. Moreover, the t.w.s. is unique, up to shift.

Instead, if

$$cg(\alpha)-f(\alpha) < 2\sqrt{\frac{\mathrm{d(D}\rho)}{\mathrm{d}u}(\alpha)}, \tag{7}$$

then no t.w.s. exists with speed c.

As for the structure of the set

$$\Gamma := \{c \ : \ \text{there exists a t.w.s. with speed } c\},$$

of the admissible wave speeds, we prove the following result.

Theorem 2 *Assume that the function $u \mapsto D(u)\rho(u)$ is differentiable at α. If $g(\alpha) > 0$, then Γ is nonempty and there exists a minimal wave speed $c^* := \min\Gamma$, satisfying*

$$\inf_{u\in(\alpha,\beta)} ⨍_\alpha^u (c^*g(s)-f(s))\mathrm{d}s \le 2\sqrt{\sup_{u\in(\alpha,\beta)} ⨍_\alpha^u \frac{D(s)\rho(s)}{s-\alpha}} \tag{8}$$

and

$$c^*g(\alpha)-f(\alpha) \ge 2\sqrt{\frac{\mathrm{d(D}\rho)}{\mathrm{d}u}(\alpha)}. \tag{9}$$

Moreover, if $\int_\alpha^u g(s)\ \mathrm{d}s \ge 0$ for every $u \in (\alpha,\beta)$, then $\Gamma = [c^*,+\infty)$.

Instead, if $g(\alpha) < 0$, then Γ is nonempty and there exists a maximal wave speed $c^{**} :=$ $\max\Gamma$, satisfying (9)-(8) *(with c^* replaced by c^{**}). Moreover, if $\int_\alpha^u g(s)\ \mathrm{d}s \le 0$ for every $u \in (\alpha,\beta)$, then $\Gamma = (-\infty, c^{**}]$.*

As we observe in Remark 3, in the particular case $g \equiv 1$, that is for the classical reaction-diffusion-convection equations, we obtain, as particular cases, the known results and inequalities (9)-(8) reduce to the well-known esimates for the minimal wave speed. Note also that the case $g(\alpha) = 0$ is not covered; as we will explain in more detail in Remark 4, this particular case corresponds to a rather complicated situation, which deserves an in-depth study.

Finally, we also provide a classification of the t.w.s.'s, between smooth or sharp, according to the values of c, g, f, D at the equilibria α, β (see Section 4: Theorem 16 and the following tables.).

As regards the technique we adopt, notice that t.w.s.'s for equation (2) are solutions of the following second-order (possibly singular) equation

$$(D(u)u')' + (cg(u)-f(u))u' + \rho(u) = 0, \tag{10}$$

where $'$ stands for derivative with respect to the wave variable $t = x - c\tau$. The boundary conditions are $u(-\infty) = \beta$ and $u(+\infty) = \alpha$.

So, we first prove that the existence of t.w.s.'s connecting the equilibria α and β is equivalent to the solvability of the following singular first order boundary value problem

$$\begin{cases} \dot{z} = f(u) - cg(u) - \dfrac{D(u)\rho(u)}{z} \\ z(\alpha^+) = z(\beta^-) = 0 \\ z(u) < 0 \text{ in } (\alpha, \beta) \end{cases} \tag{11}$$

where $z(\alpha^+) = \lim_{u\to\alpha^+} z(u)$, $z(\beta^-) = \lim_{u\to\beta^-} z(u)$, and then we study the existence and uniqueness of solutions of (11), according to the value of the constant c. With respect to the known results available in the literature in the special case $g \equiv 1$, here the possible changes of sign of the function g cause the loss of some crucial properties of the solutions of (11), in particular we have no monotonicity of the solutions with respect to c, which was a key tool for the proof of various results in those papers. Indeed, in the present general context, the set Γ of the admissible wave speeds is not a halfline in general, unless the integral function of g is non-negative. For this reason, we had to introduce new techiques in the study of the singular boundary value problem (11).

The paper is organized as follows: in Section 2 we prove some properties of the t.w.s.'s and the equivalence to problem (11). Then, Section 3 is entirely devoted to the study of the solvability of problem (11). Finally, in Section 4, we provide the classification of the t.w.s.

2 Preliminary results

We are interested in studying decreasing t.w.s.'s connecting the equilibria α and β. We will search for heteroclinic solutions of equation (10) in the following sense.

Definition 1. A travelling wave solution (t.w.s.) of (3) is a function $u \in C^1(a,b)$, with $(a,b) \subseteq \mathbb{R}$, such that $u(t) \in [\alpha,\beta]$ and $D(u)u' \in C^1(a,b)$, satisfying equation (10) in (a,b) and the following boundary conditions:

$$u(a^+) := \lim_{t\to a^+} u(t) = \beta, \quad u(b^-) := \lim_{t\to b^-} u(t) = \alpha \tag{12}$$

$$\lim_{t\to a^+} D\,(u(t))\,u'(t) = \lim_{t\to b^-} D\,(u(t))\,u'(t) = 0. \tag{13}$$

Condition (13) added to the classical boundary condition (12) is motivated by the possible occurrence of sharp solutions, that is solutions reaching the equilibria at a finite time (in this case one or both the extrema a, b are finite). However, when the existence interval is the whole real line, then condition (13) is automatically satisfied and then the previous definition reduces to the classical one, as the following result states.

Proposition 3 *Let u be a solution to* (10)*, for some $c \in \mathbb{R}$, satisfying* (12)*.*

Then, if $a = -\infty$ we have $\lim_{t\to-\infty} D\,(u(t))\,u'(t) = 0$. Similarly, if $b = +\infty$ we have $\lim_{t\to+\infty} D\,(u(t))\,u'(t) = 0$.

Proof. Assume that $b = +\infty$. Integrating equation (10) in $[0,t]$ we obtain

$$D\,(u(t))\,u'(t) = D\,(u(0))\,u'(0) - c\,(g(u(t)) - g(u(0))) + f\,(u(t)) - f\,(u(0)) - \int_0^t \rho(u(s))\;\mathrm{d}s. \tag{14}$$

Notice that the limit $\lim_{t\to+\infty}\int_0^t \rho(u(s))\mathrm{d}s = \int_0^{+\infty}\rho(u(s))\mathrm{d}s$ exists (finite or $+\infty$), since ρ is positive. So, by (14) we infer that there exists also the limit $\lambda := \lim_{t\to+\infty} D\left(u(t)\right)u'(t)$, finite or $-\infty$. Therefore, since D is bounded, if $\lambda \neq 0$ then there exists (finite or not) also the limit

$$\lim_{t\to+\infty} u'(t) = \lim_{t\to+\infty}\frac{D\left(u(t)\right)u'(t)}{D\left(u(t)\right)} < 0$$

in contradiction with the boundedness of u. Hence, necessarily $\lambda = 0$.
The proof concerning the limit as $t \to -\infty$ is analogous.

□

Remark 1 In [8] the Authors considered a weak form of definition of solutions (see [8, Definition 2.1]). Hovever, in a context of regularity of the terms involved in the equation, such a definition is equivalent to the present one (see [8, Theorem 2.10] and [9, Proposition 4.1]).

From now on, for a given solution u, let (a, b) denote the smallest interval (possibly the whole real line) such that $u(t) = \beta$ for every $t \le a$ and $u(t) = \alpha$ for every $t \ge b$.

The following result concerns the monotonicity of the solutions we are looking for and a necessary condition for their existence.

Proposition 4 *If u is a solution of* (10) *satisfying* (12)- (13)*, then $u'(t) < 0$ for every $t \in (a, b)$ and*

$$\int_\alpha^\beta \left[cg(s) - f(s)\right]\,\mathrm{d}s > 0. \tag{15}$$

Proof. Integrating equation (10) in (a, b) we have

$$\int_a^b \left(D\left(u(t)\right)u'(t)\right)'\,\mathrm{d}t + \int_a^b \left[cg(u(t)) - f(u)\right]u'(t)\,\mathrm{d}t + \int_a^b \rho(u(t))\,\mathrm{d}t = 0$$

The first integral is null by (13), so

$$0 < \int_a^b \rho(u(t))\,\mathrm{d}t = -\int_a^b \left[cg(u(t)) - f(u)\right]u'(t)\,\mathrm{d}t = \int_\alpha^\beta \left[cg(s) - f(s)\right]\,\mathrm{d}s$$

and (15) holds.

We now split the proof into some steps.

Claim 1: we have $\alpha < u(t) < \beta$ for every $t \in (a, b)$.
Indeed, if $u(t_0) = \alpha$ for some $t_0 \in (a, b)$, then $u'(t_0) = 0$ and integrating equation (10) in $[t_0, b)$ we get

$$\int_{t_0}^b (D(u(t))\,u'(t))'\,\mathrm{d}t + \int_{t_0}^b \left[cg(u(t)) - f(u(t))\right]u'(t)\,\mathrm{d}t + \int_{t_0}^b \rho(u(t))\,\mathrm{d}t = 0.$$

The first integral is null by (13), the second integral is null by (12) and so the last integral must be null and this is a contradiction, by (4). The proof that $u(t) < \beta$ for every $t \in (a, b)$ is analogous.

Claim 2: $u'(t) \neq 0$, $\forall t \in (a, b)$.

Indeed, assume by contradiction that $u'(t_0) = 0$ for some $t_0 \in (a, b)$. Since $\alpha < u(t) < \beta$, we have $D(u(t_0)) > 0$. From equation (10) we get

$$(D(u(t))\, u'(t))'_{|t=t_0} = -\rho(u(t_0)) < 0$$

so the function $(D \circ u)u'$ is strictly decreasing in a neighborhood of t_0 and it vanishes at t_0. Taking account of the sign of the function D we deduce that t_0 is a proper local maximum point for the function u. Since $u(a^-) = \beta$ and $\alpha < u(t) < \beta$, $\forall t \in (a, b)$, there exists a local minimum point $t^* < t_0$, with $t^* > a$, so $\alpha < u(t^*) < \beta$. Repeating the same consideration just made for t_0, we achieve the contradiction that t^* is a proper maximum point for the function u.

Claim 3: $u'(t) < 0$ for every $t \in (a, b)$.

Indeed, assume by contradiction $u'(t_0) > 0$ for some $t_0 \in (a, b)$ (we recall that, from the previous Claim, we have $u'(t_0) \neq 0$). Let (t_1, t_2) be the largest interval containing t_0 such that $u'(t) > 0$ for every $t \in (t_1, t_2)$. By Claim 1 we have that $t_1 = a$ and $t_2 = b$, in contradiction with (12).

□

The monotonicity we have proved in the previous proposition allows us to associate a singular first order equation to equation (3), as stated in the following result.

Theorem 5 *Equation* (3) *with boundary conditions* (12) - (13) *is equivalent to the following singular first-order boundary value problem* (11) *in the following sense: if u is a solution of* (3) *in the interval (a, b), satisfying* (12) *and* (13)*, then the function $z(u) := D(u)u'(t(u))$ is a solution of problem* (11)*, where $u \mapsto t(u)$ denotes the inverse function of u, defined in (α, β). Vice versa, if z is a solution of* (11)*, then the (unique) solution of Cauchy problem*

$$\begin{cases} u' = \dfrac{z(u)}{D(u)} \\ u(0) = \frac{1}{2}(\alpha + \beta) \end{cases} \tag{16}$$

is a solution of (3) *in its maximal existence interval (a, b), satisfying* (12) *and* (13).

Proof. Assume that $u(t)$ is a t.w.s. of equation (10) and let $t(u) : (\alpha, \beta) \to (a, b)$ be the inverse function, whose existence is ensured by Proposition 4. Define $z(u) := D(u)u'(t(u))$ for $u \in (\alpha, \beta)$. Then $z \in C^1(a, b)$ with

$$\dot{z}(u) = \frac{dz}{du} = \frac{(D(u)\, u')'}{u'(t)} = \left(D(u)\, u'\right)' \frac{D(u)}{z(u)} = -cg(u) + f(u) - \frac{D(u)\, \rho(u)}{z(u)}$$

for every $u \in (\alpha, \beta)$. Now, since $u'(t) < 0$, for every $t \in (a, b)$ we get that $z(u) < 0$ for $u \in (\alpha, \beta)$ and, by (13), we get $z(\alpha^+) = z(\beta^-) = 0$ and the proof of the necessary part is concluded.

Now assume that there exists a solution $z \in C^1(\alpha, \beta)$ of (11) and consider the unique solution of the Cauchy problem (16), defined in its maximal interval of existence (a, b) (in which we have $\alpha < u(t) < \beta$). Clearly, $u \in C^1(a, b)$ and $D(u)u' = z(u)$ for every $u \in (\alpha, \beta)$, so also $D(u)u' \in C^1(a, b)$. Furthermore, we have that u is a strictly decreasing function in (a, b) with $u(a^+) = \beta$. Indeed, if $u(a^+) < \beta$, then

$$\lim_{t \to a^+} u'(t) = \lim_{u \to u(a^+)} \frac{z(u)}{D(u)} < 0$$

in contradiction with the boundedness of u. By using the same argument, one can show that $u(b^-) = \alpha$. Now, note that

$$\begin{aligned}\left(D\left(u(t)\right)u'(t)\right)' &= \dot{z}\left(u(t)\right)u'(t) \\ &= -\left(cg(u(t)) - f(u(t))\right)u'(t) - \frac{D\left(u(t)\right)\rho\left(u(t)\right)}{z\left(u(t)\right)}u'(t) \\ &= -\left(cg(u(t)) - f(u(t))\right)u'(t) - \rho\left(u(t)\right)\end{aligned}$$

for every $t \in (a, b)$, so u is a solution of equation (3).

Finally, observe that

$$\lim_{t\to a_+} D\left(u(t)\right)u'(t) = \lim_{t\to a^+} z(u(t)) = z\left(\beta^-\right) = 0$$

and similarly

$$\lim_{t\to b_-} D\left(u(t)\right)u'(t) = \lim_{t\to b^-} z(u(t)) = z\left(\alpha^+\right) = 0$$

□

3 First order singular problem: technical lemmas

In view of the equivalence proved in Theorem 5, in this section we investigate the singular problem (11). More in general, we now consider the following differential equation

$$\dot{z} = f(u) - cg(u) - \frac{h(u)}{z} \tag{17}$$

and the related singular boundary value problem

$$\begin{cases}\dot{z} = f(u) - cg(u) - \dfrac{h(u)}{z} \\ z(\alpha^+) = z(\beta^-) = 0 \\ z(u) < 0 \text{ in } (\alpha, \beta)\end{cases} \tag{18}$$

where $f, g, h : [\alpha, \beta] \to \mathbb{R}$ are continuous functions, and the function h is assumed to satisfy

$$h(\alpha) = h(\beta) = 0, \quad h(u) > 0 \;\text{ in } (\alpha, \beta). \tag{19}$$

The next technical lemmas provide some properties of the solutions of equation (17). They essentially consist on the structure of the solution $z(u)$ in the extremes, on suitable upper bounds for $z(u)$ and about some properties of a sequence of solutions of (17). These results will allow us to prove the existence / non-existence of the solutions via upper-lower solutions arguments.

Lemma 6 *Let z be a negative solution of equation* (17)*, defined in its maximal existence interval $(\alpha', \beta') \subset (\alpha, \beta)$. Then, $\alpha' = \alpha$ and there exist, finite, the limits $z(\alpha^+)$ and $z(\beta'^-)$. Moreover, if h is differentiable at α, then z is differentiable at α and $\dot{z}(\alpha)$ is a root of the trinomial $p(t) := t^2 + (cg(\alpha) - f(\alpha))t + \dot{h}(\alpha)$. Similarly, if h is differentiable at β', then z is differentiable at β' and $\dot{z}(\beta')$ is a root of the trinomial $q(t) := t^2 + (cg(\beta') - f(\beta'))t + \dot{h}(\beta')$.*

Proof. Since $\dot{z}(u) > f(u) - cg(u)$, we deduce that $\dot{z}$ is bounded from below, so there exist both the limits $z(\alpha'^{+}) := \lim_{u\to\alpha'^{+}} z(u)$ and $z(\beta'^{-}) := \lim_{u\to\beta'^{-}} z(u)$. If $z(\alpha'^{+}) = -\infty$, then by equation (17) we get $\lim\limits_{u\to\alpha'^{+}} \dot{z}(u) = f(\alpha') - cg(\alpha') \in \mathbb{R}$, a contradiction. Similarly, one can show that $z(\beta'^{-}) > -\infty$.

Since (α', β') is the maximal existence interval of z, necessarily we have $z(\alpha'^{+}) = 0$. If $\alpha^{+} > \alpha$, then $\lim\limits_{u\to\alpha'^{+}} (f(u) - cg(u))\, z(u) - h(u) = -h(\alpha'^{+}) < 0$. So, fixed $\varepsilon \in (0, h(\alpha'))$, we have $\dot{z}_0(u) = f(u) - cg(u) - \dfrac{h(u)}{z_0(u)} > -\dfrac{\varepsilon}{z_0(u)} > 0$ in a right neighbourhood of α', a contradiction. Hence, we infer $\alpha' = \alpha$.

Let us now consider

$$L := \limsup_{u\to\alpha^{+}} \frac{z(u)}{u-\alpha}, \quad \ell := \liminf_{u\to\alpha^{+}} \frac{z(u)}{u-\alpha},$$

with $\ell \le L \le 0$ and assume, by contradiction, that $\ell < L$. Let us fix $\gamma \in (\ell, L)$. Notice that there exist a decreasing sequence $(u_n)_n$ converging to α such that

$$\frac{z(u_n)}{u_n - \alpha} = \gamma, \quad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{z(u)}{u-\alpha}\right)_{|u=u_n} \le 0. \tag{20}$$

Indeed, there exists decreasing sequences $(v_n)_n$, $(w_n)_n$ converging to α such that

$$\frac{z(v_n)}{v_n - \alpha} \to L, \quad \frac{z(w_n)}{w_n - \alpha} \to \ell.$$

Without restriction we can assume $v_n < w_n < v_{n+1}$ for every $n \in \mathbb{N}$. Fixed $\varepsilon > 0$ such that $\ell < \gamma - \varepsilon < \gamma + \varepsilon < L$ there exists an integer $\bar{n}$ such that

$$\frac{z(v_n)}{v_n - \alpha} > \gamma + \varepsilon, \quad \frac{z(w_n)}{w_n - \alpha} < \gamma - \varepsilon \quad \text{for every } n \ge \bar{n}.$$

By the continuity of the function $u \mapsto \dfrac{z(u)}{u-\alpha}$ we conclude that for every $n \ge \bar{n}$ there exists $u_n \in (v_n, w_n)$ satisfying (20).

Since

$$\frac{d}{du}\left(\frac{z(u)}{u-\alpha}\right) = \frac{\dot{z}(u)(u-\alpha) - z(u)}{(u-\alpha)^2} = \frac{1}{u-\alpha}\left(\dot{z}(u) - \frac{z(u)}{u-\alpha}\right),$$

we get

$$\dot{z}(u_n) \le \frac{z(u_n)}{u_n - \alpha} = \gamma$$

so

$$f(u_n) - cg(u_n) - \frac{h(u_n)}{z(u_n)} = f(u_n) - cg(u_n) - \frac{h(u_n)}{\gamma(u_n - \alpha)} \le \gamma.$$

Taking the limit as $n \to +\infty$, we get $f(\alpha) - cg(\alpha) - \frac{\dot{h}(\alpha)}{\gamma} \le \gamma$, that is

$$\gamma^2 - (f(\alpha) - cg(\alpha))\gamma + \dot{h}(\alpha) \le 0.$$

Similarly, we can prove that there exists a decreasing sequence v_n converging to α such that

$$\frac{z(v_n)}{v_n - \alpha} = \gamma, \quad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{z(u)}{u-\alpha}\right)_{|u=v_n} \ge 0.$$

Arguing as in the previous part, we get $\gamma^2 - (f(\alpha) - cg(\alpha))\gamma + \dot{h}(\alpha) \geq 0$, hence we conclude

$$\gamma^2 - (f(\alpha) - cg(\alpha))\gamma + \dot{h}(\alpha) = 0 \quad \text{for every } \gamma \in (\ell, L),$$

a contradiction. Therefore, $\ell = L$ and there exists the limit $\lambda := \lim_{u \to \alpha^+} \frac{z(u)}{u - \alpha}$.

Now, observe that since

$$\dot{z}(u) = f(u) - cg(u) - \frac{h(u)}{u - \alpha} \cdot \frac{u - \alpha}{z(u)}, \tag{21}$$

then $\lambda > -\infty$, otherwise $\dot{z}(u) \to f(\alpha) - cg(\alpha)$, a contradiction. If $\lambda \neq 0$ we deduce that there exists also the limit $\dot{z}(\alpha^+) := \lim_{u \to \alpha^+} \dot{z}(u)$ and its value is λ. So, taking the limit as $u \to \alpha^+$ in (21), we infer

$$\lambda^2 = (f(\alpha) - cg(\alpha))\lambda - \dot{h}(\alpha),$$

that is, $\dot{z}(\alpha)$ is a root of the trinomial $p(t)$. Instead, if $\lambda = 0$, then necessarily $\dot{h}(\alpha) = 0$ too, otherwise $\dot{z}(u) \to +\infty$, a contradiction. Hence, also when $\lambda = 0$ it is a root of the trinomial $p(t)$.

The local analysis at the point β' can be made analogously.

□

Lemma 7 *For every $r > 0$, $r < \frac{1}{2}(\beta - \alpha)$, there exists $\delta = \delta_r > 0$ such that for every negative solution z of equation* (17)*, defined in the whole interval (α, β) we have*

$$z(u) \leq -\delta \quad \textit{for every } u \in [\alpha + r, \beta - r].$$

Proof. Let us fix a real positive number $r > 0$, with $r < \frac{1}{2}(\beta - \alpha)$. Let $m =: \min\{h(u) : u \in [\alpha + r, \beta - r]\} > 0$. By the uniform continuity of the function $(u, z) \mapsto (f(u) - cg(u))z - h(u)$ in the compact rectangle $[\alpha, \beta] \times [-m, 0]$, we have that there exists a positive value $\delta = \delta_r < \min\{m, r\}$, such that if $u_0 \in [\alpha + r, \beta - r]$, then

$$(f(u) - cg(u))z - h(u) < -h(u_0) + \frac{1}{2}m \leq -\frac{1}{2}m \quad \text{whenever } |u - u_0| \leq \delta,\ |z| \leq \delta.$$

So, we deduce

$$f(u) - cg(u) - \frac{h(u)}{z} > -\frac{m}{2z} \quad \text{whenever } u_0 \in [\alpha + r, \beta - r],\ |u - u_0| < \delta,\ |z| < \delta. \tag{22}$$

For every fixed $u_0 \in [\alpha + r, \beta - r]$, let us now consider the function $\psi(u) := -\sqrt{\delta^2 - m(u - u_0)}$ for $u_0 \leq u \leq u_0 + \delta^2/m$. By (22) we deduce that $\dot{\psi}(u) < f(u) - cg(u) - \frac{h(u)}{\psi(u)}$, for every $u \in (u_0, u_0 + \delta^2/m)$, that is ψ is a lower-solution for equation (17) in the interval $(u_0, u_0 + \delta^2/m)$. Therefore, by Lemma 9 we deduce that $z_n(u_0) \leq \psi(u_0) = -\delta$ for every $n \in \mathbb{N}$, because $\delta^2/m < \delta < r$, $\psi(u_0 + \delta^2/m) = 0$, whereas z_n is defined in the whole interval (α, β). □

Remark 2 Notice that, in view of the proof of Lemma 7, for the parameter c varying in a compact set, the constant δ can be chosen not depending on c, but just on the compact set to which c belongs.

Lemma 8 *If $(z_n(u))_n$ is an sequence of negative solutions of equation* (17)*, all defined in* (α, β)*, pointwise convergent to a function $z_0(u)$ in (α, β), then we have $z_0(u) < 0$ for every $u \in (\alpha, \beta)$ and z_0 is a solution of equation* (17)*.*

Proof. By virtue of Lemma 7, by the arbitrariness of $r > 0$, we have $z_0(u) < 0$ for every $u \in (\alpha, \beta)$. Moreover, again by Lemma 7, we have, for every $r > 0$, $r < \frac{1}{2}(\beta - \alpha)$, that

$$f(u) - cg(u) \leq f(u) - cg(u) - \frac{h(u)}{z_n(u)} \leq f(u) - cg(u) + \frac{h(u)}{\delta} \quad \text{for every } u \in [\alpha + r, \beta - r],$$

for every $n \in \mathbb{N}$. So we can apply the Dominated Convergence Theorem to deduce that, for every $u, u^* \in [\alpha + r, \beta - r]$, we have

$$\begin{aligned} z_0(u) - z_0(u^*) = \lim_{n \to +\infty} (z_n(u) - z_n(u^*)) &= \lim_{n \to \infty} \int_{u^*}^{u} f(s) - cg(s) - \frac{h(s)}{z_n(s)} \, \mathrm{d}s \\ &= \int_{u^*}^{u} f(s) - cg(s) - \frac{h(s)}{z_0(s)} \, \mathrm{d}s, \end{aligned}$$

hence, $\dot{z}_0(u) = f(u) - cg(u) - \dfrac{h(u)}{z_0(u)}$ for every $u \in [\alpha + r, \beta - r]$. By the arbitrariness of $r > 0$, we derive that z_0 is a solution of equation (17) in the whole interval (α, β).

□

A key tool for our investigation is the method of lower and upper-solutions. Recall that a C^1-function z, defined in an open interval $I \subset (\alpha, \beta)$, is said to be a *lower-solution* [*upper-solution*] for equation (17) if

$$\dot{z} \leq \ [\geq] \ f(u) - cg(u) - \frac{h(u)}{z} \quad \text{for every } u \in I. \tag{23}$$

The function z is said to be a *strict lower-solution* [*strict upper-solution*] if inequality in (23) is strict for every $u \in I$.

Since the right-hand side of equation (17) is locally lipschitzian with respect to the variable z, uniformly with respect to the variable u, the following comparison result holds, as a consequence of Gronwall's Lemma.

Lemma 9 *Let z, ζ respectively be a solution and an upper-solution of equation* (17) *in an interval $I \subset (\alpha, \beta)$ and let $u_0 \in I$ be fixed. Then,*

- *if $z(u_0) \leq \zeta(u_0)$, then $z(u) \leq \zeta(u)$ for every $u \geq u_0$*
- *if $z(u_0) \geq \zeta(u_0)$, then $z(u) \geq \zeta(u)$ for every $u \leq u_0$.*

Instead, if ζ is a lower-solution, then

- *if $z(u_0) \geq \zeta(u_0)$, then $z(u) \geq \zeta(u)$ for every $u \geq u_0$*
- *if $z(u_0) \leq \zeta(u_0)$, then $z(u) \leq \zeta(u)$ for every $u \leq u_0$.*

The next results provide the main tools, based on the methods of upper and lower-solutions, in order to obtain the existence and non-existence results of the solutions of problem (18).

Proposition 10 *Assume* (19) *and suppose that there exists an upper-solution* φ *for equation* (17) *in the whole interval* (α, β)*, such that* $\varphi(\alpha^+) = 0$ *and* $\varphi(u) < 0$ *for every* $u \in (\alpha, \beta)$*.*

Then, there exists a C^1-*function* $z : (\alpha, \beta) \to \mathbb{R}$*, solution of the singular boundary value problem* (18)*, such that* $\varphi(u) < z(u) < 0$ *for every* $u \in (\alpha, \beta)$*.*

Proof. Since $\dot\varphi(u) \geq f(u) - cg(u)$, we have that $\dot\varphi$ is bounded from below, hence, there exists the limit $\varphi(\beta^-) > -\infty$. Let us split the proof into two cases.

Case 1: $\varphi(\beta^-) < 0$. For every $n \in \mathbb{N}$ let us consider the (unique) solution z_n of equation (17), satisfying the terminal condition $z(\beta) = \varphi(\beta^-)/n$. By Lemma 6, we have that z_n is defined in the whole interval $(\alpha, \beta]$. Moreover, by Lemma 9 we also have $z_n(u) \geq \varphi(u)$ for every $u \in (\alpha, \beta]$. Observe now that by the uniqueness of the solution of equation (17) passing throughout a give point, we deduce that $\varphi(u) \leq z_n(u) \leq z_{n+1}(u) < 0$ for every $n \in \mathbb{N}$ and every $u \in (\alpha, \beta)$. Put $\zeta(u) := \lim\limits_{n\in\mathbb{N}} z_n(u)$, by Lemma 8, we have that $\zeta(u) < 0$ in (α, β) and is a solution of equation (17). Moreover, since $z_n(\alpha^+) = 0$ for all $n \in \mathbb{N}$, we have $\zeta(\alpha^+) = 0$. Finally, by definition, $z_n(\beta) \to 0 = \zeta(\beta^-)$.

Case 2: $\varphi(\beta^-) = 0$. Let $(\gamma_n)_n$ be a strictly increasing seuqence in (α, β), converging to β. For every $n \in \mathbb{N}$ we consider the (unique) solution of equation (17) satisfying $z(\gamma_n) = \varphi(\gamma_n)$, defined in its maximal existence interval (α_n, β_n). By Lemma 6 we have $\alpha_n = \alpha$ for every $n \in \mathbb{N}$; moreover, moreover, by Lemma 9 we have $z_n(u) \geq \varphi(u)$ for every $u \in (\alpha, \gamma_n)$ and $z_n(u) \leq \varphi(u)$ for every $u \in (\gamma_n, \beta_n)$. Hence, since $z(\beta_n^-)$ exists finite by Lemma 6, we deduce that also $\beta_n = \beta$ for every $n \in \mathbb{N}$. Furthermore, since $z_n(\gamma_n) = \varphi(\gamma_n) \leq z_{n+1}(\gamma_n)$, by the uniqueness of the solution of equation (17) passing through a point, we deduce that $z_n(u) = z_{n+1}(u)$ for every $u \in (\alpha, \beta)$ or $z_n(u) < z_{n+1}(u)$ for every $u \in (\alpha, \beta)$. Thus, the sequence of solutions $(z_n)_n$ is increasing. Put $\zeta(u) = \lim\limits_{n\to+\infty} z_n(u)$, by Lemma 8 we conclude that $\zeta(u) < 0$ for every $u \in (\alpha, \beta)$ and ζ is a solution of equation (17). Finally, we also have $\zeta(u) \geq \varphi(u)$ for every $u \in (\alpha, \beta)$, hence $\zeta(\alpha^+) = \varphi(\alpha^+) = 0$ and $\zeta(\beta^-) = \varphi(\beta^-) = 0$.

□

Corollary 11 *Assume* (19) *and suppose that there exists a continuous negative function* $\psi : (\alpha, \beta) \to \mathbb{R}$ *such that* $\psi(\alpha^+) = 0$ *and*

$$\psi(u) \geq \int_\alpha^u \left(f(s) - cg(s) - \frac{h(s)}{\psi(s)} \right) \mathrm{d}s \quad \textit{for every } u \in (\alpha, \beta). \tag{24}$$

Then, there exists a differentiable function $z : (\alpha, \beta) \to \mathbb{R}$*, solution of the singular boundary value problem* (18)*.*

Proof. Put $\varphi(u) := \displaystyle\int_\alpha^u \left(f(s) - cg(s) - \frac{h(s)}{\psi(s)} \right) \mathrm{d}s$. Since ψ is negative, by (24) also φ is negative. Moreover, φ is differentiable, with

$$\dot\varphi(u) = f(u) - cg(u) - \frac{h(u)}{\psi(u)} \geq f(u) - cg(u) - \frac{h(u)}{\varphi(u)} \quad \text{for every } u \in (\alpha, \beta).$$

Finally, since $\varphi(\alpha^+) = 0$, we get that φ satisfies all the assumptions of Proposition 10. So, a solution z of problem (18) exists with $\varphi(u) \leq z(u) < 0$ for every $u \in (\alpha, \beta)$. □

4 First order singular problem: main tools

Now we have all the tools to prove an existence result for t.w.s.'s, in which a key role is assumed by the following constants:

$$N_c^* := \inf_{u\in(\alpha,\beta)} \fint_\alpha^u (c\,g(s) - f(s))\,\mathrm{d}s\ , \quad H^* := \sup_{u\in(\alpha,\beta)} \fint_\alpha^u \frac{h(s)}{s-\alpha}\,\mathrm{d}s, \tag{25}$$

where, as usual, $\fint_a^b v(x)\mathrm{d}x$ denotes the mean value of the function v in $[a,b]$.

Theorem 12 *Let $h(u) > 0$ in (α, β), satisfying* (19) *and differentiable at α. Suppose that*

$$N_c^* > 2\sqrt{H^*}. \tag{26}$$

Then, problem (18) *admits solutions. Instead, if*

$$c\,g(\alpha) - f(\alpha) < 2\sqrt{\dot{h}(\alpha)}, \tag{27}$$

then problem (18) *does not admit solutions.*

Proof. Let us define

$$\Psi(u) := \begin{cases} \dfrac{h(u)}{u-\alpha}, & u \in (\alpha,\beta] \\ \\ \dot{h}(\alpha), & u = \alpha. \end{cases}$$

Since $h(u)$ is differentiable at α, we have that $\Psi(u)$ is continuous in $[\alpha,\beta]$. Now consider the set

$$T := \left\{(x,u) \in \mathbb{R}^2 : \alpha \le x \le u \le \beta\right\},$$

and the functions

$$H(x,u) := \begin{cases} \dfrac{1}{u-x}\displaystyle\int_x^u \Psi(s)\,\mathrm{d}s, & \alpha \le x < u \le \beta \\ \Psi(u), & \alpha \le x = u \le \beta \end{cases}$$

$$N_c(x,u) := \begin{cases} \dfrac{1}{u-x}\displaystyle\int_x^u (cg(s) - f(s))\mathrm{d}s, & \alpha \le x < u \le \beta \\ cg(u) - f(u), & \alpha \le x = u \le \beta. \end{cases}$$

Of course, H and N_c are continuous in the compact set T.
By (26) we get $H^* < \frac{1}{4}(N_c^*)^2$ and $N_c^* > 0$, so there exists $\varepsilon > 0$ such that $N_c^* > \varepsilon$ and

$$H^* + \varepsilon < \frac{(N_c^* - \varepsilon)^2}{4}.$$

Now, by the uniform continuity of $H(x,u)$ and $N_c(x,u)$ in the compact set T, we deduce that there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$H(x,u) \le H(\alpha,u) + \varepsilon, \qquad \text{for every } x \in (\alpha, \alpha+\delta),\ u \in [x,\beta]. \tag{28}$$

$$N_c(x,u) \ge N_c(\alpha,u) - \varepsilon, \qquad \text{for every } x \in (\alpha, \alpha+\delta),\ u \in [x,\beta]. \tag{29}$$

Let $L > 0$ be a constant such that

$$H^* + \varepsilon < L < \frac{(N_c^* - \varepsilon)^2}{4}$$

and define

$$K := \frac{N_c^* - \varepsilon + \sqrt{(N_c^* - \varepsilon)^2 - 4L}}{2}.$$

Then, we get

$$K^2 - K(N_c^* - \varepsilon) = -L \tag{30}$$

and $K > 0$ since $N_c^* > \varepsilon$. Furthermore, by (28) and (25), we observe that for every $n > \frac{1}{\delta}$ and $u \in \left(\alpha + \frac{1}{n}, \beta\right)$ we have

$$H\left(\alpha + \frac{1}{n}, u\right) = \frac{1}{u - \alpha - 1/n} \int_{\alpha+1/n}^{u} \Psi(s) \, \mathrm{d}s < H^* + \varepsilon < L.$$

So, by (30) we infer that for every $n > \frac{1}{\delta}$, $u \in \left(\alpha + \frac{1}{n}, \beta\right)$ we have

$$K^2 - K(N_c^* - \varepsilon) < -\frac{1}{u - \alpha - 1/n} \int_{\alpha+1/n}^{u} \Psi(s) \, \mathrm{d}s$$

implying

$$K\left(u - \alpha - \frac{1}{n}\right) - (N_c^* - \varepsilon)\left(u - \alpha - \frac{1}{n}\right) + \int_{\alpha+1/n}^{u} \frac{\Psi(s)}{K} \, \mathrm{d}s < 0. \tag{31}$$

By (29) and (25), for every $n > \frac{1}{\delta}$, $u \in \left(\alpha + \frac{1}{n}, \beta\right)$ we have

$$(N_c^* - \varepsilon)\left(u - \alpha - \frac{1}{n}\right) < \int_{\alpha+1/n}^{u} (cg(s) - f(s)) \mathrm{d}s$$

hence, recalling the definition of $\Psi(u)$, we obtain

$$-K(u - \alpha) > -\frac{K}{n} + \int_{\alpha+1/n}^{u} \left(-cg(s) + f(s) - \frac{h(s)}{-K(s - \alpha)}\right) \mathrm{d}s.$$

Then, if we define $\ell(u) := -K(u - \alpha)$, we have that

$$\ell(u) > \ell\left(\alpha + \frac{1}{n}\right) + \int_{\alpha+1/n}^{u} \left(-cg(s) + f(s) - \frac{h(s)}{\ell(s)}\right) \mathrm{d}s, \qquad \text{for every } u \in \left(\alpha + \frac{1}{n}, \beta\right).$$

Then, taking the limit as $n \to +\infty$ we have that ℓ satisfies the assumptions of Corollary 11 and the singular problem (18) admits a solution.

Now we prove the non-existence of solutions for the singular problem (18) when (27) holds. First of all we claim that $cg(\alpha) - f(\alpha) \geq 0$. Indeed, if $cg(\alpha) - f(\alpha) < 0$, since

$$\dot{z}(u) = -cg(u) + f(u) - \frac{h(u)}{z(u)},$$

we obtain that if there exists a solution $z(u)$ of problem (18), then $\dot{z}$ should be positive in a right neighborhood of α, a contradiction. Hence, by (27), we have $\dot{h}(\alpha) > 0$.
By Lemma 6, z is differentiable at α and $\dot{z}(\alpha)$ is a root of the trinomial $p(t) = t^2 + (cg(\alpha) - f(\alpha)t + \dot{h}(\alpha)$, which is a contradiction, since by (27) this trinomial has not real roots.

□

Proposition 13 *If problem* (18) *admits a solution for some* $c \in \mathbb{R}$*, then the solution is unique.*

Proof. Assume by contradiction the existence of two different solutions z_1, z_2 of problem (18), with respect the same constant c. By the uniqueness of the solution of a generic Cauchy problem for the differential equation in (18), we get $z_1(u) \neq z_2(u)$ for every $u \in (\alpha, \beta)$. So, we have $z_1(u) < z_2(u)$ for every $u \in (\alpha, \beta)$ (or vice versa) and then

$$0 = z_1(\beta^-) - z_1(\alpha^+) = \int_\alpha^\beta \left(f(u) - cg(u) - \frac{h(u)}{z_1(u)} \right) \mathrm{d}u$$
$$< \int_\alpha^\beta \left(f(u) - cg(u) - \frac{h(u)}{z_2(u)} \right) \mathrm{d}u = z_2(\beta^-) - z_2(\alpha^+) = 0,$$

a contradiction. □

In the next result, we prove the existence of a minimal wave speed c^* in the case $g(\alpha) > 0$. As we will discuss in Remark 4, when $g(\alpha) < 0$ the dynamic admits a maximal wave speed c^{**}.

Theorem 14 *Assume that* h *is differentiable at* α *and suppose that* (26) *holds for some constant* $c \in \mathbb{R}$*. Moreover, assume that* $g(\alpha) > 0$*. Then, put*

$$\Gamma := \{c \in \mathbb{R} : \textit{ problem } (18) \textit{ admits solutions}\}.$$

We have that Γ *is bounded from below and it admits the minimum value* $c^* := \min \Gamma$*, satisfying (see* (25)*)*

$$N^*_{c^*} \leq 2\sqrt{H^*} \ , \quad c^* g(\alpha) - f(\alpha) \geq 2\sqrt{\dot{h}(\alpha)}. \tag{32}$$

Moreover, if $G(u) := \int_\alpha^u g(s) \, \mathrm{d}s \geq 0$ *for every* $u \in (\alpha, \beta)$*, then* $\Gamma = [c^*, +\infty)$*, that is problem* (18) *admits solutions if and only if* $c \geq c^*$*.*

Proof. Since (26) holds for some c, by Theorem (12) the set Γ is nonempty. Put $c^* := \inf \Gamma$. Since $g(\alpha) > 0$, condition (27) holds for every $c < 2\sqrt{\dot{h}(\alpha)}/g(\alpha)$; hence $c^* \geq 0$. Let us show that $c^* \in \Gamma$.

To this aim, note that if $c^* \notin \Gamma$ then there exists a decreasing sequence $(c_n)_n \subset \Gamma$, converging to c^*. Let z_n be the (unique) solution of problem (18) for $c = c_n$. Put

$$M := \max_{(u,c) \in [\alpha,\beta] \times [c^*, c_1]} (f(u) - cg(u)), \qquad m := \min_{(u,c) \in [\alpha,\beta] \times [c^*, c_1]} (f(u) - cg(u)). \tag{33}$$

Let us fix a decreasing sequence $(r_n)_n$ converging to 0, with $r_1 < \frac{1}{2}(\beta - \alpha)$, and put $I_k := [\alpha + r_k, \beta - r_k]$. Of course, $I_k \subset I_{k+1}$ and $(\alpha, \beta) = \cup_{k \geq 1} I_k$.

By Lemma 7, for every $n \in N$ there exists a value $\delta_{r_1} > 0$ such that

$$f(u) - c_n g(u) \leq f(u) - c_n g(u) - \frac{h(u)}{z_n(u)} \leq f(u) - c_n g(u) + \frac{h(u)}{\delta_{r_1}} \ , \quad \text{ for every } u \in I_1.$$

Therefore, we have (see (33))

$$m \leq \dot{z}_n(u) \leq M + \frac{h(u)}{\delta_{r_1}}, \quad \text{ for every } u \in I_1$$

implying that the sequence of functions $(z_n)_n$ is equicontinuous in I_1. Moreover, we also have

$$m(u-\alpha) \le \int_\alpha^u (f(s) - c_n g(s))\, \mathrm{d}s \le z_n(u) \le -\delta_{r_1}$$

so, the sequence $(z_n)_n$ is also equibounded in I_1. Hence, by the Ascoli-Arzelà Theorem, we infer the existence of a subsequence $(z_n^{(1)})_n$ uniformly convergent to a function $z_0^{(1)}$ in the interval I_1.

Now, consider the interval $I_2 \supset I_1$. By repeating the same argument, we derive that the sequence $(z_n^{(1)})_n$ admits a further subsequence $(z_n^{(2)})_n$ uniformly convergent to a function $z_0^{(2)}$ in the interval I_2. Of course, we have $z_0^{(2)}(u) = z_0^{(1)}(u)$ for every $u \in I_1$.

Proceeding in this way, for every $k \in \mathbb{N}$ we obtain that the sequence $(z_n^{(k)})_n$ admits a subsequence $(z_n^{(k+1)})_n$ uniformly convergent to a function $z_0^{(k+1)}$ in the interval I_{k+1}, with $z_0^{(k+1)} = z_0^{(k)}(u)$ for every $u \in I_k$.

Let us consider now the function $\zeta_0 : (\alpha, \beta) \to \mathbb{R}$, defined by $\zeta_0(u) = z_0^{(k)}(u)$ if $u \in I_k$. By what we proved above, the function ζ_0 is well-defined. Moreover, ζ_0 is a solution of equation (17). Indeed, fo every fixed $u^* \in (\alpha, \beta)$, fixed an integer k such that $u^* \in I_k$, we have that ζ_0 is the uniform limit of the sequence of solutions $(z_n^{(k)})_n$, hence for every $u \in I_k$ we have

$$\zeta_0(u) - \zeta_0(u^*) = \lim_{n\to+\infty} (z_n^{(k)}(u) - z_n^{(k)}(u^*)) = \lim_{n\to+\infty} \int_{u^*}^u \left(f(s) - c_n g(s) - \frac{h(s)}{z_n^{(k)}(s)} \right) \mathrm{d}s.$$

Since

$$m \le f(u) - c_n g(u) \le \dot{z}_n(u) \le f(u) - c_n g(u) + \frac{h(u)}{\delta_{r_k}} \le M + \frac{h(u)}{\delta_{r_k}}, \qquad \text{for each } u \in I_k, n \in \mathbb{N},$$

we can apply the Dominated Convergence Theorem to deduce

$$\lim_{n\to+\infty} \int_{u^*}^u \left(f(s) - c_n g(s) - \frac{h(s)}{z_n^{(k)}(s)} \right) \mathrm{d}s = \int_{u^*}^u \left(f(s) - c^* g(s) - \frac{h(s)}{\zeta_0(s)} \right) \mathrm{d}s$$

hence, ζ_0 is a solution of (17) for $c = c^*$ in the interval I_k. Therefore, ζ_0 is a solution of (17) for $c = c^*$ in the whole interval (α, β).

Observe now that put $\mathcal{C} := \max\{|c_n|, n \ge 1\}$, we have

$$z_n(u) \ge \int_\alpha^u (f(s) - c_n g(s))\mathrm{d}s \ge \int_\alpha^u (f(s) - \mathcal{C}|g(s)|)\mathrm{d}s$$

for every $n \in \mathbb{N}$ and $u \in (\alpha, \beta)$. So, also $\zeta_0(u) \ge \int_\alpha^u (f(s) - \mathcal{C}|g(s)|)\mathrm{d}s$, for every $u \in (\alpha, \beta)$, implying that $\zeta_0(\alpha^+) = 0$.

Finally, by Lemma 6, there exists the limit $\zeta_0(\beta^-) \in (-\infty, 0]$. Therefore, the function ζ_0 satisfies the assumptions of Proposition 10 and we conclude that there exists a solution of problem (18) for $c = c^*$.

Finally, let us prove that if $G(u) \ge 0$ for all $u \in (\alpha, \beta)$ then $\Gamma = [c^*, +\infty)$. Indeed, let z^* be the solution of problem (18) for $c = c^*$. Since $G(u) \ge 0$, for every $c > c^*$, we have

$$z^*(u) = \int_\alpha^u \left(f(s) - c_0 g(s) - \frac{h(s)}{z^*(s)} \right) \mathrm{d}s \ge \int_\alpha^u \left(f(s) - c g(s) - \frac{h(s)}{z^*(u)} \right) \mathrm{d}s$$

So, z^* satisfies assumption (24) of Corollary 11 and we derive the existence of a solution of problem (18) for every $c > c^*$.

□

Remark 3 Notice that when g is constant, say $g(u) \equiv 1$, then inequalities (14) reduce to the known estimate (see [23])

$$f(\alpha) + 2\sqrt{\dot{h}(\alpha)} \leq c^* \leq \sup_{u\in(\alpha,\beta]} ⨍_\alpha^u f(s)\mathrm{d}s + 2\sqrt{\sup_{u\in(\alpha,\beta]} ⨍_\alpha^u \frac{h(s)}{s-\alpha}\mathrm{d}s}.$$

Instead, when $f(u) \equiv 0$, then the necessary condition for the existence of solutions is

$$c^* g(\alpha) \geq 2\sqrt{\dot{h}(\alpha)}.$$

Remark 4 In Theorem 14 we have assumed $g(\alpha) > 0$. Notice that if $g(\alpha) < 0$, then we can reduce to the previous case simply by changing sign to c. Hence, when $g(\alpha) < 0$ we can deduce that the set Γ of the admissible wave speeds is nonempty, bounded by above and it admits maximum c^{**}. Moreover, if $\int_\alpha^u g(s)\,\mathrm{d}s \leq 0$ for every $u \in (\alpha, \beta)$, then $\Gamma = (-\infty, c^{**}]$. A more complicated situation occurs if $g(\alpha) = 0$. Indeed, if $2\sqrt{\dot{h}(\alpha)} + f(\alpha) > 0$, then by (27) no solution exists, whatever c may be. For instance, the simple equation

$$vv_\tau = v_{xx} + v(1 - v)$$

(with $D \equiv 1$ and $f \equiv 0$) does not admit t.w.s.'s, for any speed $c \in \mathbb{R}$. Instead, the case when

$$g(\alpha) = f(\alpha) + 2\sqrt{(D\rho)'(\alpha)} = 0$$

is not covered by our results and may deserve to be investigated.

We conlude this section with a result concerning the slope of the solution of problem (18) at the the extreme points. It is the key tool in order to classify the fronts. The proof is partially inspired by that proposed in [6] for an analogous result.

Theorem 15 *Assume h differentiable at α and β. Let z_c be the (unique) solution of problem* (18) *for some c. Put*

$$r_\pm(c,u) := \frac{1}{2}\left(f(u) - cg(u) \pm \sqrt{(f(u) - cg(u))^2 - 4\dot{h}(u)}\right) \qquad \text{for } c \geq c^*,\ u \in [\alpha, \beta],$$

if $g(\alpha) > 0$, we have

$$\dot{z}_c(\alpha) = \begin{cases} r_+(c,\alpha) & \text{if } c > c^* \\ r_-(c,\alpha) & \text{if } c = c^*, \end{cases} \qquad \dot{z}_c(\beta) = r_+(c,\beta) \ \text{ for every } c \geq c^*;$$

if $g(\alpha) < 0$, we have

$$\dot{z}_c(\alpha) = \begin{cases} r_+(c,\alpha) & \text{if } c < c^* \\ r_-(c,\alpha) & \text{if } c = c^*, \end{cases} \qquad \dot{z}_c(\beta) = r_+(c,\beta) \ \text{ for every } c \leq c^*;$$

Proof. First notice that by Lemma 6 we have that z_c is differentiable at α, with $\dot{z}(\alpha) \in \{r_+(c,\alpha), r_-(c,\alpha)\}$, for every $c \geq c^*$ and the same relation holds for $\dot{z}(\beta)$ too.

Assume $g(\alpha) > 0$. We first prove the assertion concerning the point α. Let us first consider the case $c > c^*$ and assume, by contradiction, $\dot{z}_c(\alpha) = r_-(c,\alpha)$. Since $g(\alpha) > 0$, the function $c \mapsto r_-(c,\alpha)$ is strictly decreasing, so we have $r_-(c,\alpha) < r_-(c^*,\alpha) \leq r_+(c^*,\alpha)$.

Therefore, we have $\dot{z}_c(\alpha) < \dot{z}_{c^*}(\alpha)$, implying $z_c(u) < z_{c^*}(u)$ in a right neighbourhood of α. Put

$$u_0 := \sup\{u \in (\alpha, \beta) : \; z_c(\xi) < z_{c^*}(\xi) \text{ for every } \xi \in (\alpha, u)\}.$$

Since $z_c(\beta^-) = z_{c^*}(\beta^-) = 0$, we have $z_c(u_0) = z_{c^*}(u_0)$ and then, by (17) we deduce

$$0 = z_c(u_0) - z_{c^*}(u_0) = (c^* - c)\int_\alpha^{u_0} g(s)\,\mathrm{d}s - \int_\alpha^{u_0} h(s)\left(\frac{1}{z_c(s)} - \frac{1}{z_{c^*}(s)}\right)\mathrm{d}s < 0,$$

a contradiction. So, $\dot{z}_c(\alpha) = r_+(c, \alpha)$ when $c > c^*$.

Let us now consider the case $c = c^*$. If $r_-(c^*, \alpha) = r_+(c^*, \alpha)$ the assertion is trivial. So, from now on assume that $r_-(c^*, \alpha) < r_+(c^*, \alpha)$ and we rename for simplicity $r_- := r_-(c^*, \alpha)$, $r_+ := r_+(c^*, \alpha)$. Fix a value K with $r_- < K < r_+ \le 0$.

Let us now prove that there exists a value $\rho > 0$ such that

$$f(u) - cg(u) - \frac{h(u)}{K(u-\alpha)} < K \quad \text{whenever } |c - c^*| < \rho, \; u - \alpha < \rho. \tag{34}$$

To this aim, first notice that $\dot{h}(\alpha) = r_+ r_-$, hence $K r_- > r_+ r_- = \dot{h}(\alpha)$, and this implies that

$$\dot{h}(\alpha)\left(\frac{1}{r_-} - \frac{1}{K}\right) < K - r_-.$$

Let $\varepsilon > 0$ be such that

$$\dot{h}(\alpha)\left(\frac{1}{r_-} - \frac{1}{K}\right) + 2\varepsilon < K - r_-. \tag{35}$$

Let $\delta > 0$ be such that

$$-\frac{h(u)}{K(u-\alpha)} < -\frac{\dot{h}(\alpha)}{K} + \varepsilon \qquad \text{for every } u \in (\alpha, \alpha + \delta). \tag{36}$$

By the continuity of the function $(c, u) \mapsto f(u) - cg(u)$, there exists a positive value $\rho < \delta$ such that

$$f(u) - cg(u) < f(\alpha) - c^* g(\alpha) + \varepsilon \quad \text{whenever } |c - c^*| < \rho, \; u - \alpha < \rho. \tag{37}$$

Finally, notice that

$$f(\alpha) - c^* g(\alpha) - \frac{\dot{h}(\alpha)}{r_-} = r_-. \tag{38}$$

So, if $|c - c^*| < \rho$ and $u - \alpha < \rho$, then

$$f(u) - cg(u) - \frac{h(u)}{K(u-\alpha)} \overset{(36)}{<} f(u) - cg(u) - \frac{\dot{h}(\alpha)}{K} + \varepsilon \overset{(37)}{<} f(\alpha) - c^* g(\alpha) - \frac{\dot{h}(\alpha)}{K} + 2\varepsilon$$

$$= f(\alpha) - c^* g(\alpha) - \frac{\dot{h}(\alpha)}{r_-} + \frac{\dot{h}(\alpha)}{r_-} - \frac{\dot{h}(\alpha)}{K} + 2\varepsilon \overset{(38)}{=} r_- + \frac{\dot{h}(\alpha)}{r_-} - \frac{\dot{h}(\alpha)}{K} + 2\varepsilon \overset{(35)}{<} K$$

and claim (34) is proved. Therefore, if we consider the function $\sigma(u) := K(u - \alpha)$, we get that it is an upper solution for equation (17) in the interval $(\alpha, \alpha + \rho)$ for every c such that $|c - c^*| < \rho$.

Let $(c_n)_n$ be an increasing sequence converging to c^* such that $|c_n - c^*| < \rho$ and let z_n be the unique solution of the Cauchy problem

$$\begin{cases} \dot{z}(u) = -c_n g(u) + f(u) - \dfrac{h(u)}{z(u)} \\ z(\beta) = -\dfrac{1}{n} \end{cases}$$

By Lemma 6, each solution z_n is defined in the whole interval (α, β), moreover, since $c_n < c^*$, necessarily we have $z_n(\alpha^+) < 0$.

Osbserve that $\inf z_n(\alpha^+) > -\infty$. Indeed, put $A_n := \{u \in (\alpha, \beta) : \; z_n(u) \geq -1\}$, we have that A_n is nonempty, so we can define $i_n := \inf A_n$. If $\inf z_n(\alpha^+) = -\infty$, we have $i_n > \alpha$ and then $z_n(i_n) = -1$ for n sufficiently large. Moreover, put

$$M_1 := \max_{(c,u)\in[c^*-\rho,\; c^*+\rho]\times[\alpha,\beta]} f(u) - cg(u), \quad M_2 := \max_{u\in[\alpha,\beta]} h(u),$$

for n sufficiently large we have

$$\dot{z}_n(u) = f(u) - c_n g(u) - \frac{h(u)}{z_n(u)} \leq M_1 + M_2 \quad \text{ for every } u \in [\alpha, i_n].$$

Hence,

$$-1 - z_n(\alpha^+) = \int_\alpha^{i_n} \dot{z}_n(u)\mathrm{d}u \leq (M_1 + M_2)(i_n - \alpha) \leq (M_1 + M_2)(\beta - \alpha)$$

implying that $z_n(\alpha^+) > -1-(M_1+M_2)(\beta-\alpha)$, a contradiction. So, $N := \inf z_n(\alpha^+) > -\infty$. Furthermore, since $\dot{z}_n(u) = f(u) - c_n g(u) - \dfrac{h(u)}{z(u)} \geq f(u) - (c^* + \rho)g(u)$ for every $u \in (\alpha, \beta)$, there exists a constant L such that $\dot{z}_n(u) \geq L$ for every $u \in (\alpha, \beta)$ and n sufficiently large. Then, since $z_n(\alpha^+) \geq N$ and $z_n(\beta^-) = -1/n$ for every n, we infer that the sequence $(z_n)_n$ is equibounded in (α, β).

Let us now consider a sequence $(\mu_k)_k$ of decreasing positive numbers, such that $2\mu_1 < \beta-\alpha$ and the corresponding sequence of intervals $I_k := [\alpha+\mu_k, \beta-\mu_k] \subset (\alpha, \beta)$. By Lemma 7 (see also Remark 2), there exists a value δ_k such that

$$z_n(u) \leq -\delta_k \quad \text{for every } u \in I_k, n \in \mathbb{N}.$$

So, for every $u \in I_k$ and $n \in \mathbb{N}$, we have

$$f(u) - c_n g(u) \leq \dot{z}_n(u) = f(u) - c_n g(u) - \frac{h(u)}{z_n(u)} \leq f(u) - c_n g(u) + \frac{h(u)}{\delta_k},$$

hence the sequence $(z_n)_n$ is equicontinuous in each interval I_k.

By the Ascoli-Arzelà Theorem, the sequence $(z_n)_n$ a subsequence, denoted $(z_n^{(1)})_n$ for simplicity, uniformly convergent in the interval I_1 to a function $\tilde{z}_1$. Considering the interval I_2, we have that the sequence $(z_n^{(1)})_n$ admits a subsequence, denoted $(z_n^{(2)})_n$ for simplicity, uniformly convergent in the interval I_2 to a function $\tilde{z}_2$. Of course, $\tilde{z}_2 = \tilde{z}_1$ in the interval I_1. Proceeding in this way we obtain a function $\tilde{z}$, defined on the whole interval (α, β), such that in each interval I_k it is the uniform limit of a subsequence $(z_n^{(k)})_n$.

Therefore, fixed a value $u \in (\alpha, \beta)$ an integer k such that $u \in I_k$, and a value $u_0 \in I_k$, we have

$$\tilde{z}(u) - \tilde{z}(u_0) = \lim_{n\to+\infty} (z_n^{(k)}(u) - z_n^{(k)}(u_0)) = \lim_{n\to+\infty} \int_{u_0}^{u} \dot{z}_n^{(k)}(s)\; \mathrm{d}s =$$

$$= \lim_{n\to+\infty} \int_{u_0}^{u} \left(f(s) - c_n g(s) - \frac{h(s)}{z_n^{(k)}(s)} \right) \mathrm{d}s = \int_{u_0}^{u} \left(f(s) - c^* g(s) - \frac{h(s)}{\tilde{z}(s)} \right)\; \mathrm{d}s$$

and this means that $\tilde{z}$ is a solution of equation (17), for $c = c^*$, in the whole interval (α, β), satisfying $\tilde{z}(\beta^-) = 0$.

Observe now that $\tilde{z}(\alpha^+) = 0$, indeed, if we assume by contradiction $\tilde{z}(\alpha^+) < 0$, then we have $\tilde{z}(u) < z_{c^*}(u)$ for every $u \in (\alpha, \beta)$, by the uniqueness of the solution of equation (17) passing through any point (u_0, z_0) with $u_0 \in (\alpha, \beta)$, $z_0 < 0$. So, for every $u \in (\alpha, \beta)$, we have

$$\tilde{z}(u) - \tilde{z}(\alpha^+) = \int_\alpha^u \left(f(s) - c^* g(s) - \frac{h(s)}{\tilde{z}(s)} \right) \mathrm{d}s < \int_\alpha^u \left(f(s) - c^* g(s) - \frac{h(s)}{z_{c^*}(s)} \right) \mathrm{d}s = z_{c^*}(u)$$

implying $\tilde{z}(u) < z_{c^*}(u) + \tilde{z}(\alpha^+)$, then $\tilde{z}(\beta^-) \le \tilde{z}(\alpha^+) < 0$, a contradiction. Hence, $\tilde{z}(\alpha^+) = 0$, so $\tilde{z}$ is a solution of boundary value problem (18) for $c = c^*$. By the uniqueness of the solution of such a problem (see Theorem 14), we get $\tilde{z} = z_{c^*}$.

Since $z_n(\alpha^+) < 0$ we can take a value $u_n \in (\alpha, \alpha + \rho)$ such that

$$z_n(u) < K(u - \alpha) = \sigma(u) \qquad \text{for every } u \in (\alpha, u_n).$$

Since σ is an upper solution for equation (17) with $c = c_n$, we infer that $z_n(u) \le K(u - \alpha)$ also for every $u \in [u_n, \alpha + \rho)$. So, we obtain $z_{c^*}(u) \le K(u - \alpha)$ for every $u \in (\alpha, \alpha + \rho)$ and therefore $\dot{z}(\alpha) \le K < r_+$, implying that $\dot{z}(\alpha) = r_-$.

Let us now prove the assertion concerning the point β.

First notice that if $\dot{h}(\beta) \neq 0$, then $r_-(c, \beta) < 0$, so necessarily we have $\dot{z}(\beta) = r_+(c, \beta)$. So, we have to prove the assertion in the case $\dot{h}(\beta) = 0$.

If $f(\beta) - cg(\beta) = 0$, then $r_-(c, \beta) = r_+(c, \beta)$ and the assertion holds true. Moreover, if $f(\beta) - cg(\beta) < 0$ then $\dot{z}(\beta) = 0$, indeed, if $\dot{z}(\beta) > 0$, then since

$$\frac{z(u)}{u - \beta} \left[\dot{z}(u) + cg(u) - f(u) \right] = 0$$

we have

$$\lim_{u \to \beta^-} \dot{z}(u) = f(\beta) - cg(\beta) < 0,$$

a contradiction. So $\dot{z}(\beta) = 0 = r_+(c, \beta)$.

Finally, if $f(\beta) - cg(\beta) > 0$, then $r_-(c, \beta) = 0$, so it suffices to show that $\dot{z}(\beta) > 0$. In order to do this, let $\eta > 0$ be such that

$$f(u) - cg(u) > \frac{1}{2}(f(\beta) - cg(\beta)) > 0 \qquad \text{for every } u \in (\beta - \eta, \beta).$$

We have

$$\dot{z}(u) = f(u) - cg(u) - \frac{h(u)}{z(u)} \ge f(u) - cg(u) > \frac{1}{2}(f(\beta) - cg(\beta)) \qquad \text{for every } u \in (\beta - \eta, \beta).$$

Therefore, by the classical Darboux Theorem (see, e.g., [24], p. 108), we infer that $\dot{z}(\beta) > 0$ and this concludes the proof in the case $g(\alpha) > 0$. The proof concerning the case $g(\alpha) < 0$ can be deduced by a change of sign of c and g.

$\square$

5 Main results and classification of t.w.s.'s

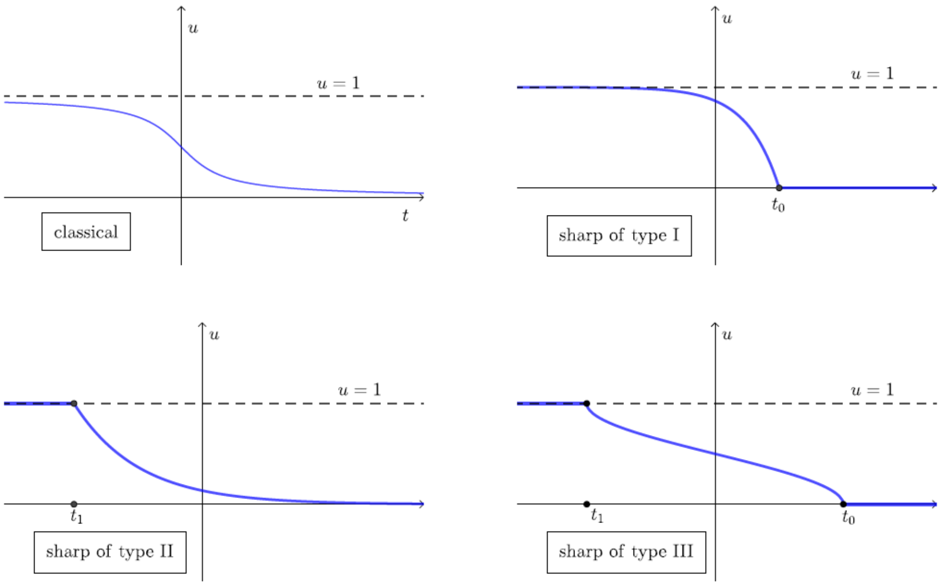
As is it easy to check, the main Theorems 1 and 2 stated in Introduction are immediate consequence of the equivalence Theorem 5, and the existence Theorems 12 and 14.

We now deal with the classification of the t.w.s.'s As we mentioned in Introduction, equation (3) may admit various types of solution, according to the structure of their maximal existence interval (a, b). More in detail, we can classify the solutions as follows.

Definition A t.w.s. is said to be

- *classical* if $\lim_{t\to a^+} u'(t) = \lim_{t\to b^-} u'(t) = 0$;
- *sharp of type (I)* if $\lim_{t\to a^+} u'(t) = 0$ and $\lim_{t\to b^-} u'(t) < 0$;
- *sharp of type (II)* if $\lim_{t\to a^+} u'(t) < 0$ and $\lim_{t\to b^-} u'(t) = 0$;
- *sharp of type (III)* if $\lim_{t\to a^+} u'(t) < 0$ and $\lim_{t\to b^-} u'(t) < 0$.

(see figure below).



Of course, when $\lim_{t\to a^+} u'(t) < 0$ then necessarily $a > -\infty$ and similarly when $\lim_{t\to b^-} u'(t) < 0$ then $b < +\infty$. In this case the dynamic present the phenomenon of finite speed of propagation, since the solution reaches (or leaves) the equilibria in a finite time. Moreover, when this happens, the solution may be not Lipchitzian, since the limit of the derivative can be infinite. We will see that this may happens when $D(\alpha) = \dot{D}(\alpha) = 0$ or $D(\beta) = \dot{D}(\beta) = 0$.

Instead, when the limits of the derivative are null, then not necessarily the corresponding extreme a or b is infinite, but the solution admits a C^1-continuation which is constant outside the interval (a, b).

The following result provides a classification of the t.w.s.'s In what follows, with a slight abuse of notation, $\dot{D}(\alpha)$ denotes the limit $\lim_{u\to\alpha} \dfrac{D(u) - D(\alpha)}{u - \alpha}$, even if it is infinite. The same holds for $\dot{D}(\beta)$.

Theorem 16 *Let u be a solution of* (3)*, defined in the interval (a, b), satisfying conditions* (12) *and* (13)*. If $g(\alpha) > 0$, then we have*

- *if $D(\alpha) \neq 0$, then $\lim_{t\to b^-} u'(t) = 0$, whatever $c \geq c^*$ may be;*
- *if $D(\alpha) = 0$ and $0 < \dot{D}(\alpha) < +\infty$, then* $$\lim_{t\to b^-} u'(t) = \begin{cases} \dfrac{f(\alpha) - cg(\alpha)}{\dot{D}(\alpha)} \leq 0 & \text{if } c = c^* \\ 0 & \text{if } c > c^*; \end{cases}$$
- *if $D(\alpha) = 0$ and $\dot{D}(\alpha) = +\infty$, then $\lim_{t\to b^-} u'(t) = 0$, whatever $c \geq c^*$ may be;*

- *if* $D(\alpha) = \dot{D}(\alpha) = 0$ *and* $f(\alpha) - cg(\alpha) \neq 0$, *then* $\lim_{t\to b^-} u'(t) = \begin{cases} -\infty & \text{if } c = c^* \\ 0 & \text{if } c > c^*. \end{cases}$

Moreover, for every $c \geq c^*$ *we have*

- *if* $D(\beta) \neq 0$, *then* $\lim_{t\to a^+} u'(t) = 0$;
- *if* $D(\beta) = 0$ *and* $-\infty < \dot{D}(\beta) < 0$, *then* $\lim_{t\to a^+} u'(t) = \max\left\{0, \dfrac{f(\beta) - cg(\beta)}{\dot{D}(\beta)}\right\}$;
- *if* $D(\beta) = 0$ *and* $\dot{D}(\beta) = -\infty$, *then* $\lim_{t\to a^+} u'(t) = 0$;
- *if* $D(\beta) = \dot{D}(\beta) = 0$ *and* $f(\beta) - cg(\beta) \neq 0$, *then*

$$\lim_{t\to a^+} u'(t) = \begin{cases} 0 & \text{if } f(\beta) - cg(\beta) < 0 \\ -\infty & \text{if } f(\beta) - cg(\beta) > 0. \end{cases}$$

Proof. Since $u'(t) = \dfrac{z(u(t))}{D(u(t))}$, we have $\lim_{t\to b^-} u'(t) = \lim_{u\to\alpha} \dfrac{z(u)}{D(u)}$. So, if $D(\alpha) \neq 0$, then $\lim_{t\to b^-} u'(t) = 0$ too. Similarly one can see that $\lim_{t\to a^+} u'(t) = 0$ when $D(\beta) \neq 0$.

Assume now that $D(\alpha) = 0$ and $\dot{D}(\alpha) \neq 0$. Put $h(u) := D(u)\rho(u)$, we have $\dot{h}(\alpha) = 0$. So, since $f(\alpha) - cg(\alpha) \leq 0$ by Theorem 12, from Theorem 15 we deduce that

$$\dot{z}(\alpha) = \begin{cases} f(\alpha) - cg(\alpha) & \text{if } c = c^* \\ 0 & \text{if } c > c^*. \end{cases}$$

Consider now the third case, when $D(\alpha) = \dot{D}(\alpha) = 0$ and $f(\alpha) - cg(\alpha) \neq 0$. First notice that $f(\alpha) - cg(\alpha) < 0$, since by Theorem 12 necessarily we have $f(\alpha) - cg(\alpha) \leq 0$.

If $c = c^*$, by Theorem 15 we have $\dot{z}(\alpha) = f(\alpha) - cg(\alpha) < 0$, hence $\lim_{t\to b^-} u'(t) = \lim_{u\to\alpha} \dfrac{z(u)}{D(u)} = -\infty$ since $\dot{D}(\alpha) = 0$. Instead, for $c > c^*$ we have $\dot{z}(\alpha) = 0$; then there exists a sequence $u_n \to \alpha$ such that $\dot{z}(u_n) \to 0$. Hence, by (17) for $h(u) = D(u)\rho(u)$ we get

$$\frac{D(u_n)}{z(u_n)} = \frac{f(u_n) - cg(u_n) - \dot{z}(u_n)}{\rho(u_n)} \to -\infty \tag{39}$$

since $f(\alpha) - cg(\alpha) < 0$.

Let us now fix a real $\varepsilon > 0$ and define $\varphi_\varepsilon(u) := -\varepsilon D(u)$. Notice that

$$\lim_{u\to\alpha^+} f(u) - cg(u) - \frac{D(u)\rho(u)}{\varphi_\varepsilon(u)} = f(\alpha) - cg(\alpha) < 0.$$

So, since $\dot{\varphi}_\varepsilon(u) = -\varepsilon\dot{D}(u) \to 0$ as $u \to \alpha^+$, there exists a value $\delta = \delta_\varepsilon > 0$ such that

$$\dot{\varphi}_\varepsilon(u) > f(u) - cg(u) - \frac{D(u)\rho(u)}{\varphi_\varepsilon(u)} \quad \text{for every } u \in (\alpha, \alpha+\delta), \tag{40}$$

that is φ_ε is a strict upper-solution for equation (17) in the interval $(\alpha, \alpha+\delta)$.

By (39) we have $-\varepsilon\dfrac{D(u_n)}{z(u_n)} > 1$ for n sufficiently large; so there exists a value $\eta = \eta_\varepsilon < \alpha + \delta$ such that $z(\eta) > \varphi_\varepsilon(\eta)$. Since φ_ε is a strict upper-solution, by Lemma 9 we derive $z(u) \geq \varphi_\varepsilon(u) = -\varepsilon D(u)$ for every $u \in (\alpha, \eta)$. Then,

$$-\varepsilon \leq \frac{z(u)}{D(u)} \leq 0 \quad \text{for every } u \in (\alpha, \eta)$$

that is $\lim_{u\to\alpha}\dfrac{z(u)}{D(u)}=0$ and this concludes the proof of the classification relative to the equilibrium α.

Assume now $D(\beta)=0$ and $\dot{D}(\beta)\neq 0$. Also in this case we have $\dot{h}(\beta)=0$, where $h(u):=D(u)\rho(u)$. So, by Theorem 15 we have

$$\dot{z}(\beta)=\frac{1}{2}((f(\beta)-cg(\beta)+|f(\beta-cg(\beta)|=\max\{0,f(\beta-cg(\beta)\}.$$

Hence, $\lim_{t\to a^+}u'(t)=\lim_{u\to\beta^-}\dfrac{z(u)}{D(u)}=\dfrac{\dot{z}(\beta)}{\dot{D}(\beta)}=\max\left\{0,\dfrac{f(\beta)-cg(\beta)}{\dot{D}(\beta)}\right\}$ for every $c\geq c^*$.

Consider now the last case, when $D(\beta)=\dot{D}(\beta)=0$ and $f(\beta)-cg(\beta)\neq 0$. First assume that $f(\beta)-cg(\beta)>0$. Since $\dot{z}(\beta)=f(\beta)-cg(\beta)>0$, we have that $\lim_{u\to\beta^-}\dfrac{z(u)}{D(u)}=-\infty$, since $\dot{D}(\beta)=0$.

Finally, if $D(\beta)=\dot{D}(\beta)=0$ and $f(\beta)-cg(\beta)<0$, then considering again the function φ_ε above defined, by a similar argument we can show that for some $\delta=\delta_\varepsilon>0$ we have

$$\dot{\varphi}_\varepsilon(u)<f(u)-cg(u)-\frac{D(u)\rho(u)}{\varphi_\varepsilon(u)}\quad\text{for every } u\in(\beta-\delta,\beta),$$

that is φ_ε is a strict lower-solution. Moreover, since $\dot{z}(\beta)=0$, we can find a sequence v_n converging to β such that $\dot{z}(v_n)\to 0$ and $n\to+\infty$. By an argument similar to what we used above, we get that $z(u_n)>\varphi_\varepsilon(u_n)$ for n sufficiently large. Since φ_ε is a lower-solution, this implies that $z(u)>\varphi_\varepsilon(u)=-\varepsilon D(u)$ in a left neighborhood of β, so $\lim_{u\to\beta^-}\dfrac{z(u)}{D(u)}=0$ and this concludes the proof. □

In light of the previous theorem we can summarize the classification of the value of the limits $\lim_{t\to a^+}u'(t)$ and $\lim_{t\to b^-}u'(t)$ in th following tables, from which one can immediate deduce the classification of the t.w.s.'s:

c	$D(\alpha)$	$\dot{D}(\alpha)$	$f(\alpha)-cg(\alpha)$	$u'(b^-)$	type of solution at α
any	$\neq 0$	any	any	0	smooth
any	0	$+\infty$	any	0	smooth
$c>c^*$	0	$\neq 0,\neq+\infty$	any	0	smooth
$c=c^*$	0	$\neq 0,\neq+\infty$	0	0	smooth
$c=c^*$	0	$\neq 0,\neq+\infty$	$\neq 0$	<0	sharp
$c>c^*$	0	0	$\neq 0$	0	smooth
$c=c^*$	0	0	$\neq 0$	$-\infty$	sharp

c	$D(\beta)$	$\dot{D}(\beta)$	$f(\beta)-cg(\beta)$	$u'(a^+)$	type of solution at β
any	$\neq 0$	any	any	0	smooth
any	0	$-\infty$	any	0	smooth
any	0	$\neq 0,\neq-\infty$	≤ 0	0	smooth
any	0	$\neq 0,\neq-\infty$	>0	<0	sharp
any	0	0	<0	0	smooth
any	0	0	>0	$-\infty$	sharp

Remark 5 Note that Theorem 16 and the consequent tables do not cover the cases when $D(\alpha) = \dot{D}(\alpha) = f(\alpha) - cg(\alpha) = 0$ and/or $D(\beta) = \dot{D}(\beta) = f(\beta) - cg(\beta) = 0$. In these special cases the solution can be smooth or not, as showed in [6, Remark 10.1] in the particular case $g(u) \equiv 1$, so further assumptions are needed in order to classify it.

Remark 6 A similar classification can be obtained in the case $g(\alpha) < 0$*, simply replacing* $c > c^*$ *with* $c < c^{**}$*.*

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