Dynamical analysis of a financial market with fundamentalists, chartists, and imitators

Dynamical Analysis of a Financial Market with Fundamentalists, Chartists, and Imitators

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Abstract

The aim of the paper is to understand the price dynamics generated by the interaction of traders relying on heterogeneous expectations in an asset pricing model. In the present work we propose a financial market populated by three types of agents – fundamentalists, chartists and imitators. The latter submit buying/selling orders according to different trading rules using a 2D Piecewise Linear (PWL) discontinuous map. Our contribution to the existing financial literature is twofold. First, we perform an analytical study of the model involving a 2D PWL discontinuous map, where mainly numerical results are provided by researchers, besides few exceptions. In particular, we investigate the bifurcations showed by the model and the large variety of dynamical behavior produced. Finally, we provide numerical simulations in order to highlight the interaction between traders with heterogeneous expectations that can lead to intricate bull and bear price dynamics.

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1. Introduction

In the standard finance literature investors are homogeneous and perfectly rational (see for example the pioneering works of Friedman (1953), Muth (1961), Lucas (1972)). More precisely, the representative agent has rational expectations about future variables and is able to solve the expectation feedback system. Models with a representative and perfect rational agent converge toward the unique stable equilibrium in the long run and every deviation from the equilibrium is only temporary. As stressed by Hommes (2013), models with a perfect rational agent fit into a linear view of a predictable economy. Moreover, in most nonlinear market equilibrium models it is not possible to compute the rational expectation equilibrium, even if the agent knew all equilibrium equations.

Empirical evidence, instead, establishes that traders are heterogeneous and boundedly rational in real life (see for example Chiarella et al. (2009), Hommes (2013), Caiani et al. (2016), Colasante et al. (2017)). This point of view aims at explaining the stylized facts (such as volatility clustering, fat tails of returns, bubbles and crashes) observed in financial markets. Pioneering works on heterogeneous interacting traders in financial markets are for example Day and Huang (1990) and De Grauwe et al. (1995), Brock and Hommes (1998), where different traders, endowed by several behavioral rules, trade in the market. Heterogeneity and bounded rationality introduce non-linearity in the model which is a further element explaining complicated dynamics of prices in the market. Indeed, non-linearity is linked to the fractions of agents trading in the market which are updated at each time (see Hommes (2013) for a survey). A huge amount of deterministic and non-linear works in the heterogeneous agents framework have been proposed. For example, Naimzada and Ricchiuti (2009, 2008) analyze a model with a switching mechanism and they show that complex dynamics can arise even if fundamentalist agents generate different fundamental values. Westerhoff (2004) and Chiarella et al. (2005) show that technical traders can switch between several financial markets. Further, Agliari et al. (2018) develop a stock market model in which participation depends upon an attractivenes measure related to the market activity and the fundamental value of the market. The authors show how the participation mechanism amplifies the occurrence of booms and busts dynamics. Others recent works of deterministic models with heterogeneous agents are, for example, those of Ter Ellen and Verschoor (2018), Hommes and LeBaron (2018) and Polach and Kukacka (2019).

Differently from previous works, our map is two-dimensional, discontinuous, triangular and characterized by two linear branches. We remember that discontinuous maps belong to the more general class of piecewise linear (PWL) maps, which can be subdivided into two groups: continuous and discontinuous. The former have been deeply analyzed and we have a great knowledge of this kind of maps. Some important works in this field are Zhusubaliyev et al. (2001), Sushko et al. (2006), Di Bernardo et al. (2008), Sushko and Gardini (2010), Avrutin et al. (2014). About the latter, there are only partial results, mainly in the one dimensional case (see, for example, Sushko and Gardini (2006), Gardini et al. (2010), Gardini and Tramontana

(2011), Tramontana et al. (2012), Tramontana and Westerhoff (2013), Sushko et al. (2015), Panchuk et al. (2018), Gu and Guo (2019)).

As we will see, in our model the analytical form of the map is due to the introduction of agents which act as imitators. In fact, this type of agents follows the more successful strategy at each time and this new mechanism yields a PWL discontinuous system. Notice that PWL maps allow to consider several peculiarity of economic and financial models since they enlarge the possible final dynamics. For example, they are characterized by a much richer class of bifurcation phenomena, i.e. the border collision bifurcations (BCB) (term introduced by Nusse and Yorke (1992), Nusse and Yorke (1995)) beyond the standard bifurcations. A border collision bifurcation can be defined as any contact between an invariant set and the border separating different regions of definition (see Avrutin et al. (2014)). A peculiarity of this kind of bifurcation is to show the sharp transition to chaos with respect to the standard one and from an economic point of view this yields more unexplored scenarios. An interesting application of these maps could be related to the research field focusing on different states or regimes of the market in a deterministic setting. To this purpose, Chiarella et al. (2012) arguments that price level and volatility tend to move together with different market states, during boom and bust periods, by considering a simple heterogeneous agent model (HAM) with Markov chain regime-dependent expectations. The paper of Ang and Bekaert (2002) finds empirical evidence that stock returns follow a complicated process with multiple regimes, while Guidolin and Timmermann (2007a) and Guidolin and Timmermann (2007b) show that asset returns switch among four states and investor's behavior varies with such states. There are only few papers analyzing different states of the market as proposed by previous authors (Manzan and Westerhoff (2007) and Huang et al. (2010), for example). To this regard, we would like to cite, in particular, Gallegati et al. (2011) and Huang and Zheng (2012) which resume the contribution of Kindleberger (2005), where series of typical patters of speculative bubbles and crashes in the world history are listed. This last paper is able to identify a general pattern followed by most of these bubbles and crashes. Moreover, Rosser (2000) groups the different crises documented by Kindleberger (2005) in three different types according to their depth and length. It is possible to classify them as *sudden crisis*, *smooth crisis*, and *disturbing crisis*. It is important to stress the fact that not only a switching from bull to bear regions (and vice versa) of the market is possible, but also that the price could either stay in the same regime or escape from one regime to another.

To sum up, in this work we study the price dynamics of a financial market with heterogeneous interacting agents. The model includes an imitation component i.e., beyond fundamentalists and chartists, we consider imitators' traders. Our assumption produces a two dimensional piecewise linear discontinuous map, which allows us to widen the analysis of financial markets since it includes a larger variety of bifurcation phenomena and, then, new economic scenarios have to be investigated (as we will see in Section 4).

The contribution of the paper to the existing literature is twofold. From an analytical point of view, we analyze the local properties of the system and we focus on the study of the two-cycle. Moreover, we demonstrate that, when it exists, the two-cycle is symmetric with respect to the bisector and this result simplifies the study. The analytical results are important since two dimensional PWL discontinuous maps which have been studied by an analytical point of view are not much. Secondly, we deeply investigate the economic scenarios arising in our model by numerical simulations, following the work of Nusse and Yorke (1992). To this purpose, we analyze the consequences of different trading reaction of agents finding that chartists play a predominant role in increasing instability in the market, while imitators amplify this effect. Moreover, we investigate the occurrence of the *center bifurcation*, a kind of bifurcation which typically occurs in the case of discontinuous maps, and its economic implications.

The paper is organized as follows. In Section 2, we develop the model. In Section 3, we study some analytical properties of the system (such as fixed points, their local stability and periodic points). In Section 4, we carry out numerical simulations in order to support the analytical results and to show how complexity is exhibited by our model, as a consequence of movements in the parameters of interest. Section 5 concludes.

2. The model

We propose a financial market populated by three types of agents (fundamentalists, chartists and imitators) using a two dimensional piecewise linear discontinuous triangular map and the dynamic equation of price as in Day and Huang (1990). The model includes a *market maker* who adjusts the price basing on *order imbalances*, chartists or technical traders who bet on the persistence of bull and bear markets (i.e. markets where prices are overvalued or undervalued, respectively), two types of fundamentalists who believe in mean reversion (i.e. they expect prices return towards fundamental values).

The market maker adjusts the price at time $(t+1)$ following this rule:

$$P_{t+1} = P_t + a(D_t^f + D_t^c + nD_t^i) \tag{1}$$

where $a > 0$ is a parameter used to adjust the price. We assume $a = 1$. The demand of the chartists is:

$$D_t^C := c(P_t - P_{t-1}) \tag{2}$$

with $c > 0$.

Now, summing and subtracting the fundamental value, we obtain:

$$D_t^c = c(x_t - x_{t-1}) \tag{3}$$

where $x_t = P_t - F$.

Differently from Tramontana (2013) who introduce a non-linear trading rule for chartists, we assume a simple linear forecasting rule for them. We remember that we are using a PWL discontinuous model set up then in the model there is not non-linearity elements. Chartists have not information about the fundamental price so it is obvious for them to consider the price at time $(t-1)$. Chartists are not sophisticated traders and they decide to submit buying order if the deviation of price at time $(t-1)$ with respect to t is greater than zero; in this case they believe on the persistence of bull market. Otherwise, if $P_t - P_{t-1} < 0$, chartists submit selling orders because they believe in the persistence of bear market.

The parameter c is a positive reaction parameter. Chartists submit buying orders when prices are above the fundamental value, while they submit selling

order when prices are below the fundamental value.
The demand of fundamentalists is:

$$D_t^f := f(F - P_t) \tag{4}$$

with $f > 0$.
Remembering the deviation from the fundamental value ($x_t = P_t - F$) and substituting it into the demand function of fundamentalists we arrive to:

$$D_t^f = -f x_t \tag{5}$$

Therefore, fundamentalists submit buying order when prices are below F since in this case they believe that the market is undervalued; while they submit selling orders when prices are above F because they believe that the market is overvalued. We consider the same demand function of Tramontana (2013) for fundamentalists, and f (as for chartists) represents a positive reaction parameter.

The other new ingredient that we introduce in the model is the demand function of *imitators*. We follow the same approach as in Tramontana (2013) (though we consider a linear demand function for chartists): we use a simple forecasting rule which is known in the economic literature but it has not been applied in the framework we propose. Imitators use a very simple heuristic rule to form their expectations, in fact they look only at P_t and P_{t-1}: if P_t is closer than P_{t-1} to the fundamental value, then they conclude that fundamentalists' strategy has been successful and they imitate them at time $(t + 1)$.

Hence, the demand function of imitators is given by:

$$D_t^i = \begin{cases} i(F - P_t) + \mu_1 & \text{if} \quad |F - P_t| \geq |F - P_{t-1}| \\ -i(F - P_t) + \mu_2 & \text{if} \quad |F - P_t| < |F - P_{t-1}| \end{cases} \tag{6}$$

Considering $x_t = P_t - F$, we obtain:

$$D_t^i = \begin{cases} -ix_t + \mu_1 & \text{if} \quad |x_t| \geq |x_{t-1}| \\ ix_t + \mu_2 & \text{if} \quad |x_t| < |x_{t-1}| \end{cases} \tag{7}$$

where, differently from Tramontana (2013), we introduce the parameters μ_1 and μ_2 which represent the costs of imitation, that is the costs to imitate the best strategy. Moreover, these parameters are allowed to assume negative values meaning that imitators are able to freely acquire relevant information about the strategy to imitate. Then, the more these parameters are negative the simpler is acquiring information for imitators. Now we have to distinguish different possible scenarios which can appear when these three kinds of agents interact. We start by considering imitators. We obtain two different systems:

System 1

If $\{x_t \geq |x_{t-1}|\} \cup \{x_t \leq -|x_{t-1}|\}$ then:

$$\begin{cases} D_t^f = -fx_t \\ D_t^c = c(x_t - x_{t-1}) \\ D_t^i = -ix_t + \mu_1 \end{cases}$$

System 2

If $\{-|x_{t-1}| < x_t < |x_{t-1}|\}$ then:

$$
\begin{cases}
D_t^f = -f x_t \\
D_t^c = c(x_t - x_{t-1}) \\
D_t^i = i x_t + \mu_2
\end{cases}
$$

We put together the different demand functions into the price equation, so that we can obtain the complete model involving imitators. The price equation written in terms of deviation from the fundamental value we introduced before is:

$$
x_{t+1} = x_t + (D_t^f + D_t^c + nD_t^i) \tag{8}
$$

where

$$
\begin{aligned}
D_t^c &= D_t^c(x_{t-1}, x_t) \\
D_t^f &= D_t^f(x_t) \\
D_t^i &= D_t^i(x_t)
\end{aligned}
$$

therefore

$$
x_{t+1} = x_t + (D_t^f(x_t) + D_t^c(x_{t-1}, x_t) + nD_t^i(x_t)) \tag{9}
$$

As in Tramontana and Westerhoff (2013) we focus on the role of imitators, in other terms we assume that the number of fundamentalists and chartists are both normalized to one, that is fundamentalists and chartists are presented in the market in equal number (both normalized to one). Differently, the number of imitators (n) is not fixed. For example, a value of the parameter n equal to 2 means that imitators are twice the number of fundamentalists or chartists, that is a half of the total number of traders. Therefore, according to the previous considerations, we assume $n \in (0, +\infty)$ which implies that there are not constraints for agents participation trading.

3. The dynamical system

Basing on the previous analytical equations, we get our final dynamical system. Equation (9) defines a second-order difference equation, since it is characterized by a time delay (price of tomorrow depends on yesterday and today). Hence, by introducing the state variable $y_t = x_{t-1}$, we obtain the following two-dimensional dynamical system:

$$\begin{cases} x_{t+1} = x_t + (D_t^f(x_t + D_t^c(x_t, y_t) + nD_t^i(x_t)) \\ y_{t+1} = x_t \end{cases}$$

We study the complete model with the presence of imitators, as a consequence it is defined by the map $T = T_1 \cup T_2$, where:

$$T_1 : \begin{cases} x_{t+1} = f_1(x_t, y_t) = (1 - f + c - ni)x_t - cy_t + n\mu_1 & \text{if} \quad |x_t| \geq |y_t| \\ y_{t+1} = g_1(x_t) = x_t \end{cases} \tag{10}$$

$$T_2 : \begin{cases} x_{t+1} = f_2(x_t, y_t) = (1 - f + c + ni)x_t - cy_t + n\mu_2 & \text{if} \quad |x_t| < |y_t| \\ y_{t+1} = g_2(x_t) = x_t \end{cases} \tag{11}$$

with $f, c, n, i, \mu_1, \mu_2 > 0$.

Our map is two dimensional, piecewise linear and discontinuous,[1] describing the evolution of prices (expressed by deviations from the fundamental). It is composed by two maps, T_1 defined in the region $R_1 = \{(x_t, y_t) : -|x_t| \leq y_t \leq |x_t|\}$ and T_2 defined in $R_2 = \{(x_t, y_t) : y_t > |x_t|\} \cup \{(x_t, y_t) : y_t < -|x_t|\}$. Hence, the border which separates the state space into two regions is $\Gamma = \{(x_t, y_t) : y_t = x_t\} \cup \{(x_t, y_t) : y_t = -x_t\}$.

Moreover, the second component of the map T_i $(i = 1, 2)$ does not depend on y_t hence T_i has a triangular form (about triangular maps see Gardini and Mira (1993), Kolyada (1992), Kolyada and Sharkovsky (1991)). In what follow we study the fixed points and we carry out the local stability analysis of systems T_1 and T_2.

3.1. Fixed points and local stability analysis

The following proposition defines the fixed points owned by the system.

Proposition 1. *Map T admits a real fixed point for any range of the parameter values, defined by $E^\star = \left(\frac{n\mu_1}{f+ni}, \frac{n\mu_1}{f+ni}\right)$. Moreover, if $f \neq ni$ then map T admits a virtual fixed point given by $E^0 = \left(\frac{-n\mu_2}{ni-f}, \frac{-n\mu_2}{ni-f}\right)$.*

Proof. The equilibrium points (or steady states) of map T are the solutions of the algebraic system $T_1(x, y) = (x, y)$ if and only if $(x, y) \in R_1$ and of $T_2(x, y) = (x, y)$ if and only if $(x, y) \in R_2$. After simple algebra we obtain $E^\star$ and E^0. □

[1]Notice that the presence of imitators in the model is responsible for the discontinuity of the final map, as explained in the previous section.

Notice that for $f \neq ni$ the system $T_2(x,y) = (x,y)$ admits the unique solution E^0 which does not belong to the region R_2, such a fixed point is called *virtual* steady state.

Moreover, both the fixed points belong to the border Γ for any choice of the parameter values. Given this fact, the local stability analysis of the steady state $E^\star$ depends on both the linear maps T_1 and T_2.

Let us firstly consider T_1, the Jacobian matrix is defined as follows:

$$J_1 = \begin{pmatrix} 1 - f + c - ni & -c \\ 1 & 0 \end{pmatrix} \tag{12}$$

whose trace and determinant are $\tau_1 = 1 + c - f - ni$ and $\delta_1 = c$, respectively.

Hence, we can prove the following proposition on the stability of the linear map T_1.

Proposition 2. *By looking at map T_1, two possibilities arise:*

1. *Let $(1 + c - f - ni)^2 - 4c < 0$ then the steady state of T_1 is a focus: attracting for $c < 1$, otherwise (for $c > 1$) it is an unstable focus.*
2. *Let $(1 + c - f - ni)^2 - 4c > 0$ then the steady state of T_1 is a node: attracting for $c > \frac{f+ni}{2} - 1$ and $c < 1$, otherwise for $c < \frac{f+ni}{2} - 1$ it is a saddle point.*

Proof. According to the well-known triangle of stability in the plane (τ, δ) (see Medio and Lines (2001)), we obtain that if $\tau_1^2 \geq 4\delta_1$ then eigenvalues are real (a repeated real eigenvalues for $\tau_1^2 = 4\delta_1$). For $\tau_1^2 > 4\delta_1$ the fixed point is a node, while if $\tau_1^2 < 4\delta_1$ the fixed point is a focus. Moreover, it is attracting when the following conditions hold:[2]

$$\begin{cases} c > \frac{f+ni}{2} - 1 \\ c < 1 \end{cases}.$$

Then the proposition is proved. □

Thus, under the assumption $(1+c-f-ni)^2 - 4c < 0$ the fixed point loses stability for $c = 1$ when the pair of complex conjugate eigenvalues crosses the unit circle, giving rise to a center bifurcation.[3] This kind of bifurcation typically occurs in the families of two-dimensional piecewise linear maps. We refer to Sushko and Gardini (2008) for a detailed description of similarities and differences from the corresponding Neimark-Sacker smooth bifurcation. Differently, when $(1+c-f-ni)^2 - 4c > 0$, the fixed point loses stability since a real eigenvalue crosses -1.

Following the same steps, it is possible to attain to the stability conditions for the linear map T_2. Even if T_2 has not equilibrium points in the region of definition R_2, its eigenvalues are important in understanding the dynamics around the steady state $E^\star$ since it belongs to the border Γ separating the regions R_1 and R_2.

In this case the Jacobian matrix is defined by:

$$J_2 = \begin{pmatrix} 1-f+c+ni & -c \\ 1 & 0 \end{pmatrix} \tag{13}$$

[2]Notice that this system requires $f + ni < 4$, otherwise it would be an empty set.

[3]Although previous conditions are not sufficient for the existence of the associated bifurcations (especially in our case in which the fixed point belongs to the border), their combination with numerical simulations becomes strong evidence, as we will see in the next section.

whose trace and determinant are $\tau_2 = 1 + c - f + ni$ and $\delta_2 = c$, respectively.

Consequently, the following proposition holds:

Proposition 3. *By looking at map T_2, the following conditions hold:*

1. *Let $(1 + c - f + ni)^2 - 4c < 0$ then the steady state of T_2 is a focus: attracting for $c < 1$, otherwise (for $c > 1$) it is an unstable focus.*
2. *Let $(1 + c - f + ni)^2 - 4c > 0$ then the steady state of T_2 is a node: attracting for $c > \frac{f-ni}{2} - 1$ and $f - ni > 0$, a saddle point for $c < \frac{f-ni}{2} - 1$ or $f - ni < 0$.*

Proof. Following the same steps of the proof of Proposition 2, one can obtain the result. □

Thanks to Propositions 2 and 3, it is possible to understand some important properties of the whole system $T = T_1 \cup T_2$. More precisely, given the fact that both the maps are linear, when the fixed point is stable for T_1 and T_2 (i.e. if $c < 1$ and $c > \frac{f-ni}{2} - 1$ and $f - ni > 0$) both the maps are contracting.

In the case of an unstable node for T_1 and T_2, i.e.:

$$\begin{cases} f - ni > 0 \\ c < \frac{f-ni}{2} \\ (1 + c - f - ni)^2 - 4c > 0 \\ (1 + c - f + ni)^2 - 4c > 0 \end{cases} \quad \text{or} \quad \begin{cases} f - ni < 0 \\ c < \frac{f+ni}{2} \\ (1 + c - f - ni)^2 - 4c > 0 \end{cases}$$

we have divergent dynamics.

Differently, when the equilibrium point is an unstable focus, i.e.:

$$\begin{cases} c > 1 \\ (1 + c - f - ni)^2 - 4c < 0 \\ (1 + c - f + ni)^2 - 4c < 0 \end{cases}$$

we can have even bounded dynamics.

Since we are interested in the origin of complicated dynamics, we would like to study the focus for T_1, the following proposition better investigates this case.[4]

Proposition 4. *If $f < ni$ and $c \in (c_1^\star, c_2^\star)$ with $c_{1,2}^\star = (1+f+ni) \pm 2\sqrt{f + ni}$, then the steady state is a focus for T_1 and a saddle for T_2.*

Proof. Define $f(c) = (1 + c - f - ni)^2 - 4c$ and $g(c) = (1 + c - f + ni)^2 - 4c$. Both functions f and g have a unique minimum point given by $c_f = f+1+ni$ and $c_g = f + 1 - ni$, respectively. Hence, the condition $g(c_g) > 0$ (which is verified for $f > ni$) guarantees $(1 + c - f + ni)^2 - 4c > 0$ $\forall c$, so that the steady state of T_2 is a node. Notice that, according to Proposition 3, it is a saddle point being $f - ni < 0$.

Differently, $f(c_f)$ is always negative. Since $f(0) > 0$ and $\lim_{c \to +\infty} f(c) > 0$ then $c_1^\star > 0$ and $c_2^\star > 0$ do exist such that $f(c_1^\star) = f(c_2^\star) = 0$. By solving $f(c) = 0$ we obtain $c_{1,2}^\star = (1+f+ni) \pm 2\sqrt{f + ni}$. As a consequence, $f(c) < 0$ for $c \in (c_1^\star, c_2^\star)$ and the fixed point of T_1 is a focus.

□

[4] See Sushko and Gardini (2006) for further details on the role of the maps T_1 and T_2 in the analysis of System T.

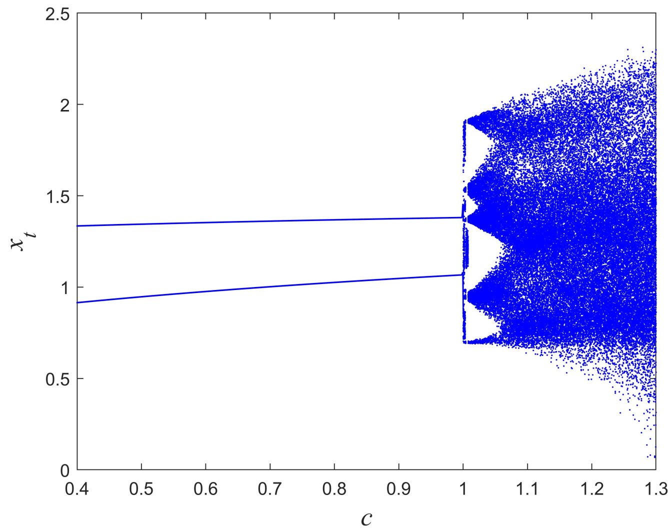

Figure 1: Bifurcation diagram with respect to c for $f = 0.3$, $n = 0.7$, $i = 0.8$, $\mu_1 = 0.8$, $\mu_2 = 0.5$ and i.c. $x_0 = 0.3$, $y_0 = 0.5$

Studying the map T numerically we get interesting bifurcation diagrams. In Figure 1 we plot the bifurcation diagram of x with respect to $c \in (c_1^\star, c_2^\star)$ when the hypothesis of Proposition 4 holds, showing complexity for $c > 1$ when the focus of T_1 loses stability.

3.2. Periodic points

Given the stability analysis of the previous section, we would like to analyse periodic points which arise for some parameter values, in particular we focus on investigating the occurrence of a 2−cycle.

We firstly consider its localization on the plane, according to the following proposition.

Proposition 5. *When a two-cycle $c_2 = \{(x_1, y_1), (x_2, y_2)\}$ exists, its points are symmetric with respect to the line $\{y = x\} \subset \Gamma$.*

Proof. Trivially, a two-cycle $c_2 = \{(x_1, y_1), (x_2, y_2)\}$ is such that

$$\begin{cases} T(x_1, y_1) = (x_2, y_2) \\ T(x_2, y_2) = (x_1, y_1) \end{cases}.$$

This means that

$$\begin{cases} y_2 = x_1 \\ y_1 = x_2 \end{cases}.$$

$\square$

Hence, when the $2-$period cycle is inside the region $R = R_1 \cup R_2$, we have that: $(x_1, y_1) \in R_1$(respectively R_2)$\Leftrightarrow$ $(x_2, y_2) \in R_2$(respectively R_1).

As a consequence, we can suppose $(x_1, y_1) \in R_1$ in order to obtain conditions under which the $2-$cycle has to exist, as proved in the following result.

Proposition 6. *Let $k_1 = 1-f+c-ni$ and $k_2 = 1-f+c+ni$. For $(1+c)^2 - k_1k_2 \neq 0$, $k_2\mu_1 - k_1\mu_2 + (1+c)(\mu_2 - \mu_1) \neq 0$ and $k_2\mu_1 + k_1\mu_2 + (1+c)(\mu_2 + \mu_1) \neq 0$ system T admits a cycle of period two defined as $c_2 = \{(x_1, y_1), (x_2, y_2)\}$ with:*

$$\begin{cases} x_1 = \frac{k_2 n\mu_1 + (1+c)n\mu_2}{(1+c)^2 - k_1k_2} \\ y_1 = x_2 \end{cases} \quad and \quad \begin{cases} x_2 = \frac{k_1 n\mu_2 + (1+c)n\mu_1}{(1+c)^2 - k_1k_2} \\ y_2 = x_1 \end{cases}.$$

Proof. According to Proposition 5 we can suppose $(x_1, y_1) \in R_1$ and $(x_2, y_2) \in R_2$ consequently:

$$\begin{cases} (x_1, y_1) = T_2(x_2, y_2) \\ (x_2, y_2) = T_1(x_1, y_1) \end{cases} \quad that \ \ is \quad \begin{cases} x_1 = k_2x_2 - cx_1 + n\mu_2 \\ x_2 = k_1x_1 - cx_2 + n\mu_1 \end{cases}$$

which admits a unique solution for $(1+c)^2 - k_1k_2 \neq 0$. In this case, the solution is

$$\begin{cases} x_1 = \frac{k_2 n\mu_1 + (1+c)n\mu_2}{(1+c)^2 - k_1k_2} \\ y_1 = x_2 \end{cases} \quad \text{and} \quad \begin{cases} x_2 = \frac{k_1 n\mu_2 + (1+c)n\mu_1}{(1+c)^2 - k_1k_2} \\ y_2 = x_1 \end{cases}.$$

Condition $k_2\mu_1 + k_1\mu_2 + (1+c)(\mu_2 + \mu_1) \neq 0$ guarantees $x_1 \neq -x_2$ in order to exclude the case in which c_2 belongs to the line $\{y = -x\} \subset R_1$.[5] Similarly, if $k_2\mu_1 - k_1\mu_2 + (1+c)(\mu_2 - \mu_1) \neq 0$ then c_2 does not belong to the line $\{y = x\}$. □

Thanks to Proposition 6, it is possible to obtain conditions under which the points of c_2 collides with the border Γ. In other words, c_2 is a border crossing periodic point since it crosses the border for some parameter values. Conditions under which the contact of the periodic point with Γ arises are defined in the following result.

Corollary 1. *Let* $k_1 = 1-f+c-ni$, $k_2 = 1-f+c+ni$ *and* $(1+c)^2 - k_1k_2 \neq 0$, *then the points of the* $2-$*cycle collide with the border* Γ *for* $k_2\mu_1 - k_1\mu_2 + (1+c)(\mu_2 - \mu_1) = 0$ *or* $k_2\mu_1 + k_1\mu_2 + (1+c)(\mu_2 + \mu_1) = 0$.

Proof. This result directly comes from Proposition 6. □

Notice that, even though the equilibrium point lies on the border for any choice of the parameters, the system is able to generate border-collision bifurcations due to some periodic orbit. In the next section we investigate the non-smooth bifurcations, particularly the border-collision bifurcation phenomena.

[5]In this case both the points of the cycle belong to R_1 and the periodic point can be obtained by applying T_1.

4. Numerical analysis and border-collision phenomena

In this section we mainly explore the border-collision bifurcations which occur when a trajectory collides with the boundary separating regions in which the system changes its definition. This kind of bifurcations has been explored for piecewise smooth maps that are continuous across the border (Banerjee et al. (2000), Zhusubaliyev et al. (2001), Di Bernardo et al. (2008), Avrutin et al. (2016)). Differently, our map is piecewise discontinuous. Only in recent years piecewise smooth maps which are discontinuous on the border have been applied to explain bifurcations in Economics and Finance (Huang et al. (2010), Tramontana et al. (2010), Tramontana et al. (2011), Tramontana and Westerhoff (2016), Gu (2017)), and the theory for understanding such phenomena is not completely available yet.
In order to describe, qualitatively, the different scenarios which can occur, we follow Nusse and Yorke (1992) and we make use of the terminology according to which a *period p attractor* is an attracting periodic orbit of period p.

In order to perform the numerical analysis, we take into account three cases that are, in our opinion, economically interesting. Indeed, we are focusing on the role of the costs of imitation (μ_1 and μ_2) and the bifurcation diagrams we have generated are linked to this assumption, in detail we take as reference parameter the cost of imitation μ_1. Further, in our simulations we have considered what occurs when fundamentalists are more aggressive than chartists and vice versa and what is the role played by imitators in both cases. In Figure 2 panel A, the bifurcation diagram with respect to μ_1 shows the occurrence of a border collision bifurcation from a period 3 to a 1-piece

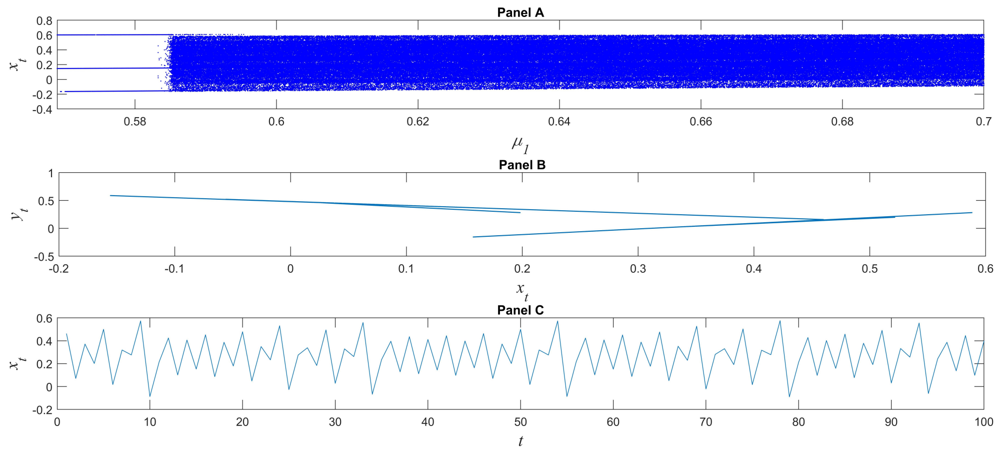

Figure 2: In Panel A the bifurcation diagram with respect to μ_1 is represented for $f = 1.45$, $c = 0.27$, $n = 1$, $i = 0.975$, $\mu_2 = 0.4401$ and i.c. $x_0 = -0.5$, $y_0 = 2$. Panel B shows the attractor in the (x,y) plane for $\mu_1 = 0.6$ while other parameter values as in Panel A. Panel C is the trajectory in the plane with parameter values as in Panel A and $\mu_1 = 0.65$.

chaotic attractor for[6] $\mu_1 \cong 0.59$. In panel B, the chaotic strange attractor is plotted for $\mu_1 = 0.6$. Notice that this economic example describes the situation in which fundamentalists are more aggressive than chartists ($f = 1.45$, $c = 0.27$) while the number of imitators are equal to that of fundamentalists and chartists and they are enough active (and more aggressive than chartists) in the market. Although fundamentalists are the most aggressive in the market, instability prevails when the cost faced by imitators to imitate fundamentalists is too high.

Another interesting situation is represented in Figure 3 panel A in which the bifurcation diagram with respect to μ_1 shows a border collision bifurca-

[6]Notice that in our model prices x_t can be negative since they are expressed as deviations from the fundamental value.

tion from a 2-cycle to a q-piece chaotic attractor for $\mu_1 \cong 0.16$. We underline that, according to the parameter values considered in this numerical simulation, the analytical condition of Corollary 1 is verified (i.e. $\mu_1 = \frac{-(k_1+1+c)\mu_2}{k_2+1+c}$, with $k_{1,2} = 1-f+c\mp ni$) and the points of the $2-$cycle collide with the border Γ, as proved in the previous section. In order to better investigate what happens after this bifurcation, panel B shows the attractor for $\mu_1 = 0.5$. Notice that now chartists are more aggressive than fundamentalists and imitators are the most responsive in the market ($i = 2.1$). An interesting element of this scenario is the cost of imitation (μ_1 or μ_2). More precisely, in this simulation we leave these parameters to take all real values. The purpose is to consider that a positive value of $\mu_i, i = 1, 2$ means a positive cost to imitate the best strategy (fundamentalist or chartists' strategy), while a negative value of the parameter μ_i means that imitators dispose of all relevant information to take their decision or differently, the market subsumes these information and imitators are able to obtain them easily. Looking at Figure 3, imitators follow indifferently both chartists and fundamentalists until they can replicate their strategy freely (represented by a negative value of $\mu_{1,2}$) or with very low costs, otherwise they trade randomly causing a period of higher instability in the market. The last scenario is described[7] in Figure 4, which seems to be characterized by a border collision bifurcation from a 2-cycle to a q-cycle. In the market after a period of stability represented by

[7]We stress the fact that the periodic cycles represented in the diagram could be of the higher order depending on the precision of the simulation. In fact, the points which appear in the figure become lines if we increase the precision. Nevertheless, we prefer to present this diagram for its major clearness.

the stable 2-cycle, a period of uncertainty is born. In this period the market faces different degree of imitation due to the higher costs sustained by imitators to follow fundamentalists causing the existence of several level of price.

Finally, we conclude the paragraph with a novelty in the financial market literature, that is the *center bifurcation* occurring in our model. This kind of bifurcation has been studied by Sushko et al. (2003) and Sushko and Gardini (2008), and it is a peculiarity of piecewise maps. The center bifurcation is associated to points having complex eigenvalues, in particular describing the transition of a fixed point to an unstable focus and the appearance of an attracting closed invariant curve. There is a certain similarity with the Neimark-Sacker bifurcation associated to the smooth maps, but in the piecewise linear maps the periodicity regions may be classified with respect to the rotation numbers and the boundaries of these periodicity regions are border collision bifurcations instead of saddle-node bifurcation curves (see Sushko and Gardini (2008) for a detailed analysis). About our model, we propose the attractors on the plane (x, y). In particular, Figures 5 and 6 show what happens at the bifurcation value[8] $c = 1$ and immediately after the center bifurcation, respectively. It stresses the predominant role of chartists with respect to the other two agents presented in the market in order to generate complicated dynamics, while imitators amplify the consequences of the chartists' trading strategies. Most fully, both panels (b) of Figures (5) and (6) show the trajectories associated to the respectively attractors. In these

[8] We remember that the parameter c represents the reaction of chartists.

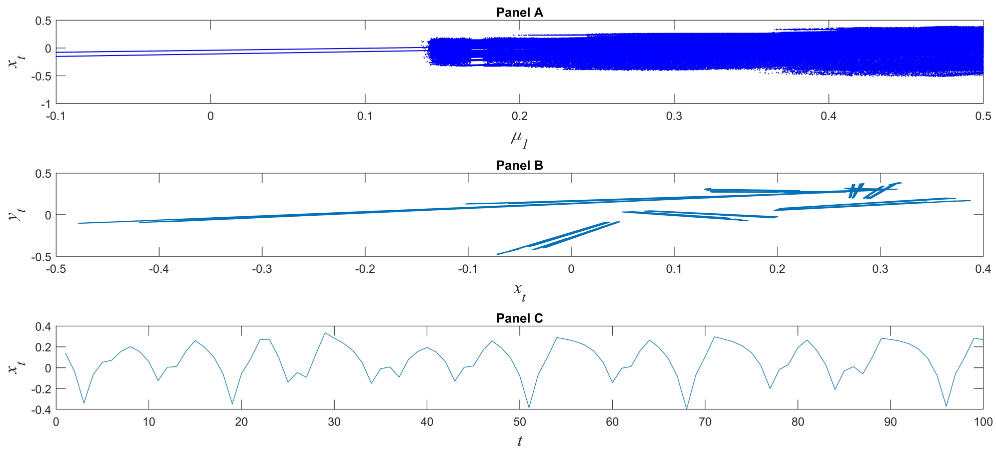

Figure 3: In Panel A the bifurcation diagram with respect to μ_1 is represented for $f = 0.4$, $c = 0.95$, $n = 0.38$, $i = 2.1$, $\mu_2 = -0.3$ and i.c. $x_0 = 0.5$, $y_0 = 0.8$. Panel B shows the attractor in the (x,y) plane for $\mu_1 = 0.5$ while other parameter values as in Panel A. Panel C is the trajectory in the plane with parameter values as in Panel A and $\mu_1 = 0.4$.

panels we can observe the occurrence of market regimes. In detail, in Figure (5) we observe a cyclical dynamic of price but inside this kind of behaviour we can observe price micro-oscillations before and after the large one. These small oscillations, although of short period, represent specific regimes of the market where the price stays inside two resistances. Moreover, from both Panels (b) of Figures (5) and (6) we can note that oscillations are more pronounced in the bull regime (increasing price trend) than in the bear one (decreasing price trend). We leave the analytical treatment of this kind of bifurcation for a further research, but in our opinion it was important to remark the economic meaning described by this scenario.

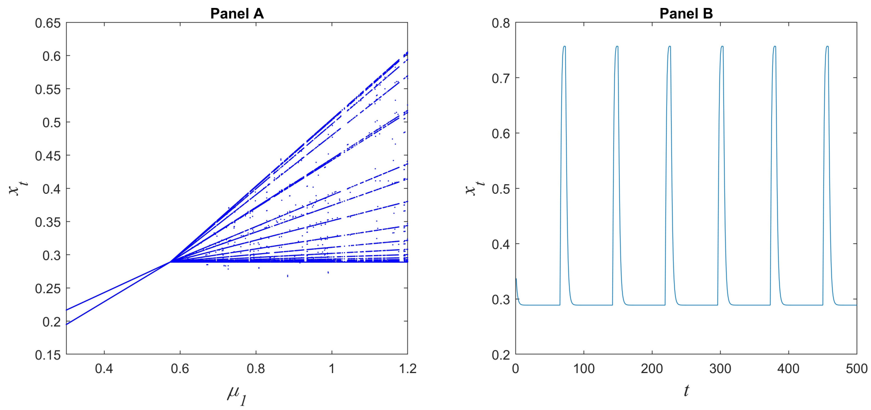

Figure 4: On the left, bifurcation diagram with respect to μ_1 for $f = 0.337$, $c = 0.1953$, $n = 0.2$, $i = 0.3005$, $\mu_2 = 0.4$ and i.c. $x_0 = -0.5$, $y_0 = 0.3$. On the right, trajectory in the plane with the same parameter values of bifurcation diagram and $\mu_1 = 1.5$.

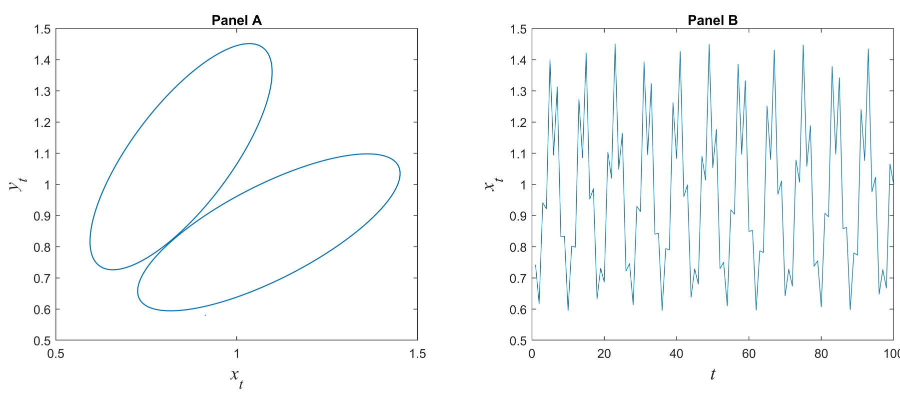

Figure 5: In Panel(a) the attractor in the plane (x, y) is represented when $c = 1$, $f = 0.4$, $n = 0.7$, $i = 0.8$, $\mu_1 = 0.8$, $\mu_2 = 0.5$ and i.c. $x_0 = 0.25$, $y_0 = 0.24$. Panel(b) shows the trajectory in the plane with the same parameter values of Panel(a) and $c = 1$.

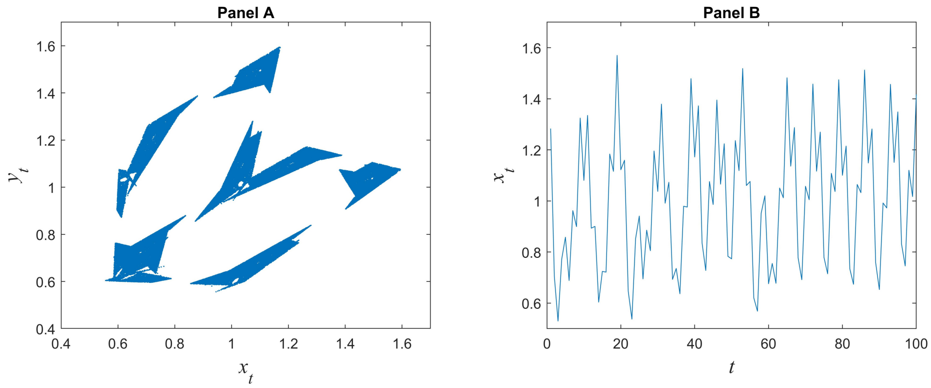

Figure 6: In Panel(a) the attractor in the plane (x, y) is represented when $c = 1.05$, $f = 0.4$, $n = 0.7$, $i = 0.8$, $\mu_1 = 0.8$, $\mu_2 = 0.5$ and i.c. $x_0 = 0.25$, $y_0 = 0.24$. Panel(b) shows the trajectory in the plane with the same parameter values of Panel(a) and $c = 1.1$.

5. Conclusions

In this paper we studied the dynamics of a financial market populated by three different kinds of agents: fundamentalists, chartists, and imitators. Our contribution is aimed at applying the theory of discontinuous maps to financial markets. Most fully, as we have stressed in the previous sections, a peculiarity of these kind of maps is the occurrence of the border collision bifurcations which lead to a sharp transition to chaos generating new interesting economic scenarios. In the first part of the work a full analytical treatment of the model has been provided. We also focus on the study of the two-cycle demonstrating that when it exists it is always symmetric with respect to the bisector. The second part of the paper concerns with numerical simulations. We find that there is a coexistence of prices that suddenly arise when a border collision bifurcation appears. Coexistence of prices means that agents face a large list of prices and they are not able to decide which is the best one for them. In our model, imitators are the agents who should do the difference in the market because, following fundamentalists or chartists, they give rise to a majority and, consequently, they impact on the market dynamics in a bull or bear direction. Without a defined trend in the market, agents lose confidence in their beliefs (trading rules) and they start to trade almost randomly. Simulation results show that the relative degree of aggressiveness of fundamentalists and chartists, through affecting the information costs for imitators, affects the stability of the market, from a stable market characterized by a fixed point to a more complex scenario. Finally, our model is able to show the occurrence of the *center bifurcation*, a kind of bifurcation typical for discontinuous maps. We have focused on the economic meaning of this bifurcation leaving the analytical part for further developments. In detail, when a center bifurcation occurs in our model the price faces different regimes where it remains for a short period between two resistances (that is two bounds). Moreover we have seen that these regimes are more pronounced in the case of bull dynamics than in the bear ones.

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