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CONCENTRATION PHENOMENA FOR A CLASS OF FRACTIONAL KIRCHHOFF EQUATIONS IN $\mathbb{R}^N$ WITH GENERAL NONLINEARITIES

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Abstract. In this paper we study the following class of fractional Kirchhoff problems:

$$\begin{cases} \varepsilon^{2s} M(\varepsilon^{2s-N}[u]_s^2)(-\Delta)^s u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $N \geq 2$, $(-\Delta)^s$ is the fractional Laplacian, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function, $M : [0,\infty) \to \mathbb{R}$ is a Kirchhoff function satisfying suitable conditions and $f : \mathbb{R} \to \mathbb{R}$ fulfills Berestycki-Lions type assumptions of subcritical or critical type. Using suitable variational arguments, we prove the existence of a family of positive solutions (u_ε) which concentrates at a local minimum of V as $\varepsilon \to 0$.

1. Introduction

1.1. **Main results.** In this paper we deal with the following class of fractional Kirchhoff problems:

$$\begin{cases} \varepsilon^{2s} M(\varepsilon^{2s-N}[u]_s^2)(-\Delta)^s u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $N \geq 2$, M is a Kirchhoff function, V is a positive potential and f is a continuous nonlinearity. The nonlocal operator $(-\Delta)^s$ appearing in (1.1) is the so called fractional Laplacian operator defined for smooth functions $u : \mathbb{R}^N \to \mathbb{R}$ by

$$(-\Delta)^s u(x) = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where $C(N,s)$ is a positive normalizing constant, and $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space of functions $u \in L^2(\mathbb{R}^N)$ such that

$$[u]_s^2 := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \sqrt{[u]_s^2 + |u|_2^2}.$$

We recall that Fiscella and Valdinoci [31] proposed for the first time a stationary fractional Kirchhoff model in a bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions and involving a critical nonlinearity:

$$\begin{cases} M\left([u]_s^2\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where M is a continuous Kirchhoff function whose prototype is given by $M(t) = a + bt$ with $a > 0$ and $b \geq 0$, $\lambda > 0$ is a parameter and f is a continuous function with subcritical growth.
Their model generalizes in the fractional context the well-known Kirchhoff model introduced by Kirchhoff [44] as an extension of the classical d'Alembert wave equation. For some interesting existence and multiplicity results for Kirchhoff problems in the classic setting, we refer to [2,27,28,35,45,50] and the references therein. In the fractional framework, after the pioneering work [31], many authors focused on fractional Kirchhoff problems set in bounded domains or in the whole space and involving nonlinearities with subcritical or critical growth; see for instance [10,30,42,43,46] and the references therein for unperturbed problems (that is when

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$\varepsilon = 1$ in (1.1)), and [9, 11] for some existence and multiplicity results for perturbed problems (that is when $\varepsilon > 0$ is sufficiently small).

On the other hand, when $M(t) \equiv 1$, equation (1.1) boils down to a nonlinear fractional Schrödinger equation of the type

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = h(x, u) \text{ in } \mathbb{R}^N, \tag{1.3}$$

proposed by Laskin [40] as a result of expanding the Feynman path integral, from the Brownian like to the Lévy like quantum mechanical paths. Equation (1.3) has been object of investigation in these last two decades and several existence and multiplicity results have been obtained under different conditions on V and h; see [5,7,21,25,26] and the references therein. In a particular way, a great attention has been devoted to the existence and concentration phenomenon as $\varepsilon \to 0$ of positive solutions to (1.3); see [3,6,22,29,34,36,39,47]. Motivated by the above works, the goal of this paper is to study the existence and concentration of positive solutions to (1.1) under very general assumptions on the Kirchhoff function M and the nonlinearity f. We always suppose that $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function which satisfies the following conditions due to del Pino and Felmer [23]:

$(V1)$ $V_1 := \inf_{x\in\mathbb{R}^N} V(x) > 0$,

$(V2)$ there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

We also set $\mathcal{M} := \{x \in \Lambda : V(x) = V_0\}$. Without loss of generality, we may assume that $0 \in \mathcal{M}$.

Concerning the Kirchhoff function M, we suppose that $M : [0,\infty) \to \mathbb{R}_+$ is continuous and such that:

$(M1)$ there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$,

$(M2)$ $\liminf_{t\to\infty} \left[\widehat{M}(t) - (1 - \frac{2s}{N})M(t)t\right] = \infty$, where $\widehat{M}(t) := \int_0^t M(\tau)\, d\tau$,

$(M3)$ $M(t)/t^{\frac{2s}{N-2s}} \to 0$ as $t \to \infty$,

$(M4)$ M is nondecreasing in $[0,\infty)$,

$(M5)$ $t \mapsto M(t)/t^{\frac{2s}{N-2s}}$ is nonincreasing in $(0,\infty)$.

We note that, if $s = 1$, the above assumptions have been used in [28]. Clearly, $M(t) = m_0 + bt$, with $b \geq 0$, satisfies $(M1)$-$(M5)$ when $b = 0$, $N \geq 2$ and $s \in (0, 1)$, and $N = 3$, $s \in (\frac{3}{4}, 1)$ whenever $b > 0$.

In the first part of the paper, we require that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t \leq 0$ and fulfills the following Beresticky-Lions type assumptions [12]:

(f_1) $\lim_{t\to0} \frac{f(t)}{t} = 0$,

(f_2) $\limsup_{t\to\infty} \frac{f(t)}{t^p} < \infty$ for some $p \in (1, 2^*_s - 1)$, where $2^*_s := \frac{2N}{N-2s}$ is the fractional critical exponent,

(f_3) there exists $T > 0$ such that $F(T) > \frac{V_0}{2}T^2$, where $F(t) := \int_0^t F(\tau)\, d\tau$.

The first main result of this work can be stated as follows:

Theorem 1.1. *Assume that $(V1)$-$(V2)$, (M_1)-(M_5) and (f_1)-(f_3) are satisfied. When $s \in (0, \frac{1}{2}]$, we also assume that $f \in C^{0,\alpha}_{loc}(\mathbb{R})$ for some $\alpha \in (1 - 2s, 1)$. Then, for small $\varepsilon > 0$, there exists a positive solution u_ε to (1.1). Moreover, there exists a maximum point $x_\varepsilon \in \mathbb{R}^N$ of u_ε such that $\lim_{\varepsilon\to0} dist(x_\varepsilon, \mathcal{M}) = 0$, and for any such x_ε, $v_\varepsilon(x) = u_\varepsilon(\varepsilon\, x + x_\varepsilon)$ converges, up to a subsequence, in $H^s(\mathbb{R}^N)$ to a least energy solution of the limiting problem*

$$M([u]_s^2)(-\Delta)^s u + V_0 u = f(u) \text{ in } \mathbb{R}^N.$$

In particular, there exists a constant $C > 0$, independent of $\varepsilon > 0$, such that

$$u_\varepsilon(x) \leq \frac{C\, \varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - x_\varepsilon|^{N+2s}} \quad \forall x \in \mathbb{R}^N.$$

Remark 1.1. *The restrictions on the regularity on f are only used to obtain the better regularity of solutions to (1.1) which guarantees the Pohozaev identity (see Proposition 1.1 in [16]).*

In the second part of this paper, we consider (1.1) by requiring that f satisfies the following Beresticky-Lions type assumptions of critical growth [52], that is f fulfills (f_1) and

(f_2') $\lim_{t\to\infty} \frac{f(t)}{t^{2^*_s-1}} = 1$,

(f_3') there exist $\lambda > 0$ and $p < 2_s^*$ such that

$$f(t) \geq t^{2_s^*-1} + \lambda t^{p-1} \quad \forall t \geq 0,$$

where $\lambda > 0$ is such that

- $p \in (2, 2_s^*)$ and $\lambda > 0$ if $N \geq 4s$,
- $p \in (\frac{4s}{N-2s}, 2_s^*)$ and $\lambda > 0$ if $2s < N < 4s$,
- $p \in (2, \frac{4s}{N-2s}]$ and $\lambda > 0$ is sufficiently large if $2s < N < 4s$.

Then, the second main result of this paper is the following:

Theorem 1.2. *Assume that (V_1)-(V_2), (M_1)-(M_5) and (f_1), (f_2')-(f_3') are satisfied. When $s \in (0, \frac{1}{2}]$, we also assume that $f \in C^{0,\alpha}_{loc}(\mathbb{R})$ for some $\alpha \in (1-2s, 1)$. Then, for small $\varepsilon > 0$, there exists a positive solution u_ε to (1.1). Moreover, there exists a maximum point $x_\varepsilon \in \mathbb{R}^N$ of u_ε such that $\lim_{\varepsilon \to 0} dist(x_\varepsilon, \mathcal{M}) = 0$, and for any such x_ε, $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges, up to a subsequence, in $H^s(\mathbb{R}^N)$ to a least energy solution of*

$$M([u]_s^2)(-\Delta)^s u + V_0 u = f(u) \text{ in } \mathbb{R}^N.$$

1.2. **State of the art and methodology.** We point out that Theorem 1.1 and Theorem 1.2 can be seen as the nonlocal fractional counterpart of Theorem 1.1 in [28] and Theorem 1.1 in [50], respectively. We recall that in [28] Figueiredo et al. refined some arguments developed in [13, 15, 17], in which the authors studied the existence and concentration of positive solutions for the nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \tag{1.4}$$

and involving general subcritical nonlinearities. More precisely, Byeon and Jeanjean [13] explored what are the essential features on f which guarantee the existence of localized ground states. To do this, the authors developed a new variational approach which consists in searching solutions of (1.4) in a neighborhood of the set of the least energy solution of the limiting problem associated with (1.4) whose mass stays close to $\mathcal{M}$; see [14, 15, 17] for more details. Subsequently, motivated by [28, 52], Zhang et al. [50] extended the result in [28] when f is a general critical nonlinearity by applying a suitable truncation argument.

The purpose of this work is to generalize the results in [28, 50] to the fractional setting $s \in (0, 1)$. For the sake of completeness, we start to mention some recent results in the case $M(t) \equiv 1$, that is when (1.1) reduces to the fractional Schrödinger equation (1.3). Seok [47] proved the existence of multi-peak solutions to (1.3) by assuming (f_1)-(f_3) and extending in the nonlocal framework the result in [14]. In [47], the author did not introduce a penalization term as in [13, 14] but proved a kind of intersection lemma by using degree theory after transforming (1.3) into a degenerate elliptic problem via the extension method [20]. In [39] Jin et al. considered (1.3) under conditions (f_1), (f_2')-(f_3') and constructed a family of positive solutions to (1.3) which concentrates at a local minimum of V as $\varepsilon \to 0$. The authors combined the extension method, a truncation argument inspired by [50] with the result in [47]. Simultaneously, He [34] obtained the same result by applying the extension method and combining the penalization methods developed in [17] and [23], respectively. We stress that this last approach has been previously used by Gloss [32] to extend the result in [13] to a p-Laplacian problem involving a general subcritical nonlinearity.

We note that the results in [34, 39, 47] improve the previous ones obtained in [3, 6, 36] in which the authors, motivated by [23], considered nonlinearities satisfying the Ambrosetti-Rabinowitz condition [4] and by requiring that $\frac{f(t)}{t}$ is strictly increasing for $t > 0$. Indeed, under assumptions (f_1)-(f_3) or (f_1), (f_2')-(f_3'), the Nehari method developed in the above mentioned papers does not work and it is very hard to verify the Palais-Smale compactness condition in this situation; see [8] for more details.

Concerning fractional Kirchhoff problems, to our knowledge, only few papers deal with the existence and concentration behavior of positive solutions as $\varepsilon \to 0$. In fact, motivated by [3, 6, 36], in [9, 11, 37] the authors studied the existence and concentration phenomena to (1.1) when $M(t) = a + bt$, $N = 3$ and $s \in (\frac{3}{4}, 1)$. However, the nonlinearities in [9, 11, 37] are less general than the ones presented here.

In this paper, by using suitable variational methods, we improve the results in [9, 11, 37] by considering a more general class of fractional Kirchhoff problems in the whole space $\mathbb{R}^N$, with $N \geq 2$. More precisely, after realizing (1.1) as a local linear degenerate elliptic equation in $\mathbb{R}^{N+1}_+$ together with a nonlinear Neumann boundary condition on $\partial\mathbb{R}^{N+1}_+$, we take inspiration by the penalization approach in [13, 23, 32] and some arguments used in [3, 9, 11, 28, 34, 39, 50], to obtain the existence of a family of positive solutions which concentrates around a local minimum of the potential $V(x)$, as $\varepsilon \to 0$. We emphasized that, making use of the extension method, several techniques used in the case $s = 1$ cannot be directly adapted in our setting because we have to take care of the traces terms of the involved functions and to work with weighted Lebesgue spaces. Moreover, due to the presence of the Kirchhoff term, our analysis is much more delicate and intriguing with respect to the case $M(t) \equiv 1$ and $s \in (0, 1)$ discussed above. For instance, if (u_ε) is a bounded sequence in $H^s(\mathbb{R}^N)$ of solutions to (1.1) such that $u_\varepsilon(\varepsilon x + x_\varepsilon) \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, then u is solution to the limiting problem $\alpha_0(-\Delta)^s u + V(x_0)u = f(u)$ in $\mathbb{R}^N$, where $\alpha_0 := \lim_{\varepsilon \to 0} M([u_\varepsilon]_s^2)$, and in general it is complicated to verify that $\alpha_0 = M([u]_s^2)$. Therefore, some refined estimates will be needed to overcome these difficulties; see Lemma 5.1 and Lemma 5.3.

As far as we know, these are the first existence results for (1.1) under local assumptions on the potential V and general nonlinearities f with subcritical or critical growth.

The paper is organized as follows. In section 2 we introduce the notations and we recall some useful results. In section 3 we study the limiting Kirchhoff problem associated with (1.1) by assuming (f_1)-(f_3). The critical limiting Kirchhoff problem is considered in section 4. In section 5 we provide the proof of Theorem 1.1. The last section is devoted to the proof of Theorem 1.2.

2. PRELIMINARIES

In this section we fix the notations and collect some preliminary results for future references. For more details we refer to [19, 20, 24, 25, 43].

We denote the upper half-space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ := \{(x, y) \in \mathbb{R}^{N+1} : y > 0\}.$$

For $p \in [1, \infty]$, let $L^p(\mathbb{R}^N)$ be the set of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$|u|_p := \begin{cases} \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{1/p} < \infty & \text{if } p < \infty, \\ \operatorname{esssup}_{x \in \mathbb{R}^N} |u(x)| & \text{if } p = \infty. \end{cases}$$

Let $\mathcal{D}^{s,2}(\mathbb{R}^N)$, with $s \in (0, 1)$, be the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[u]_s := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx dy \right)^{\frac{1}{2}}.$$

Then (see [24]) the embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \subset L^{2^*_s}(\mathbb{R}^N)$ is continuous and

$$|u|_{2^*_s} \leq c(N, s)[u]_s \quad \forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left([u]_s^2 + |u|_2^2\right)^{\frac{1}{2}}.$$

Then, $H^s(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*_s)$ and compactly in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^*_s)$; see [24]. We also define the fractional radial Sobolev space

$$H^s_{\rm rad}(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}.$$

It is well-known (see [41]) that $H^s_{\rm rad}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_s)$.

Let us define $X^s(\mathbb{R}^{N+1}_+)$ as the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ under the norm

$$\|u\|_{X^s(\mathbb{R}^{N+1}_+)} := \left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla u|^2 \, dx dy \right)^{\frac{1}{2}}.$$

Then (see [18]) there exists a linear trace operator $\mathrm{Tr} : X^s(\mathbb{R}^{N+1}_+) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that

$$\sqrt{\kappa_s}[\mathrm{Tr}(u)]_s \leq \|u\|_{X^s(\mathbb{R}^{N+1}_+)} \text{ for any } u \in X^s(\mathbb{R}^{N+1}_+),$$

where $\kappa_s := 2^{1-2s}\Gamma(1 - s)/\Gamma(s)$. In what follows, we set $u(\cdot, 0) := \mathrm{Tr}(u)$.

Denote by

$$B^+_R(x_0, y_0) := \{(x, y) \in \mathbb{R}^{N+1}_+ : |(x, y) - (x_0, y_0)| < R\}$$

the open ball in $\mathbb{R}^{N+1}_+$ with center $(x_0,y_0)\in\mathbb{R}^{N+1}_+$ and radius $R>0$, and

$$\Gamma^0_R(z_0):=\{(x,0)\in\partial\mathbb{R}^{N+1}_+ : |x-z_0|<R\}$$

the ball in $\mathbb{R}^N$ with center $z_0\in\mathbb{R}^N$ and radius $R>0$.
We denote by $X^s_0(B^+_R(0,0))$, with $R>0$, the completion of $C^\infty_c(B^+_R(0,0)\cup\Gamma^0_R(0))$ under the norm

$$\|u\|_{X^s_0(B^+_R(0,0))}:=\left(\iint_{B^+_R(0,0)} y^{1-2s}|\nabla u|^2\,dxdy\right)^{\frac{1}{2}}.$$

Note that if $w\in X^s_0(B^+_R(0,0))$ then its extension by zero outside $B^+_R(0,0)$ can be approximated by functions with compact support in $\overline{\mathbb{R}^{N+1}_+}$. Moreover, for all $r\in[1,2^*_s]$ and $u\in X^s_0(B^+_R(0,0))$ it holds (see [18])

$$C(r,s,N,R)\left(\int_{\Gamma^0_R(0)}|u(\cdot,0)|^r\,dx\right)^{\frac{2}{r}}\le\iint_{B^+_R(0,0)} y^{1-2s}|\nabla u|^2\,dxdy.$$

We define

$$X^{1,s}(\mathbb{R}^{N+1}_+):=\left\{u\in X^s(\mathbb{R}^{N+1}_+):\int_{\mathbb{R}^N}u^2(x,0)\,dx<\infty\right\}$$

equipped with the norm

$$\|u\|_{X^{1,s}(\mathbb{R}^{N+1}_+)}:=\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla u|^2\,dxdy+\int_{\mathbb{R}^N}u^2(x,0)\,dx\right)^{\frac{1}{2}}.$$

Finally, we consider

$$X^{1,s}_{\mathrm{rad}}(\mathbb{R}^{N+1}_+):=\{u\in X^{1,s}(\mathbb{R}^{N+1}_+):u(x,y)=u(|x|,y)\}.$$

The following Sobolev inequality holds true:

Lemma 2.1. *[18] For every $u\in X^{1,s}(\mathbb{R}^{N+1}_+)$ it holds for some positive constant $S(s,N)>0$*

$$S(s,N)\left(\int_{\mathbb{R}^N}|u(x,0)|^{2^*_s}\,dx\right)^{\frac{2}{2^*_s}}\le\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla u|^2\,dxdy.$$

For all $r\in(1,\infty)$, we define the weighted Lebesgue space $L^r(\mathbb{R}^{N+1}_+,y^{1-2s})$ endowed with the norm

$$\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|u|^r\,dxdy.$$

We recall the following useful result proved in [25]:

Lemma 2.2. *[25]*

(*i*) *There exists a constant $C>0$ such that for all $w\in X^s(\mathbb{R}^{N+1}_+)$ it holds*

$$\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|w|^{2\gamma}\,dxdy\right)^{\frac{1}{2\gamma}}\le C\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2\,dxdy\right)^{\frac{1}{2}},$$

where $\gamma:=1+\frac{2}{N-2s}$.

(*ii*) *Let $R>0$ and $\mathcal{T}$ be a subset of $X^s(\mathbb{R}^{N+1}_+)$ such that*

$$\sup_{w\in\mathcal{T}}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2\,dxdy<\infty.$$

Then, $\mathcal{T}$ is compact in $L^2(B^+_R(0,0),y^{1-2s})$.

The fractional Laplacian $(-\Delta)^s$ may be defined for $u:\mathbb{R}^N\to\mathbb{R}$ belonging to the Schwartz space of rapidly decaying functions by

$$(-\Delta)^s u(x)=C(N,s)P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

where

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-\cos(x_1)}{|x|^{N+2s}}\,dx\right)^{\frac{1}{2}}.$$

It can be also defined using Fourier transform by

$$\mathcal{F}((-\Delta)^s u(k)) = |k|^{2s}\mathcal{F}u(k).$$

It is well-known (see [24]) that for all $u \in H^s(\mathbb{R}^N)$

$$|(-\Delta)^{\frac{s}{2}} u|_2^2 = \int_{\mathbb{R}^N} |k|^{2s}|\mathcal{F}u(k)|^2\,dk = \frac{1}{2}C(N,s)[u]_s^2.$$

In [20], it is showed that one can see $(-\Delta)^s$ by considering it as the Dirichlet to Neumann operator associated to the s-harmonic extension in the half-space, paying the price to add a new variable. More precisely, for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ there exists a unique function $U \in X^s(\mathbb{R}^{N+1}_+)$ solving the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(\cdot,0) = u & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N. \end{cases}$$

The function U is called the s-harmonic extension of u and possesses the following properties:

(i)

$$\frac{\partial U}{\partial \nu^{1-2s}} := -\lim_{y\to 0} y^{1-2s}\frac{\partial U}{\partial y}(x,y) = \kappa_s(-\Delta)^s u(x) \text{ in distribution sense,}$$

(ii) $\sqrt{\kappa_s}[u]_s = \|U\|_{X^s(\mathbb{R}^{N+1}_+)} \leq \|V\|_{X^s(\mathbb{R}^{N+1}_+)}$ for all $V \in X^s(\mathbb{R}^{N+1}_+)$ such that $V(\cdot,0) = u$.

(iii) $U \in C^\infty(\mathbb{R}^{N+1}_+) \cap L^2(K, y^{1-2s})$ for any compact set $K \subset \overline{\mathbb{R}^{N+1}_+}$,

$$U(x,y) = \int_{\mathbb{R}^N} P_s(x-z,y)u(z)\,dz$$

where

$$P_s(x,y) := p_{N,s}\frac{y^{2s}}{(|x|^2+y^2)^{\frac{N+2s}{2}}}$$

and $p_{N,s}$ is a positive constant such that $\int_{\mathbb{R}^N} P_s(x,y)\,dx = 1$ for all $y > 0$.

Using the change of variable $x \mapsto \varepsilon\, x$, it is possible to prove that (1.1) is equivalent to the following problem

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V_\varepsilon(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $V_\varepsilon(x) := V(\varepsilon\, x)$. Then, in view of the previous facts, problem (2.1) can be realized in a local manner through the nonlinear boundary value problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})}\frac{\partial w}{\partial \nu^{1-2s}} = \kappa_s[-V_\varepsilon w(\cdot,0) + f(w(\cdot,0))] & \text{in } \mathbb{R}^N. \end{cases} \tag{2.2}$$

For simplicity we will drop the constant κ_s from the second equation in (2.2).

3. Subcritical limiting problems

We begin by modifying f as in [12]. Let $\hat{f} : \mathbb{R} \to \mathbb{R}$ be defined as follows:

(i) if $f(t) > 0$ for all $t \geq \widehat{T}$, put $\hat{f}(t) := f(t)$,

(ii) if there exists $\tau_0 \geq \widehat{T}$ such that $f(\tau_0) = 0$, we put

$$\hat{f}(t) := \begin{cases} f(t) & \text{for } t < \tau_0, \\ 0 & \text{for } t \geq \tau_0, \end{cases}$$

where $\widehat{T} := \sup\{t \in [0,T] : f(t) > V_0 t\}$.

Note that $\hat{f}$ satisfies the same assumptions as f and

$$0 \leq \liminf_{t\to\infty} \frac{\hat{f}(t)}{t^p} \leq \limsup_{t\to\infty} \frac{\hat{f}(t)}{t^p} < \infty.$$

Moreover, if (ii) occurs and u is a solution to (1.1) with $\hat{f}(t)$, then we can use $(u-\tau_0)_+$ as test function to deduce that $u \leq \tau_0$ in $\mathbb{R}^N$, that is u is a solution to (1.1) with $f(t)$. From now on, we replace f by $\hat{f}$ and keep the same notation $f(t)$.
In this section we focus on the following limiting problem associated with (2.2):

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})}\frac{\partial w}{\partial \nu^{1-2s}} = -V_0 w(\cdot,0) + f(w(\cdot,0)) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

To obtain our results we take inspiration by some arguments used in [28, 35]. Firstly, we show that the solutions of (3.1) satisfy a Pohozaev identity.

Lemma 3.1. *Assume that $(M1)$ holds and $u \in X^{1,s}(\mathbb{R}^{N+1}_+)$ is a solution to (3.1). Then u satisfies the following Pohozaev type identity:*

$$P(u) := \frac{N-2s}{2} M(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} - N\int_{\mathbb{R}^N} F(u(x,0)) - \frac{V_0}{2}u^2(x,0)\, dx = 0.$$

Proof. Put $\alpha_0 := M(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. Then u is a solution to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla u) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{\alpha_0}\frac{\partial u}{\partial \nu^{1-2s}} = -V_0 u(\cdot,0) + f(u(\cdot,0)) & \text{in } \mathbb{R}^N. \end{cases}$$

Arguing as in [5, 7, 16, 21], we deduce that u satisfies the following Pohozaev identity

$$\frac{N-2s}{2}\alpha_0\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} - N\int_{\mathbb{R}^N} F(u(x,0)) - \frac{V_0}{2}u^2(x,0)\, dx = 0$$

which implies the thesis. □

In order to find weak solutions to (3.1), we look for critical points of the energy functional $L_{V_0} : X^{1,s}(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ defined as

$$L_{V_0}(u) := \frac{1}{2}\widehat{M}\left(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) + \frac{1}{2}\int_{\mathbb{R}^N} V_0 u^2(x,0)\, dx - \int_{\mathbb{R}^N} F(u(x,0))\, dx.$$

From (f_1)-(f_2), it is easy to check that $L_{V_0} \in C^1(X^{1,s}(\mathbb{R}^{N+1}_+),\mathbb{R})$. Moreover, we see that L_{V_0} possesses a nice geometric structure.

Lemma 3.2. *Assume $(M1)$-$(M3)$. Then, L_{V_0} has a mountain pass geometry.*

Proof. By $(M1)$, (f_1), (f_2) and $H^s(\mathbb{R}^N) \subset L^{p+1}(\mathbb{R}^N)$ we have

$$\begin{aligned} L_{V_0}(u) &\geq \frac{m_0}{2}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \frac{V_0}{2}|u(\cdot,0)|_2^2 - \varepsilon\,|u(\cdot,0)|_2^2 - C_\varepsilon |u(\cdot,0)|_{p+1}^{p+1} \\ &\geq c_1\|u\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+)} - c_2\|u\|^{p+1}_{X^{1,s}(\mathbb{R}^{N+1}_+)}. \end{aligned}$$

Hence, there exist $\rho, \delta > 0$ such that $L_{V_0}(u) \geq \delta$ for $\|u\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} = \rho$.
Now, for all $R > 0$ we define

$$w_R(x,y) := \begin{cases} T & \text{if } (x,y) \in B^+_R(0,0), \\ T\left(R+1-\sqrt{|x|^2+y^2}\right) & \text{if } (x,y) \in B^+_{R+1}(0,0) \setminus B^+_R(0,0), \\ 0 & \text{if } (x,y) \in \mathbb{R}^{N+1}_+ \setminus B^+_{R+1}(0,0). \end{cases}$$

It is clear that $w_R \in X^{1,s}_{\mathrm{rad}}(\mathbb{R}^{N+1}_+)$. Note that, by (f_3), for $R > 0$ large enough it holds

$$\int_{\mathbb{R}^N} F(w_R(x,0)) - \frac{V_0}{2}w_R^2(x,0)\, dx \geq 1.$$

Now, fix such an $R>0$ and consider $w_{R,\theta}(x,y):=w_R(x/e^\theta,y/e^\theta)$. Then,

$$\begin{aligned}L_{V_0}(w_{R,\theta})&=\frac{1}{2}\widehat{M}(e^{(N-2s)\theta}\|w_R\|^2_{X^s(\mathbb{R}^{N+1}_+)})-e^{N\theta}\int_{\mathbb{R}^N}F(w_R(x,0))-\frac{V_0}{2}w_R^2(x,0)\,dx\\&\leq\frac{1}{2}\widehat{M}(e^{(N-2s)\theta}\|w_R\|^2_{X^s(\mathbb{R}^{N+1}_+)})-e^{N\theta}\to-\infty\text{ as }\theta\to\infty\end{aligned}$$

because $(M3)$ yields

$$e^{-N\theta}\widehat{M}(e^{(N-2s)\theta}\|w_R\|^2_{X^s(\mathbb{R}^{N+1}_+)})\to 0\text{ as }\theta\to\infty.$$

□

In view of Lemma 3.2 we can define the minimax level

$$c_{V_0}:=\inf_{\gamma\in\Gamma_{V_0}}\max_{t\in[0,1]}L_{V_0}(\gamma(t))\tag{3.2}$$

and

$$\Gamma_{V_0}:=\{\gamma\in C([0,1],X^{1,s}(\mathbb{R}^{N+1}_+)):\gamma(0)=0,L_{V_0}(\gamma(1))<0\}.\tag{3.3}$$

Obviously, $c_{V_0}>0$. We can also note that

$$c_{V_0}=c_{V_0,\mathrm{rad}},\tag{3.4}$$

where

$$c_{V_0,\mathrm{rad}}:=\inf_{\gamma\in\Gamma_{V_0,rad}}\max_{t\in[0,1]}L_{V_0}(\gamma(t)),$$

and

$$\Gamma_{V_0,\mathrm{rad}}:=\{\gamma\in C([0,1],X^{1,s}_{\mathrm{rad}}(\mathbb{R}^{N+1}_+)):\gamma(0)=0,L_{V_0}(\gamma(1))<0\}.$$

Indeed, $c_{V_0}\leq c_{V_0,\mathrm{rad}}$ by the definitions. For the opposite inequality, take $\gamma\in\Gamma_{V_0}$ and consider $\gamma_\varepsilon(t):=\rho_\varepsilon*\gamma(t)$, where $\rho_\varepsilon\in C^\infty_c(\mathbb{R}^{N+1}_+)$ is a standard mollifier. Then, $\gamma_\varepsilon\in C([0,1],X^{1,s}(\mathbb{R}^{N+1}_+))$, $\gamma_\varepsilon(0)=0$ and $\gamma_\varepsilon(t)\in C^\infty(\mathbb{R}^{N+1}_+)\cap X^{1,s}(\mathbb{R}^{N+1}_+)$ for all $t\in[0,1]$. Since

$$\sup_{t\in[0,1]}\|\gamma_\varepsilon(t)-\gamma(t)\|_{X^{1,s}(\mathbb{R}^{N+1}_+)}\to 0\text{ as }\varepsilon\to 0,$$

we deduce that

$$\max_{t\in[0,1]}L_{V_0}(\gamma_\varepsilon(t))\to\max_{t\in[0,1]}L_{V_0}(\gamma(t))\text{ as }\varepsilon\to 0.$$

Now, let $\phi^*_\varepsilon(t)$ be the symmetric decreasing rearrangement of $\gamma_\varepsilon(t)(\cdot,0)\in H^s(\mathbb{R}^N)$, and denote by $\gamma^*_\varepsilon(t)$ the solution of

$$\begin{cases}-\operatorname{div}(y^{1-2s}\nabla\gamma^*_\varepsilon(t))=0 & \text{in }\mathbb{R}^{N+1}_+,\\ \gamma_\varepsilon(t)(\cdot,0)=\phi^*_\varepsilon(t) & \text{in }\mathbb{R}^N.\end{cases}$$

Since $\gamma^*_\varepsilon(t)$ is the s-harmonic extension of $\phi^*_\varepsilon(t)$, and using the trace inequality and Theorem 9.2 in [1] we have

$$\|\gamma^*_\varepsilon(t)\|_{X^s(\mathbb{R}^{N+1}_+)}=[\phi^*_\varepsilon(t)]_s\leq[\gamma_\varepsilon(t)(\cdot,0)]_s\leq\|\gamma_\varepsilon(t)\|_{X^s(\mathbb{R}^{N+1}_+)}.$$

On the other hand, for all $G:\mathbb{R}\to\mathbb{R}$ continuous

$$\int_{\mathbb{R}^N}G(\gamma^*_\varepsilon(t)(\cdot,0))\,dx=\int_{\mathbb{R}^N}G(\phi^*_\varepsilon(t))\,dx=\int_{\mathbb{R}^N}G(\gamma_\varepsilon(t)(\cdot,0))\,dx.$$

Observing that $\widehat{M}$ is strictly increasing (by $(M1)$), we obtain that $L_{V_0}(\gamma^*_\varepsilon(t))\leq L_{V_0}(\gamma_\varepsilon(t))$ for all $t\in[0,1]$. Moreover, since $\gamma_\varepsilon(\cdot,0)\in C^\infty(\mathbb{R}^N)$, we have that $\gamma_\varepsilon(\cdot,0)$ is co-area regular (see [1]) and using Theorem 9.2 in [1] we deduce that $\phi^*_\varepsilon\in C([0,1],H^s_{\mathrm{rad}}(\mathbb{R}^N))$ and consequently $\gamma^*_\varepsilon\in C([0,1],X^{1,s}_{\mathrm{rad}}(\mathbb{R}^{N+1}_+))$. In conclusion, $\gamma^*_\varepsilon\in\Gamma_{V_0,\mathrm{rad}}$ and (3.4) holds true.

Now we prove the existence of a Palais-Smale sequence of L_{V_0} with an extra property related to the Pohozaev identity; see [28,35,38].

Proposition 3.1. *There exists a sequence $(w_n)\subset X^{1,s}_{\mathrm{rad}}(\mathbb{R}^{N+1}_+)$ such that*

$$L_{V_0}(w_n)\to c_{V_0},\ L'_{V_0}(w_n)\to 0,\ P(w_n)\to 0.\tag{3.5}$$

Proof. Let $\widetilde{L}_{V_0}(\theta, u) := (L_{V_0} \circ \Phi)(\theta, u)$ for $(\theta, u) \in \mathbb{R} \times X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+)$, where $\Phi(\theta, u) := u(\frac{x}{e^\theta}, \frac{y}{e^\theta})$. Here $\mathbb{R} \times X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+)$ is equipped with the standard norm

$$\|(\theta, u)\|_{\mathbb{R}\times X^{1,s}(\mathbb{R}^{N+1}_+)} := (|\theta|^2 + \|u\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+)})^{\frac{1}{2}}.$$

It follows from Lemma 3.2 that $\widetilde{L}_{V_0}$ has a mountain pass geometry, so we can define the mountain pass level of $\widetilde{L}_{V_0}$

$$\tilde{c}_{V_0} := \inf_{\tilde{\gamma}\in\widetilde{\Gamma}_{V_0}} \max_{t\in[0,1]} \widetilde{L}_{V_0}(\widetilde{\gamma}(t))$$

where

$$\widetilde{\Gamma}_{V_0} := \{\widetilde{\gamma} \in C([0,1], \mathbb{R} \times X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+)) : \widetilde{\gamma}(0) = (0,0), \widetilde{L}_{V_0}(\widetilde{\gamma}(1)) < 0\}.$$

It is easy to show that $\widetilde{c}_{V_0} = c_{V_0}$ (see [7, 38]). Then, by the general minimax principle (see Theorem 2.8 in [49]), we deduce that there exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+)$ such that, as $n \to \infty$,

(i) $(L_{V_0} \circ \Phi)(\theta_n, u_n) \to c_{V_0}$,
(ii) $(L_{V_0} \circ \Phi)'(\theta_n, u_n) \to 0$ in $(\mathbb{R} \times X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+))'$,
(iii) $\theta_n \to 0$.

Indeed, if we take $\varepsilon = \varepsilon_n = \frac{1}{n^2}$, $\delta = \delta_n = \frac{1}{n}$ in Theorem 2.8 in [49], (i) and (ii) follow by (a) and (c) in Theorem 2.8 in [49]. In view of (3.2), (3.3), for $\varepsilon = \varepsilon_n := \frac{1}{n^2}$, we can find $\gamma_n \in \Gamma_{V_0}$ such that $\sup_{t\in[0,1]} L_{V_0}(\gamma_n(t)) \le c_{V_0} + \frac{1}{n^2}$. Set $\tilde{\gamma}_n(t) := (0, \gamma_n(t))$. Then

$$\sup_{t\in[0,1]} (L_{V_0} \circ \Phi)(\tilde{\gamma}_n(t)) = \sup_{t\in[0,1]} L_{V_0}(\gamma_n(t)) \le c_{V_0} + \frac{1}{n^2}.$$

From (b) of Theorem 2.8 in [49], there exists $(\theta_n, u_n) \in \mathbb{R} \times X^{1,s}(\mathbb{R}^{N+1}_+)$ such that

$$\mathrm{dist}_{\mathbb{R}\times X^{1,s}(\mathbb{R}^{N+1}_+)}((\theta_n, u_n), (0, \gamma_n(t))) \le \frac{2}{n},$$

that is (iii) holds true. Here, we used the notation

$$\mathrm{dist}_{\mathbb{R}\times X^{1,s}(\mathbb{R}^{N+1}_+)}((\theta, u), A) := \inf_{(\tau,v)\in\mathbb{R}\times X^{1,s}(\mathbb{R}^{N+1}_+)} (|\theta - \tau|^2 + \|u - v\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+)})^{\frac{1}{2}},$$

for $A \subset \mathbb{R} \times H^s(\mathbb{R}^N)$. Now, for $(h, w) \in \mathbb{R} \times X^{1,s}(\mathbb{R}^{N+1}_+)$, it holds

$$\langle (L_{V_0} \circ \Phi)'(\theta_n, u_n), (h, w)\rangle = \langle L'_{V_0}(\Phi(\theta_n, u_n)), \Phi'(\theta_n, w)\rangle + P(\Phi(\theta_n, u_n))h. \tag{3.6}$$

Then, choosing $h = 1$ and $w = 0$ in (3.6), we deduce that

$$P(\Phi(\theta_n, u_n)) \to 0.$$

On the other hand, for every $v \in X^{1,s}(\mathbb{R}^{N+1}_+)$, taking $w(x, y) = v(e^{\theta_n}x, e^{\theta_n}y)$ and $h = 0$ in (3.6), it follows from (ii) and (iii) that

$$\langle L'_{V_0}(\Phi(\theta_n, u_n)), v\rangle = o(1)\|v(e^{\theta_n}x, e^{\theta_n}y)\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} = o(1)\|v\|_{X^{1,s}(\mathbb{R}^{N+1}_+)}.$$

Consequently, $w_n := \Phi(\theta_n, u_n)$ is the sequence that fulfills the desired properties. □

Lemma 3.3. *Every sequence (w_n) satisfying (3.5) is bounded in $X^{1,s}(\mathbb{R}^{N+1}_+)$.*

Proof. Using (3.5) we see that

$$\begin{aligned} c_{V_0} + o_n(1) &= L_{V_0}(w_n) - \frac{1}{N}P(w_n) \\ &= \frac{1}{2}\widehat{M}\left(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - \left(\frac{N-2s}{2N}\right) M\left(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}. \end{aligned}$$

From $(M2)$ we deduce that $(\|w_n\|_{X^s(\mathbb{R}^{N+1}_+)})$ is bounded in $\mathbb{R}$. On the other hand, $P(w_n) = o_n(1)$ and (f_1)-(f_2) yield

$$\frac{N-2s}{2} M\left(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) \|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} + N\frac{V_0}{2}|w_n(\cdot,0)|_2^2 = N\int_{\mathbb{R}^N} F(w_n(x,0))\,dx + o_n(1)$$
$$\leq N\delta|w_n(\cdot,0)|_2^2 + NC_\delta|w_n(\cdot,0)|_{2^*_s}^{2^*_s} + o_n(1).$$

Choosing $\delta > 0$ sufficiently small and using $(M1)$ and the boundedness of $(|w_n(\cdot,0)|_{2^*_s})$, we can infer that $(|w_n(\cdot,0)|_2)$ is bounded in $\mathbb{R}$. In conclusion, (w_n) is bounded in $X^{1,s}(\mathbb{R}^{N+1}_+)$. □

Lemma 3.4. *There exist a sequence $(x_n) \subset \mathbb{R}^N$ and constants $R > 0$, $\beta > 0$ such that*

$$\int_{\Gamma^0_R(x_n)} w_n^2(x,0)\,dx \geq \beta,$$

where (w_n) is the sequence given in Proposition 3.1.

Proof. Assume by contradiction that the thesis is not true. Then, by the vanishing Lions-type lemma (see Lemma 3.3 in [36]), we deduce that

$$w_n(\cdot,0) \to 0 \text{ in } L^q(\mathbb{R}^N) \quad \forall q \in (2, 2^*_s). \tag{3.7}$$

Consequently, by (f_1)-(f_2), we have

$$\int_{\mathbb{R}^N} f(w_n(x,0))w_n(x,0)\,dx = o_n(1).$$

Recalling that $\langle L'_{V_0}(w_n), w_n\rangle = o_n(1)$, we get

$$M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} + V_0|w_n(\cdot,0)|_2^2 = o_n(1)$$

and using $(M1)$ we obtain that

$$\|w_n\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} \to 0.$$

Therefore, $L_{V_0}(w_n) \to 0$ and this leads to a contradiction because $c_{V_0} > 0$. □

Now we define

$$\mathcal{T}_{V_0} := \{u \in X^{1,s}(\mathbb{R}^{N+1}_+) \setminus \{0\} : L'_{V_0}(u) = 0, \max_{\mathbb{R}^N} u(\cdot,0) = u(0,0)\},$$
$$b_{V_0} := \inf_{u\in\mathcal{T}_{V_0}} L_{V_0}(u),$$

and

$$\mathcal{S}_{V_0} := \{u \in \mathcal{T}_{V_0} : L_{V_0}(u) = b_{V_0}\}.$$

Lemma 3.5. *Assume $(M1)$-$(M5)$. Then there exists $u \in \mathcal{S}_{V_0}$.*

Proof. Let (w_n) be the sequence given by Lemma 3.1. Set $\tilde{w}_n(x,y) := w_n(x+x_n,y)$ where (x_n) is given in Lemma 3.4. By Lemma 3.3, we know that (w_n) is bounded in $X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+)$, that is $\|w_n\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} \leq C$ for all $n \in \mathbb{N}$. Hence $\tilde{w}_n \rightharpoonup \tilde{w}$ in $X^{1,s}_{rad}(\mathbb{R}^{N+1}_+)$ and $\tilde{w}_n(\cdot,0) \to \tilde{w}(\cdot,0)$ in $L^q(\mathbb{R}^N)$ for any $q \in (2,2^*_s)$, for some $\tilde{w} \in X^{1,s}_{\rm rad}(\mathbb{R}^{N+1}_+) \setminus \{0\}$. Then, $\tilde{w}$ is a weak solution to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla\tilde{w}) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{\alpha_0}\frac{\partial\tilde{w}}{\partial\nu^{1-2s}} = -V_0\tilde{w}(\cdot,0) + f(\tilde{w}(\cdot,0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.8}$$

where

$$\alpha_0 := \lim_{n\to\infty} M(\|\tilde{w}_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) = \lim_{n\to\infty} M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \leq M(C^2) < \infty.$$

Note that the last inequality is due to $(M4)$.
Clearly, by Fatou's Lemma, we have

$$0 < m_0 \leq M(\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \leq \alpha_0. \tag{3.9}$$

In what follows, we prove that

$$\alpha_0 = M(\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)}),$$

and thus $\tilde{w}$ is a weak solution to (1.1). Since $\tilde{w}$ solves (3.8) and using the regularity assumptions on f, we deduce that $\tilde{w}$ satisfies the following Pohozaev identity [7,16,21]:

$$\frac{N-2s}{2}\alpha_0\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)} - N\int_{\mathbb{R}^N}\left(F(\tilde{w}(x,0)) - \frac{V_0}{2}\tilde{w}^2(x,0)\right)dx = 0. \tag{3.10}$$

Now, we apply Lemma 2.4 in [21] with $X = H^s_{\mathrm{rad}}(\mathbb{R}^N)$, $P(t) = f(t)t$, $p_1 = 2$ and $p_2 = 2^*_s$ to see that

$$\begin{aligned}\alpha_0\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)} + V_0|\tilde{w}(\cdot,0)|_2^2 &\leq \liminf_{n\to\infty}[M(\|\tilde{w}_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|\tilde{w}_n\|^s_{X^s(\mathbb{R}^{N+1}_+)} + V_0|\tilde{w}_n(\cdot,0)|_2^2]\\ &\leq \limsup_{n\to\infty}[M(\|\tilde{w}_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|\tilde{w}_n\|^s_{X^s(\mathbb{R}^{N+1}_+)} + V_0|\tilde{w}_n(\cdot,0)|_2^2]\\ &= \limsup_{n\to\infty}[M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} + V_0|w_n(\cdot,0)|_2^2]\\ &= \limsup_{n\to\infty}\int_{\mathbb{R}^N} f(w_n(x,0))w_n(x,0)\,dx\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} f(\tilde{w}_n(x,0))\tilde{w}_n(x,0)\,dx\\ &= \int_{\mathbb{R}^N} f(\tilde{w}(x,0))\tilde{w}(x,0)\,dx\\ &= \alpha_0\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)} + V_0|\tilde{w}(\cdot,0)|_2^2\end{aligned}$$

which implies that $\|\tilde{w}_n\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} \to \|\tilde{w}\|_{X^{1,s}(\mathbb{R}^{N+1}_+)}$ and thus $\tilde{w}_n \to \tilde{w}$ in $X^{1,s}(\mathbb{R}^{N+1}_+)$. Hence, $\alpha_0 = M(\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. Therefore, by $L_{V_0}(w_n) = L_{V_0}(\tilde{w}_n) \to c_{V_0}$ and $L'_{V_0}(w_n) = L'_{V_0}(\tilde{w}_n) \to 0$, we have that $L_{V_0}(\tilde{w}) = c_{V_0}$ and $L'_{V_0}(\tilde{w}) = 0$. Since $\tilde{w} \neq 0$, we deduce that $c_{V_0} \geq b_{V_0}$.

Now, let $w \in X^{1,s}(\mathbb{R}^{N+1}_+) \setminus \{0\}$ be any solution to (3.1). Define

$$\gamma(t) := \begin{cases} w(\frac{x}{t}, \frac{y}{t}) & \text{for } t > 0,\\ 0 & \text{for } t = 0.\end{cases}$$

Using the fact that w satisfies the Pohozaev identity (see Lemma 3.1), we get

$$L_{V_0}(\gamma(t)) = \frac{1}{2}\widehat{M}\left(t^{N-2s}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - t^N\left(\frac{N-2s}{2N}\right)M\left(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)},$$

and differentiating with respect to t we obtain

$$\frac{d}{dt}L_{V_0}(\gamma(t)) = \frac{N-2s}{2}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}t^{N-2s-1}\left[M(t^{N-2s}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - t^{2s}M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})\right].$$

By $(M5)$ and using a change of variable, we observe that $t \mapsto M(t^{N-2s}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})/t^{2s}$ is nonincreasing in $(0,\infty)$, so we have

$$\frac{d}{dt}L_{V_0}(\gamma(t)) > 0 \quad \forall t \in (0,1),\ \frac{d}{dt}L_{V_0}(\gamma(t)) < 0 \quad \forall t \in (1,\infty),$$

which implies that

$$\max_{t\geq 0} L_{V_0}(\gamma(t)) = L_{V_0}(\gamma(1)) = L_{V_0}(w).$$

Moreover, noting that $(M1)$ and $(M3)$ yield

$$\lim_{t\to\infty}\frac{\widehat{M}(t^{N-2s})}{t^N} = \left[\frac{\infty}{\infty}\right] = \lim_{t\to\infty}\frac{M(t^{N-2s})}{(t^{N-2s})^{\frac{2s}{N-2s}}}\frac{N-2s}{N} = 0,$$

we deduce

$$L_{V_0}(\gamma(t)) = \frac{t^N}{2}\left[\frac{1}{t^N}\widehat{M}\left(t^{N-2s}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - \left(\frac{N-2s}{2N}\right)M\left(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] \to -\infty,$$

as $t \to \infty$. Then there exists $\tau > 0$ sufficiently large such that $L_{V_0}(\gamma(\tau)) < 0$. After a suitable scale change in t, we obtain that $\gamma \in \Gamma_{V_0}$. By the definition of c_{V_0}, we see that $L_{V_0}(w) \geq c_{V_0}$. Since w is arbitrary, we have that $b_{V_0} \geq c_{V_0}$ and this implies that $b_{V_0} = c_{V_0}$.

Choosing $u^- = \min\{u, 0\}$ as test function in the weak formulation of (3.1) we can deduce that $u \geq 0$ in $\mathbb{R}^N$. By (f_1)-(f_2) and using a Moser iteration argument (see [7, 21]), we obtain that $u \in L^\infty(\mathbb{R}^N)$. By the growth assumptions on f and in view of the Hölder regularity results in [48], we deduce that $u \in C^{0,\beta}(\mathbb{R}^N)$ (see [7, 16, 21]). From the Harnack inequality [19, 33] we conclude that $u > 0$ in $\mathbb{R}^N$. □

Remark 3.1. *For $m > 0$, we use the notation*

$$L_m(u) = \frac{1}{2}\widehat{M}(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}) + \frac{m}{2}|u(\cdot,0)|_2^2 - \int_{\mathbb{R}^N} F(u(x,0))\,dx$$

and denote by c_m the corresponding mountain pass level. It is standard to verify that if $m_1 > m_2$ then $c_{m_1} > c_{m_2}$.

In what follows, we aim to show that $\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$. To do this we begin by giving some auxiliary results. Let us consider the following fractional elliptic problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}} = -V_0 w(\cdot,0) + f(w(\cdot,0)) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.11}$$

If w is a solution to (3.11), then it satisfies the Pohozaev identity (see [5, 7, 16, 21, 51])

$$\frac{N-2s}{2}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} - N\int_{\mathbb{R}^N} F(u(x,0)) - \frac{V_0}{2}u^2(x,0)\,dx = 0. \tag{3.12}$$

Let

$$\mathcal{E}_{V_0}(u) = \frac{1}{2}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \frac{V_0}{2}|u(\cdot,0)|_2^2 - \int_{\mathbb{R}^N} F(u(x,0))\,dx,$$

$$\tilde{b}_{V_0} := \inf_{u\in\widetilde{\mathcal{T}}_{V_0}} \mathcal{E}_{V_0}(u),$$

$$\widetilde{\mathcal{T}}_{V_0} = \{u \in X^{1,s}(\mathbb{R}^{N+1}_+)\setminus\{0\} : \mathcal{E}'_{V_0}(u) = 0, \max_{\mathbb{R}^N} u(\cdot,0) = u(0,0)\},$$

and

$$\widetilde{\mathcal{S}}_{V_0} = \{u \in \widetilde{\mathcal{T}}_{V_0} : \mathcal{E}_{V_0}(u) = \tilde{b}_{V_0}\}.$$

Next we show that it is possible to define a map which relates the ground state solutions of (3.11) to the ones for (3.1). We first prove the following result for the Kirchhoff functions.

Lemma 3.6. *Assume that $M \in C([0,\infty))$ and $M(t) \geq 0$. Then, $(M5)$ is equivalent to*
$(M6)$ $t \mapsto \widehat{M}(t) - \left(1 - \frac{2s}{N}\right) M(t)t$ is nondecreasing in $[0,\infty)$.

Proof. We argue as in Lemma 2.17 in [28]. Let $(M5)$ be in force. Then, for $0 \leq t_1 < t_2$ we have

$$\begin{aligned} \widehat{M}(t_2) - \left(1 - \frac{2s}{N}\right) M(t_2)t_2 &= \widehat{M}(t_1) + \int_{t_1}^{t_2} \frac{M(t)}{t^{\frac{2s}{N-2s}}} t^{\frac{2s}{N-2s}}\,dt - \left(1 - \frac{2s}{N}\right) M(t_2)t_2 \\ &\geq \widehat{M}(t_1) + \frac{M(t_2)}{t_2^{\frac{2s}{N-2s}}}\int_{t_1}^{t_2} t^{\frac{2s}{N-2s}}\,dt - \left(1 - \frac{2s}{N}\right) M(t_2)t_2 \\ &= \widehat{M}(t_1) - \left(1 - \frac{2s}{N}\right)\frac{M(t_2)}{t_2^{\frac{2s}{N-2s}}} t_1^{\frac{N}{N-2s}} \\ &\geq \widehat{M}(t_1) - \left(1 - \frac{2s}{N}\right) M(t_1)t_1. \end{aligned} \tag{3.13}$$

The other implication is obtained as in the case $s = 1$ with small modifications, so we omit the details. □

Lemma 3.7. *Assume $(M1)$-$(M5)$. Then, $\mathcal{S}_{V_0} \neq \emptyset$ and there exists an injective map $T : \widetilde{\mathcal{S}}_{V_0} \to \mathcal{S}_{V_0}$.*

Proof. By [7, 16, 21] we know that $\widetilde{\mathcal{S}}_{V_0} \neq \emptyset$. Let $\phi \in \widetilde{\mathcal{S}}_{V_0}$ and define

$$t_\phi := \inf\left\{t > 0 : t^{2s} = M(t^{N-2s}\|\phi\|^2_{X^s(\mathbb{R}^{N+1}_+)})\right\}.$$

In what follows we verify that $t_\phi \in (0,\infty)$. Since $\mathcal{T}_{V_0} \neq \emptyset$ by Lemma 3.5, we can find $w \in \mathcal{T}_{V_0}$ and put $\alpha^{2s} := M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. Set $w_\alpha(x,y) = w(\alpha x, \alpha y)$ and note that w_α is a weak solution to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w_\alpha) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w_\alpha}{\partial \nu^{1-2s}} = -V_0 w_\alpha(\cdot,0) + f(w_\alpha(\cdot,0)) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.14}$$

By (4.3) we get

$$\frac{s}{N}\|\phi\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \mathcal{E}_{V_0}(\phi) \le \mathcal{E}_{V_0}(w_\alpha) = \frac{s}{N}\|w_\alpha\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \frac{s}{N}\alpha^{2s-N}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}$$

that is $\alpha^{N-2s}\|\phi\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le \|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}$. Using $(M4)$ we have

$$M(\alpha^{N-2s}\|\phi\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \le M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}) = \alpha^{2s}.$$

From $(M1)$ and the continuity of M, there is $t_0 \in (0,\alpha]$ such that $t_0^{2s} = M(t_0^{N-2s}\|\phi\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. Consequently, $0 < m_0 \le t_\phi^{2s} \le \alpha^{2s}$ and t_ϕ is well-defined.

At this point, for $u \in \mathcal{T}_{V_0}$, we define

$$(Tu)(x,y) := u(x/t_u, y/t_u).$$

Since

$$t_u^{2s} = M(t_u^{N-2s}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}),$$

we see that Tu is a solution to (3.1). Using $t_u \le \alpha$ and $\alpha^{N-2s}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le \|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}$ we get $\|Tu\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le \|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}$. On the other hand, we observe that for all $u \in X^{1,s}(\mathbb{R}^{N+1}_+)$ such that $P(u) = 0$ it holds

$$L_{V_0}(u) = \frac{1}{2}\left[\widehat{M}(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1 - \frac{2s}{N}\right) M(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right].$$

Then, from Lemma 3.6 and $(M5)$, we deduce that $L_{V_0}(Tu) \le L_{V_0}(w)$. By the arbitrariness of $w \in \mathcal{T}_{V_0}$, we infer that $Tu \in \mathcal{S}_{V_0}$. Hence, $\mathcal{S}_{V_0} \neq \emptyset$ and $T : \widetilde{\mathcal{S}}_{V_0} \to \mathcal{S}_{V_0}$ is well-defined.

Finally, we show that T is injective. Let $u_1, u_2 \in \widetilde{\mathcal{S}}_{V_0}$ be such that $Tu_1 = Tu_2$. Then, $u_1(x,y) = u_2(\alpha x, \alpha y)$ for some $\alpha > 0$. Since $u_1(\cdot,0)$ and $u_2(\cdot,0)$ are nontrivial solutions of $(-\Delta)^s u + V_0 u = f(u)$ in $\mathbb{R}^N$, we deduce that $\alpha^{2s}(-\Delta)^s u_2(\alpha x, 0) = (-\Delta)^s u_1(x,0) = (-\Delta)^s u_2(\alpha x, 0)$ which implies that $(\alpha^{2s}-1)(-\Delta)^s u_2(\cdot,0) = 0$ in $\mathbb{R}^N$. Hence, $\alpha = 1$ and $u_1 \equiv u_2$. □

Proposition 3.2. *$\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$.*

Proof. Let $(w_n) \subset \mathcal{S}_{V_0}$ and set $v_n(x,y) := w_n(\alpha_n x, \alpha_n y)$ where

$$\alpha_n^{2s} := M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}).$$

Then, v_n is a solution to (3.11). Now we prove that $v_n \in \widetilde{\mathcal{S}}_{V_0}$ and that there exists $C_0 > 0$ such that $m_0 \le \alpha_n^{2s} \le C_0^{2s}$ for all $n \in \mathbb{N}$. Note that $m_0 \le \alpha_n^{2s}$ thanks to $(M1)$. Now, by Lemma 3.1 we have

$$\begin{aligned} b_{V_0} &= L_{V_0}(w_n) - \frac{1}{N}P(w_n) \\ &= \frac{1}{2}\left[\widehat{M}\left(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - \left(1 - \frac{2s}{N}\right) M\left(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|w_n\|^2_x\right]. \end{aligned}$$

In light of $(M2)$ we deduce that $\|w_n\|_{X^s(\mathbb{R}^{N+1}_+)}$ is bounded and then (α_n) is bounded.

Take $\phi_n \in \widetilde{\mathcal{S}}_{V_0}$. Proceeding as in the proof of Lemma 3.7 and using $(M6)$ we can see that $\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le \|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}$, $t_n \le \alpha_n$ and $b_{V_0} = L_{V_0}(\phi_{n,t_n}) \le L_{V_0}(w_n) = b_{V_0}$, where

$$t_n := \inf\left\{t \in (0,\infty) : t^{2s} = M(t^{N-2s}\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\right\}$$

and $\phi_{n,t_n}(x,y) := \phi_n(\frac{x}{t_n}, \frac{y}{t_n}) = T(\phi_n)$. Moreover, $L_{V_0}(\phi_{n,t_n}) = b_{V_0} = L_{V_0}(w_n)$. At this point, if we show that

$$\|\phi_n\|_{X^s(\mathbb{R}^{N+1}_+)} = \|v_n\|_{X^s(\mathbb{R}^{N+1}_+)}, \tag{3.15}$$

then we have

$$\mathcal{E}_{V_0}(\phi_n) = \frac{s}{N}\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \frac{s}{N}\|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \mathcal{E}_{V_0}(v_n),$$

where we used (4.3). Hence we deduce that $v_n \in \widetilde{\mathcal{S}}_{V_0}$. Next, we prove that (3.15) holds true. Assume by contradiction that $\|v_n\|_{X^s(\mathbb{R}^{N+1}_+)} > \|\phi_n\|_{X^s(\mathbb{R}^{N+1}_+)}$. Taking into account that $t_n \leq \alpha_n$ and $\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \alpha_n^{N-2s}\|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}$, we get

$$\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)} = t_n^{N-2s}\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} < \alpha_n^{N-2s}\|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}.$$

On the other hand, using $P(\phi_{n,t_n}) = 0 = P(w_n)$, we infer that

$$\begin{aligned}&\frac{1}{2}\left\{\widehat{M}(\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1-\frac{2s}{N}\right)M(\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right\}\\ &= L_{V_0}(\phi_{n,t_n}) = L_{V_0}(w_n) = \frac{1}{2}\left\{\widehat{M}(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1-\frac{2s}{N}\right)M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right\}.\end{aligned}$$

By $(M5)$, $(M6)$ in Lemma 3.7 and (3.13), it is easy to see that for any $\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)} \leq t_1 < t_2 \leq \|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}$ it holds

$$\widehat{M}(t_1) - \left(1-\frac{2s}{N}\right)M(t_1)t_1 = \widehat{M}(t_2) - \left(1-\frac{2s}{N}\right)M(t_2)t_2$$

and

$$\frac{M(t_1)}{t_1^{2s/(N-2s)}} = \frac{M(t_2)}{t_2^{2s/(N-2s)}}. \tag{3.16}$$

Otherwise, we have $L_{V_0}(\phi_{n,t_n}) < L_{V_0}(w_n)$, that is a contradiction. Moreover, in view of (3.16), we get

$$M(t) = k_0 t^{\frac{2s}{N-2s}} \text{ in } [\|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)}, \|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}],$$

for some $k_0 > 0$. By the definitions of α_n and t_n, and using $t_n^{N-2s}\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \|\phi_{n,t_n}\|^2_{X^s(\mathbb{R}^{N+1}_+)}$, we deduce that

$$\begin{aligned}t_n^{2s} &= M(t_n^{N-2s}\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) = k_0 t_n^{2s}\|\phi_n\|^{\frac{4s}{N-2s}}_{X^s(\mathbb{R}^{N+1}_+)}\\ \alpha_n^{2s} &= M(\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) = M(\|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}) = k_0\alpha_n^{2s}\|v_n\|^{\frac{4s}{N-2s}}_{X^s(\mathbb{R}^{N+1}_+)}\end{aligned}$$

which gives $\|\phi_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} = k_0^{-\frac{N-2s}{2}} = \|v_n\|^2_{X^s(\mathbb{R}^{N+1}_+)}$ and this is a contradiction.

Now, observing that $w_n(x,y) = v_n(\alpha_n^{-1}x, \alpha_n^{-1}y)$, it is enough to prove that v_n has a convergent subsequence in $X^{1,s}(\mathbb{R}^{N+1}_+)$. Since $\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$ (see Proposition 2.6 in [47]) we obtain the thesis. □

4. CRITICAL LIMITING PROBLEMS

In this section we extend the previous results for the following critical limiting problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{1}{M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})}\frac{\partial w}{\partial \nu^{1-2s}} = -V_0 w(\cdot,0) + f(w(\cdot,0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where f satisfies (f_1), (f_2') and (f_3'). The study of (4.1) will be done following some arguments used in [50]. In order to find weak solutions to (4.1), we look for critical points of the energy functional $L_{V_0} : X^{1,s}(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ given by

$$L_{V_0}(u) := \frac{1}{2}\widehat{M}\left(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) + \frac{1}{2}\int_{\mathbb{R}^N} V_0 u^2(x,0)\,dx - \int_{\mathbb{R}^N} F(u(x,0))\,dx.$$

We define

$$\mathcal{T}_{V_0} := \left\{u \in X^{1,s}(\mathbb{R}^{N+1}_+)\setminus\{0\} : L'_{V_0}(u) = 0, \max_{\mathbb{R}^N} u(\cdot,0) = u(0,0)\right\},$$

$$b_{V_0} := \inf_{u\in\mathcal{T}_{V_0}} L_{V_0}(u),$$

and

$$\mathcal{S}_{V_0} := \{u \in \mathcal{T}_{V_0} : L_{V_0}(u) = b_{V_0}\}.$$

We consider the following elliptic critical problem:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w}{\partial \nu^{1-2s}} = -V_0 w(\cdot,0) + f(w(\cdot,0)) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.2}$$

Any solution w to (4.2) satisfies the following Pohozaev identity (see [5, 39, 51])

$$\frac{N-2s}{2}\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} - N\int_{\mathbb{R}^N} F(u(x,0)) - \frac{V_0}{2}u^2(x,0)\,dx = 0. \tag{4.3}$$

Let us define

$$\mathcal{E}_{V_0}(u) := \frac{1}{2}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \frac{V_0}{2}|u(\cdot,0)|_2^2 - \int_{\mathbb{R}^N} F(u(x,0))\,dx,$$

$$\tilde{b}_{V_0} := \inf_{u\in\widetilde{\mathcal{T}}_{V_0}} \mathcal{E}_{V_0}(u),$$

where

$$\widetilde{\mathcal{T}}_{V_0} := \left\{u \in X^{1,s}(\mathbb{R}^{N+1}_+)\setminus\{0\} : \mathcal{E}'_{V_0}(u) = 0, \max_{\mathbb{R}^N} u(\cdot,0) = u(0,0)\right\},$$

and

$$\widetilde{\mathcal{S}}_{V_0} := \{u \in \widetilde{\mathcal{T}}_{V_0} : \mathcal{E}_{V_0}(u) = \tilde{b}_{V_0}\}.$$

In what follows, we show that $\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$. Arguing as in the proof of Lemma 3.7 and in view of results in [5, 51], we obtain that:

Lemma 4.1. *Assume (M1)-(M5). Then, $\mathcal{S}_{V_0} \neq \emptyset$ if $\widetilde{\mathcal{S}}_{V_0} \neq \emptyset$. Moreover, there exists an injective map $T : \widetilde{\mathcal{S}}_{V_0} \to \mathcal{S}_{V_0}$. In particular, for any $u \in \widetilde{\mathcal{S}}_{V_0}$,*

$$(Tu)(x,y) := u(x/t_u, y/t_u)$$

where $t_u := \inf\left\{t \in (0,\infty) : t^{2s} = M(t^{N-2s}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})\right\}.$

Lemma 4.2. *Assume that $\widetilde{\mathcal{S}}_{V_0} \neq \emptyset$. Then $\mathcal{S}_{V_0} \neq \emptyset$. Moreover, for any $v \in \mathcal{S}_{V_0}$ there exists $u \in \widetilde{\mathcal{S}}_{V_0}$ such that $v(x,y) = u(x/h_v, y/h_v)$, where $h_v^{2s} = M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})$.*

Proof. By the definition of T, we know that $\mathcal{S}_{V_0} \neq \emptyset$ if $\widetilde{\mathcal{S}}_{V_0} \neq \emptyset$. Let $v \in \mathcal{S}_{V_0}$. Thus v satisfies (4.1) and $L_{V_0}(v) = b_{V_0}$. Define $u(x,y) := v(hx,hy)$ where $h^{2s} := M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. Then, u solves (4.2). Now, we show that $u \in \widetilde{\mathcal{S}}_{V_0}$. To do this, we prove that $\mathcal{E}_{V_0}(u) = \tilde{b}_{V_0}$. Using the Pohozaev identity, we know that

$$\mathcal{E}_{V_0}(u) = \frac{s}{N}\left[\frac{M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})}{(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})^{\frac{2s}{N-2s}}}\right]^{\frac{2s-N}{2s}}.$$

Let $\tilde{u} \in \widetilde{\mathcal{S}}_{V_0}$. Then $\tilde{v} := T\tilde{u} = u(x/t_{\tilde{u}}, y/t_{\tilde{u}}) \in \mathcal{S}_{V_0}$, where $t_{\tilde{u}}$ is defined as in Lemma 4.1. By Lemma 3.1 (which holds even if replace (f_2)-(f_3) by (f_2')-(f_3')), we obtain that

$$\begin{aligned} L_{V_0}(\tilde{v}) &= \frac{1}{2}\left[\widehat{M}(\|\tilde{v}\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1-\frac{2s}{N}\right) M(\|\tilde{v}\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|\tilde{v}\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] = b_{V_0} \\ L_{V_0}(v) &= \frac{1}{2}\left[\widehat{M}(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1-\frac{2s}{N}\right) M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] = b_{V_0}. \end{aligned} \tag{4.4}$$

On the other hand, by the proof of Lemma 3.6 and $(M5)$, it is easy to see that if for some $0 \leq t_1 < t_2$ it holds

$$\widehat{M}(t_1) - \left(1-\frac{2s}{N}\right) M(t_1)t_1 = \widehat{M}(t_2) - \left(1-\frac{2s}{N}\right) M(t_2)t_2$$

then

$$\frac{M(t_1)}{t_1^{2s/(N-2s)}} = \frac{M(t_2)}{t_2^{2s/(N-2s)}}.$$

Hence, by (4.4), it follows that

$$\mathcal{E}_{V_0}(u) = \frac{s}{N}\left[\frac{M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})}{(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})^{\frac{2s}{N-2s}}}\right]^{\frac{2s-N}{2s}} = \frac{s}{N}\|\tilde{u}\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \tilde{b}_{V_0}$$

that is $u \in \widetilde{\mathcal{S}}_{V_0}$. □

Lemma 4.3. *Assume that $\widetilde{\mathcal{S}}_{V_0} \neq \emptyset$. Then there exist $C, c > 0$ (independent of v) such that $c \leq h_v \leq C$ for all $v \in \mathcal{S}_{V_0}$, where h_v is given in Lemma 4.2.*

Proof. Fix $v \in \mathcal{S}_{V_0}$. Then $h_v^{2s} = M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})$. From $(M1)$ we have that $h_v^{2s} \geq m_0$. On the other hand, by Lemma 3.1, we see that for all $v \in \mathcal{S}_{V_0}$,

$$L_{V_0}(v) = \frac{1}{2}\left[\widehat{M}(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \left(1-\frac{2s}{N}\right) M(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)})\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] = b_{V_0}.$$

Thus, in view of $(M2)$, we infer that $\sup_{v\in\mathcal{S}_{V_0}} h_v < \infty$. □

Now, we recall the following result (see [5,34,39]):

Lemma 4.4. *Assume that (f_1), (f_2')-(f_3') hold true. Then:*
(i) there exists $u \in \widetilde{\mathcal{S}}_{V_0}$ such that $u(\cdot,0) \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and radially symmetric;
(ii) $\widetilde{\mathcal{S}}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$.

As a consequence of Lemma 4.2, Lemma 4.3 and Lemma 4.4, we obtain that:

Proposition 4.1. *Under the assumptions of Theorem 1.2 we have that:*
(i) there exists $u \in \mathcal{S}_{V_0}$ such that $u(\cdot,0) \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and radially symmetric;
(ii) $\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$.

5. Proof of Theorem 1.1

In light of Section 2, to study (2.2) we look for critical points of the functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ defined as

$$I_\varepsilon(u) = \frac{1}{2}\widehat{M}\left(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)u^2(x,0)\,dx - \int_{\mathbb{R}^N} F(u(x,0))\,dx$$

where

$$X_\varepsilon := \left\{u \in X^{1,s}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} V_\varepsilon(x)u^2(x,0)\,dx < \infty\right\}$$

endowed with the norm

$$\|u\|_\varepsilon := \left(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \int_{\mathbb{R}^N} V_\varepsilon(x)u^2(x,0)\,dx\right)^{\frac{1}{2}}.$$

It follows from (V_1) that $X_\varepsilon \subset X^{1,s}(\mathbb{R}^{N+1}_+)$ and

$$\|u\|^2_{X^{1,s}(\mathbb{R}^{N+1}_+)} \leq \max\{1, V_1^{-1}\}\|u\|^2_\varepsilon \quad \forall u \in X_\varepsilon.$$

We denote by $(X_\varepsilon)^{-1}$ the dual space of X_ε endowed with the norm $\|T\|_{(X_\varepsilon)^{-1}} := \sup\{Tu : u \in X_\varepsilon, \|u\|_\varepsilon \leq 1\}$. In order to obtain some convergence results and consequently results of existence for small $\varepsilon > 0$, we need to modify $f(t)$ once more. Namely, as in [23, 32], we consider the following Carathéodory function

$$g(x,t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\widehat{f}(t) \quad \text{ for } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

and we write $G(x,t) := \int_0^t g(x,\tau)\, d\tau$, where χ_Λ denotes the characteristic function of Λ, and

$$\widehat{f}(t) := \begin{cases} f(t) & \text{for } t < a, \\ \min\{f(t), \frac{V_1}{2}t\} & \text{for } t \geq a, \end{cases}$$

where $a \in (0, \tau_0)$ is such that $|f(t)| \leq \frac{V_1}{2}t$ for $t \in (0, a]$. By (f_1)-(f_2), it is easy to check that:

- $\lim_{t\to 0} \frac{g(x,t)}{t} = \lim_{t\to 0} \frac{f(t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$,
- $\limsup_{t\to\infty} \frac{g(x,t)}{t^p} \leq \limsup_{t\to\infty} \frac{f(t)}{t^p} < \infty$, for all $x \in \mathbb{R}^N$.

Therefore, we consider the following modified problem:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla u) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{M(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})} \frac{\partial u}{\partial \nu^{1-2s}} = -V_\varepsilon u(\cdot, 0) + g_\varepsilon(\cdot, u(\cdot, 0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{5.1}$$

where we set $g_\varepsilon(x,t) := g(\varepsilon x, t)$. Obviously, if u_ε is a positive solution of (5.1) satisfying $u_\varepsilon(x,0) \leq a$ for $x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$, then u_ε is indeed a solution of (2.2). Now, inspired by [13, 17, 28, 32], we define

$$J_\varepsilon(u) := P_\varepsilon(u) + Q_\varepsilon(u)$$

where

$$P_\varepsilon(u) := \frac{1}{2}\widehat{M}\left(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x) u^2(x,0)\, dx - \int_{\mathbb{R}^N} G_\varepsilon(x, u(x,0))\, dx$$

and

$$Q_\varepsilon(u) := \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2(x,0)\, dx - 1\right)^2_+$$

with

$$\chi_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \Lambda_\varepsilon := \dfrac{\Lambda}{\varepsilon}, \\ \varepsilon^{-1} & \text{if } x \notin \Lambda_\varepsilon. \end{cases}$$

The functional Q_ε will act as a penalization to force the concentration phenomena to occur inside Λ. This type of penalization was first introduced in [17]. Clearly, $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ and its differential is given by:

$$\begin{aligned} \langle J'_\varepsilon(u), v\rangle &= M(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla u \nabla v\, dxdy + \int_{\mathbb{R}^N} V_\varepsilon(x) u(x,0) v(x,0)\, dx \\ &\quad - \int_{\mathbb{R}^N} g_\varepsilon(x, u(x,0)) v(x,0)\, dx + 4\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2(x,0)\, dx - 1\right)^2_+ \int_{\mathbb{R}^N} \chi_\varepsilon(x) u(x,0) v(x,0)\, dx \end{aligned}$$

for all $u, v \in X_\varepsilon$. We stress that a critical point of P_ε is a weak solution to (5.1). In order to find solutions concentrating in Λ as $\varepsilon \to 0$, we look for critical points of J_ε for which Q_ε is zero.

Let $\delta := \dfrac{1}{10} dist\{\mathcal{M}, \mathbb{R}^N \setminus \Lambda\}$. By (f_3) we can choose $\beta \in (0, \delta)$ sufficiently small such that

$$F(T) > \frac{V(x)}{2}T^2 \quad \text{ for all } x \in \mathcal{M}^{5\beta}, \tag{5.2}$$

where

$$\mathcal{M}^\beta := \{z \in \mathbb{R}^N : \inf_{w\in\mathcal{M}} |z - w| \leq \beta\}.$$

Define a nonincreasing function $\phi_0 \in C^\infty(\mathbb{R}_+)$ such that $0 \le \phi \le 1$, $\phi_0 = 1$ in $[0,1]$, $\phi_0 = 0$ in $[2,\infty)$ and $|\phi_0'|_\infty \le C$. In what follows, we look for solutions to (5.1) near the set

$$E_\varepsilon := \Big\{\phi_0(\sqrt{|\,\varepsilon\, x - x'|^2 + \varepsilon^2\, y^2}/\beta) W(x - (x'/\,\varepsilon), y) : x' \in \mathcal{M}^\beta, W \in \mathcal{S}_{V_0}\Big\}.$$

Fix $W^* \in \mathcal{S}_{V_0}$ and define for $t > 0$ and $(x,y) \in \mathbb{R}^{N+1}_+$

$$W_{\varepsilon,t}(x,y) := \phi_0\left(\frac{\varepsilon}{\beta}\sqrt{|x|^2 + y^2}\right) W^*\left(\frac{x}{t}, \frac{y}{t}\right).$$

Next we show that J_ε has a mountain pass geometry [4]. Indeed, by $(M1)$, (V_1), (f_1), (f_2) and $\mathrm{Tr}(X_\varepsilon) \subset L^q(\mathbb{R}^N)$ for all $q \in [2, 2^*_s]$, we have

$$\begin{aligned} J_\varepsilon(u) &\ge \frac{m_0}{2}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x) u^2(x,0)\, dx - \varepsilon\, |u(\cdot,0)|_2^2 - C_\varepsilon |u(\cdot,0)|^{2^*_s}_{2^*_s} \\ &\ge c_1\|u\|^2_\varepsilon - c_2\|u\|^{2^*_s}_\varepsilon. \end{aligned}$$

Hence, there exist $\rho, \delta > 0$ such that $J_\varepsilon(u) \ge \delta$ for $\|u\|_\varepsilon = \rho$.
On the other hand, using the fact that W^* satisfies the Pohozaev identity and $(M3)$, we have

$$\begin{aligned} &L_{V_0}\left(W^*\left(\frac{\cdot}{t}, \frac{\cdot}{t}\right)\right) \\ &= \frac{t^N}{2}\left[\frac{1}{t^N}\widehat{M}\left(t^{N-2s}\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - \left(\frac{N-2s}{N}\right) M\left(\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] \to -\infty, \end{aligned}$$

as $t \to \infty$. Then there exists $t_0 > 0$ such that

$$L_{V_0}\left(W^*\left(\frac{\cdot}{t}, \frac{\cdot}{t}\right)\right) < -2 \quad \forall t \ge t_0. \tag{5.3}$$

Now we prove the following result:

Lemma 5.1. *It holds*

$$\sup_{t \in [0,t_0]} |J_\varepsilon(W_{\varepsilon,t}) - L_{V_0}(W^*_t)| \to 0 \text{ as } \varepsilon \to 0,$$

where $W^*_t(x,y) := W^*(\frac{x}{t}, \frac{y}{t})$ *for* $t > 0$, *and* $W^*_0 \equiv W_{\varepsilon,0} \equiv 0$.

Proof. Since $supp(W_{\varepsilon,t}(\cdot,0)) \subset \Lambda_\varepsilon$ and $supp(\chi_\varepsilon) \subset \mathbb{R}^N \setminus \Lambda_\varepsilon$, we have $Q(W_{\varepsilon,t}) = 0$ and $G_\varepsilon(x, W_{\varepsilon,t}(x,0)) = F(W_{\varepsilon,t}(x,0))$ for all $\varepsilon, t \ge 0$ and $x \in \mathbb{R}^N$. Hence, for all $t \in (0, t_0]$

$$\begin{aligned} |J_\varepsilon(W_{\varepsilon,t}) - L_{V_0}(W^*_t)| \le &\frac{1}{2}|\widehat{M}(\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \widehat{M}(\|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)})| + \frac{1}{2}\int_{\mathbb{R}^N} |V_\varepsilon(x)\phi_0(\varepsilon\,|x|/\beta) - V_0|(W^*_t(x,0))^2\, dx \\ &+ \int_{\mathbb{R}^N} |F(W_{\varepsilon,t}(x,0)) - F(W^*_t(x,0))|\, dx. \end{aligned}$$

Note that as $\varepsilon \to 0$

$$\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)} + o(1) \text{ uniformly in } t \in [0, t_0]. \tag{5.4}$$

Indeed,

$$\begin{aligned} \|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)} &= \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\phi_0(\varepsilon\sqrt{|x|^2+y^2}/\beta)|^2\left(W^*\left(\frac{x}{t}, \frac{y}{t}\right)\right)^2 dxdy \\ &+ \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\phi_0(\varepsilon\sqrt{|x|^2+y^2}/\beta)|^2\left|\nabla W^*\left(\frac{x}{t}, \frac{y}{t}\right)\right|^2 dxdy \\ &+ 2\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla\phi_0(\varepsilon\sqrt{|x|^2+y^2}/\beta)\nabla W^*\left(\frac{x}{t}, \frac{y}{t}\right)\phi_0(\varepsilon\sqrt{|x|^2+y^2}/\beta) W^*\left(\frac{x}{t}, \frac{y}{t}\right) dxdy \\ &=: A_{\varepsilon,t} + B_{\varepsilon,t} + C_{\varepsilon,t}. \end{aligned}$$

Now, by Lemma 2.2, for any $t \in (0, t_0]$ we have

$$
\begin{aligned}
A_{\varepsilon,t} &\leq C\,\varepsilon^2 \iint_{B^+_{\frac{2\beta}{\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{\varepsilon}}(0,0)} y^{1-2s}\left(W^*\left(\frac{x}{t},\frac{y}{t}\right)\right)^2 dxdy \\
&\leq C\,\varepsilon^2 \left[\iint_{B^+_{\frac{2\beta}{\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{\varepsilon}}(0,0)} y^{1-2s}\left(W^*\left(\frac{x}{t},\frac{y}{t}\right)\right)^{2\gamma} dxdy\right]^{\frac{1}{\gamma}} \left[\iint_{B^+_{\frac{2\beta}{\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{\varepsilon}}(0,0)} y^{1-2s}\,dxdy\right]^{1-\frac{1}{\gamma}} \\
&\leq C\,\varepsilon^2 \left[\iint_{B^+_{\frac{2\beta}{\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{\varepsilon}}(0,0)} y^{1-2s}\left(W^*\left(\frac{x}{t},\frac{y}{t}\right)\right)^{2\gamma} dxdy\right]^{\frac{1}{\gamma}} \left[\int_{\frac{\beta}{\varepsilon}}^{\frac{2\beta}{\varepsilon}} r^{N+1-2s}\,dr\right]^{1-\frac{1}{\gamma}} \\
&\leq C \left[\iint_{B^+_{\frac{2\beta}{\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{\varepsilon}}(0,0)} y^{1-2s}\left(W^*\left(\frac{x}{t},\frac{y}{t}\right)\right)^{2\gamma} dxdy\right]^{\frac{1}{\gamma}} \\
&\leq C \left[\iint_{B^+_{\frac{2\beta}{t\,\varepsilon}}(0,0)\setminus B^+_{\frac{\beta}{t\,\varepsilon}}(0,0)} t^{N+2-2s} y^{1-2s}\left(W^*(x,y)\right)^{2\gamma} dxdy\right]^{\frac{1}{\gamma}} \\
&\leq C \left[\iint_{\mathbb{R}^{N+1}_+\setminus B^+_{\frac{\beta}{t_0\,\varepsilon}}(0,0)} t_0^{N+2-2s} y^{1-2s}\left(W^*(x,y)\right)^{2\gamma} dxdy\right]^{\frac{1}{\gamma}} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.5}
$$

On the other hand, for $t \in (0, t_0]$, using that $0 \leq \phi_0 \leq 1$ and ϕ_0 is nonincreasing we get

$$
\begin{aligned}
&\left| B_{\varepsilon,t} - \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla W^*\left(\frac{x}{t},\frac{y}{t}\right)|^2\,dxdy \right| \\
&\leq \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}[1-(\phi_0(\varepsilon\sqrt{|x|^2+y^2}/\beta))^2]\left|\nabla W^*\left(\frac{x}{t},\frac{y}{t}\right)\right|^2 dxdy \\
&= \iint_{\mathbb{R}^{N+1}_+} t^{N-2s} y^{1-2s}[1-(\phi_0(\varepsilon\, t\sqrt{|x|^2+y^2}/\beta))^2]\left|\nabla W^*(x,y)\right|^2 dxdy \\
&\leq \iint_{\mathbb{R}^{N+1}_+} t_0^{N-2s} y^{1-2s}[1-(\phi_0(\varepsilon\, t_0\sqrt{|x|^2+y^2}/\beta))^2]\left|\nabla W^*(x,y)\right|^2 dxdy \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Since Hölder's inequality yields $C_{\varepsilon,t} \leq A^{1/2}_{\varepsilon,t} B^{1/2}_{\varepsilon,t}$, we deduce that

$$
\sup_{t\in[0,t_0]} C_{\varepsilon,t} \to 0 \text{ as } \varepsilon \to 0.
$$

Therefore (5.4) holds true.

Now, noting that $\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)}, \|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)} \leq C$ for all $t \in [0, t_0]$ and $\varepsilon > 0$ sufficiently small, and using $\widehat{M}(t_2) - \widehat{M}(t_1) = \int_{t_1}^{t_2} M(\tau)\,d\tau$ and $(M4)$, we see that

$$
\left|\widehat{M}\left(\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - \widehat{M}\left(\|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\right| \leq M(C)\left|\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)} - \|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right|
$$

which together with (5.4) implies that

$$
\widehat{M}\left(\|W_{\varepsilon,t}\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) = \widehat{M}\left(\|W^*_t\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) + o(1) \text{ uniformly in } t \in [0, t_0].
$$

On the other hand, recalling that (see [26]) $W^*(\cdot, 0)$ has the following polynomial type-decay

$$
0 < W^*(x, 0) \leq \frac{C}{1+|x|^{N+2s}} \quad \forall x \in \mathbb{R}^N,
$$

we have

$$0 < W_t^*(x,0) \leq \frac{Ct_0^{N+2s}}{t_0^{N+2s}+|x|^{N+2s}} \quad \forall x \in \mathbb{R}^N, t \in (0,t_0], \tag{5.6}$$

which together with $0 \leq V_\varepsilon(x)\phi_0(\varepsilon\,|x|/\beta) \leq \max_{x\in\overline{\Gamma^0_{2\beta}}(0)} V(x)$ and $\phi_0(\varepsilon\,\cdot) \to 1$ as $\varepsilon \to 0$, implies that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,t_0]} \left| \int_{\mathbb{R}^N} [V_\varepsilon(x)\phi_0(\varepsilon\,|x|/\beta) - V_0](W_t^*(x,0))^2\, dx \right| = 0.$$

Finally, observing that

$$F(a+b) - F(a) = b\int_0^1 f(a+\tau b)\, d\tau,$$

it follows from (f_1) and (f_2) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F(W_{\varepsilon,t}(x,0)) - F(W_t^*(x,0))|\, dx \\
&\leq \int_{\mathbb{R}^N} |W_{\varepsilon,t}(x,0) - W_t^*(x,0)| \int_0^1 |f(W_t^*(x,0) + \tau(W_{\varepsilon,t}(x,0) - W_t^*(x,0))|\, d\tau dx \\
&\leq C\int_{\mathbb{R}^N} |W_{\varepsilon,t}(x,0) - W_t^*(x,0)|[|W_t^*(x,0)| + |W_{\varepsilon,t}(x,0) - W_t^*(x,0)| \\
&\quad + |W_t^*(x,0)|^{2^*_s-1} + |W_{\varepsilon,t}(x,0) - W_t^*(x,0)|^{2^*_s-1}]\, dx.
\end{aligned}$$

Taking into account $W_{\varepsilon,t}(x,0) - W_t^*(x,0) = (\phi_0(\varepsilon\,|x|/\beta)-1)W_t^*(x,0)$, (5.6) and $\phi_0(\varepsilon\,\cdot) \to 1$ as $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,t_0]} \left| \int_{\mathbb{R}^N} F(W_{\varepsilon,t}(x,0)) - F(W_t^*(x,0))\, dx \right| = 0.$$

□

Notice that from (5.3) and Lemma 5.1 there exists ε_0 sufficiently small such that

$$|J_\varepsilon(W_{\varepsilon,t_0}) - L_{V_0}(W^*_{t_0})| \leq -L_{V_0}(W_{t_0}) - 2 \quad J_\varepsilon(W_{\varepsilon,t_0}) < -2 \quad \text{for}\, \varepsilon \in (0,\varepsilon_0).$$

Therefore, we can define the minimax level

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t))$$

where

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = 0, \gamma(1) = W_{\varepsilon,t_0}\}.$$

Lemma 5.2. $\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}$.

Proof. We first prove that

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq c_{V_0}. \tag{5.7}$$

Since $W_{\varepsilon,t} \to 0$ in X_ε as $t \to 0$, and setting

$$\gamma_\varepsilon(\tau) := W_{\varepsilon,\tau t_0} \text{ for } \tau \in (0,1],\ \gamma_\varepsilon(0) = 0, \tag{5.8}$$

we see that $\gamma_\varepsilon \in \Gamma_\varepsilon$ and thus

$$c_\varepsilon \leq \max_{t\in[0,1]} J_\varepsilon(\gamma_\varepsilon(t)) = \max_{t\in[0,t_0]} J_\varepsilon(W_{\varepsilon,t}). \tag{5.9}$$

By Lemma 5.1, Pohozaev Identity and $(M5)$ we deduce that

$$\begin{aligned}
\max_{t\in[0,t_0]} J_\varepsilon(W_{\varepsilon,t}) &= \max_{t\in[0,t_0]} L_{V_0}\left(W^*\left(\frac{\cdot}{t},\frac{\cdot}{t}\right)\right) + o(1) \\
&= \max_{t\in[0,t_0]} \left[\frac{1}{2}\widehat{M}\left(t^{N-2s}\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right) - t^N\left(\frac{N-2s}{2N}\right) M\left(\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right)\|W^*\|^2_{X^s(\mathbb{R}^{N+1}_+)}\right] + o(1) \\
&\leq L_{V_0}(W^*) + o(1) = c_{V_0} + o(1).
\end{aligned}$$

Next, we show that

$$\liminf_{\varepsilon\to 0} c_\varepsilon \geq c_{V_0}. \tag{5.10}$$

Assume by contradiction that $\liminf_{\varepsilon\to 0} c_\varepsilon < c_{V_0}$. Then there exist $\alpha>0$, $\varepsilon_n\to 0$ and $\gamma_n\in\Gamma_{\varepsilon_n}$ such that $\max_{t\in[0,1]} J_{\varepsilon_n}(\gamma_n(t)) < c_{V_0}-\alpha$. Take ε_n such that

$$\frac{V_0}{2}\,\varepsilon_n[1+(1+c_{V_0})^2] < \min\{\alpha,1\} \text{ and } P_{\varepsilon_n}(\gamma_n(1)) < -2.$$

Denoting ε_n by ε and γ_n by γ, since $P_\varepsilon(\gamma(0))=0$, we can find $t_0\in(0,1)$ such that

$$P_\varepsilon(\gamma(t_0)) = -1 \text{ and } P_\varepsilon(\gamma(t)) \quad \forall t\in[0,t_0].$$

Hence,

$$Q_\varepsilon(\gamma(t)) \le J_\varepsilon(\gamma(t)) + 1 < c_{V_0} - \alpha + 1 < c_{V_0} + 1$$

and consequently

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \gamma(t)^2\,dx \le \varepsilon[1+(1+c_{V_0})^2] \text{ for } t\in[0,t_0].$$

Since $G(x,t)\le F(t)$ we obtain for $t\in[0,t_0]$

$$\begin{aligned} P_\varepsilon(\gamma(t)) &\ge L_{V_0}(\gamma(t)) - \frac{V_0}{2}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\gamma(t)^2\,dx \\ &\ge L_{V_0}(\gamma(t)) - \frac{V_0}{2}\,\varepsilon[1+(1+c_{V_0})^2] \end{aligned}$$

which yields

$$L_{V_0}(\gamma(t_0)) \le \frac{V_0}{2}\,\varepsilon[1+(1+c_{V_0})^2] - 1 < 0.$$

On the other hand, the mountain pass level corresponds to the least energy level (see Lemma 3.5), so we have

$$\max_{t\in[0,t_0]} L_{V_0}(\gamma(t)) \ge c_{V_0}.$$

From

$$c_{V_0} - \alpha > \max_{t\in[0,1]} L_{V_0}(\gamma(t)) \ge \max_{t\in[0,t_0]} P_\varepsilon(\gamma(t))$$

we get

$$c_{V_0} - \alpha > c_{V_0} - \frac{V_0}{2}\,\varepsilon[1+(1+c_{V_0})^2] > c_{V_0} - \alpha$$

and this gives a contradiction.
Now, we define

$$d_\varepsilon := \max_{t\in[0,1]} J_\varepsilon(\gamma_\varepsilon(t)), \tag{5.11}$$

where γ_ε is given in (5.8). Then, by (5.7), (5.9) and (5.10) we see that $c_\varepsilon\le d_\varepsilon$ and

$$\lim_{\varepsilon\to 0} d_\varepsilon = \lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}.$$

This ends the proof of lemma. □

Now we use the notations

$$J_\varepsilon^b := \{w\in X_\varepsilon : J_\varepsilon(w)\le b\},$$

and for $A\subset X_\varepsilon$

$$A^b := \{w\in X_\varepsilon : \inf_{v\in A}\|w-v\|_\varepsilon \le b\}.$$

The next lemma will be crucial to prove the main result of this work.

Lemma 5.3. *There exists $d_0>0$ such that for any (ε_n) and (w_{ε_n}) with*

$$\lim_{n\to\infty}\varepsilon_n = 0,\ w_{\varepsilon_n}\in E_{\varepsilon_n}^{d_0},\ \lim_{n\to\infty} J_{\varepsilon_n}(w_{\varepsilon_n}) \le c_{V_0},\ \lim_{n\to\infty}\|J'_{\varepsilon_n}(w_{\varepsilon_n})\|_{(X_{\varepsilon_n})^{-1}} = 0,$$

there exists, up to a subsequence, $(z_n)\subset\mathbb{R}^N$, $x_0\in\mathcal{M}$ and $W\in\mathcal{S}_{V_0}$ such that

$$\lim_{n\to\infty}|\varepsilon_n z_n - x_0| = 0 \text{ and } \lim_{n\to\infty}\|w_{\varepsilon_n} - \phi_0(\varepsilon_n\sqrt{|x-z_n|^2+y^2}/\beta)W(x-z_n,y)\|_{\varepsilon_n} = 0.$$

Proof. For simplicity, we write ε instead of ε_n and the same will be done for the subsequences. By the definition of $E_\varepsilon^{d_0}$ and the compactness of $\mathcal{S}_{V_0}$ and $\mathcal{M}^\beta$, there exist $W_0 \in \mathcal{S}_{V_0}$ and $(x_\varepsilon) \subset \mathcal{M}^\beta$ such that for all $\varepsilon > 0$ small enough

$$\left\| w_\varepsilon - \phi_0\left(\frac{\varepsilon}{\beta}\sqrt{\left|x - \frac{x_\varepsilon}{\varepsilon}\right|^2 + y^2}\right) W_0\left(x - \frac{x_\varepsilon}{\varepsilon}, y\right)\right\|_\varepsilon \leq 2d_0, \tag{5.12}$$

and, as $\varepsilon \to 0$,

$$x_\varepsilon \to x_0 \in \mathcal{M}^\beta.$$

In what follows, we prove that there exist $(w_{\varepsilon,1}), (w_{\varepsilon,2}) \subset X_\varepsilon$, $(k_\varepsilon), (j_\varepsilon) \subset \mathbb{N}$ such that

- (i) $k_\varepsilon \leq \sqrt{\beta_\varepsilon/5\,\varepsilon}$ and $k_\varepsilon \to \infty$ as $\varepsilon \to 0$, $0 \leq j_\varepsilon \leq k_\varepsilon - 1$, $|w_{\varepsilon,1}|, |w_{\varepsilon,2}| \leq |w_\varepsilon|$,
- (ii) $w_{\varepsilon,1} = w_\varepsilon$ in $B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+(5j_\varepsilon+1)k_\varepsilon}(\frac{x_\varepsilon}{\varepsilon}, 0)$, $\quad w_{\varepsilon,2} = w_\varepsilon$ in $\mathbb{R}^{N+1}_+ \setminus B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+(5j_\varepsilon+4)k_\varepsilon}(\frac{x_\varepsilon}{\varepsilon}, 0)$
- (iii) $supp(w_{\varepsilon,1}) \subset \overline{B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+(5j_\varepsilon+2)k_\varepsilon}(\frac{x_\varepsilon}{\varepsilon}, 0)}$, $\quad supp(w_{\varepsilon,2}) \subset \mathbb{R}^{N+1}_+ \setminus B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+(5j_\varepsilon+3)k_\varepsilon}(\frac{x_\varepsilon}{\varepsilon}, 0)$,
- (iv) $\|w_\varepsilon - w_{\varepsilon,1} - w_{\varepsilon,2}\|_\varepsilon \to 0$ as $\varepsilon \to 0$,
- (v) $\|w_\varepsilon\|_{X_0^s(B_{j_\varepsilon,\varepsilon})} \to 0$ and

$$\iint_{B_{j_\varepsilon,\varepsilon}} y^{1-2s}|w_\varepsilon|^{2\gamma}\, dxdy \to 0 \text{ as } \varepsilon \to 0,$$

where

$$B_{j_\varepsilon,\varepsilon} := \overline{B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+5(j_\varepsilon+1)k_\varepsilon}\left(\frac{x_\varepsilon}{\varepsilon}, 0\right)} \setminus B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+5j_\varepsilon k_\varepsilon}\left(\frac{x_\varepsilon}{\varepsilon}, 0\right),$$

and

$$\int_{\Gamma_{j_\varepsilon,\varepsilon}} V_\varepsilon(x)|w_\varepsilon(x,0)|^2\, dx \to 0 \text{ as } \varepsilon \to 0,$$

where

$$\Gamma_{j_\varepsilon,\varepsilon} := \overline{\Gamma^0_{(\frac{2\beta_\varepsilon}{\varepsilon})+5(j_\varepsilon+1)k_\varepsilon}\left(\frac{x_\varepsilon}{\varepsilon}\right)} \setminus \Gamma^0_{(\frac{2\beta_\varepsilon}{\varepsilon})+5j_\varepsilon k_\varepsilon}\left(\frac{x_\varepsilon}{\varepsilon}\right).$$

Let $k_\varepsilon \in \mathbb{N}$ be such that $k_\varepsilon \leq \sqrt{\frac{\beta}{5\,\varepsilon}}$ and $k_\varepsilon \to \infty$ as $\varepsilon \to 0$, and put $\widetilde{w}_\varepsilon(x,y) := w_\varepsilon(x + \frac{x_\varepsilon}{\varepsilon}, y)$. By (5.12), Lemma 2.2-$(i)$ and $\phi_0(\varepsilon\sqrt{|x|^2 + y^2}/\beta) = 0$ in $\mathbb{R}^{N+1}_+ \setminus B^+_{\frac{2\beta}{\varepsilon}}(0,0)$ we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+ \setminus B^+_{\frac{2\beta}{\varepsilon}}(0,0)} y^{1-2s}|\nabla \tilde{w}_\varepsilon|^2\, dxdy + \int_{\mathbb{R}^N \setminus \Gamma^0_{\frac{2\beta}{\varepsilon}}(0)} V(\varepsilon\, x + x_\varepsilon)|\tilde{w}_\varepsilon(x,0)|^2\, dx \\
&+ \left(\iint_{\mathbb{R}^{N+1}_+ \setminus B^+_{\frac{2\beta}{\varepsilon}}(0,0)} y^{1-2s}|\tilde{w}_\varepsilon|^{2\gamma}\, dxdy\right)^{\frac{1}{\gamma}} \leq Cd_0.
\end{aligned} \tag{5.13}$$

For all $j = 0, 1, \ldots, k_\varepsilon - 1$, we set

$$\tilde{B}_{j,\varepsilon} := \overline{B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+5(j+1)k_\varepsilon}}(0,0) \setminus B^+_{(\frac{2\beta_\varepsilon}{\varepsilon})+5jk_\varepsilon}(0,0) \text{ and } \tilde{\Gamma}_{j,\varepsilon} := \overline{\Gamma^0_{(\frac{2\beta_\varepsilon}{\varepsilon})+5(j+1)k_\varepsilon}}(0) \setminus \Gamma^0_{(\frac{2\beta_\varepsilon}{\varepsilon})+5jk_\varepsilon}(0).$$

From (5.13) we deduce that

$$\begin{aligned}
&\sum_{j=0}^{k_\varepsilon-1} \iint_{\tilde{B}_{j,\varepsilon}} y^{1-2s}|\nabla \tilde{w}_\varepsilon|^2\, dxdy + \sum_{j=0}^{k_\varepsilon-1} \int_{\tilde{\Gamma}_{j,\varepsilon}} V(\varepsilon\, x + x_\varepsilon)|\tilde{w}_\varepsilon(x,0)|^2\, dx \\
&+ \sum_{j=0}^{k_\varepsilon-1} \left(\iint_{\tilde{B}_{j,\varepsilon}} y^{1-2s}|\tilde{w}_\varepsilon|^{2\gamma}\, dxdy\right)^{\frac{1}{\gamma}} \leq Cd_0.
\end{aligned}$$

Hence, there exists $j_\varepsilon \in \{0, 1, \dots, k_\varepsilon - 1\}$ such that

$$\iint_{\tilde{B}_{j_\varepsilon,\varepsilon}} y^{1-2s}|\nabla \tilde{w}_\varepsilon|^2\, dxdy + \int_{\tilde{\Gamma}_{j_\varepsilon,\varepsilon}} V(\varepsilon x + x_\varepsilon)|\tilde{w}_\varepsilon(x,0)|^2\, dx + \left(\iint_{\tilde{B}_{j_\varepsilon,\varepsilon}} y^{1-2s}|\tilde{w}_\varepsilon|^{2\gamma}\, dxdy\right)^{\frac{1}{\gamma}} \leq Cd_0/k_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{5.14}$$

Define two cut-off functions $(\xi_{\varepsilon,1})$ and $(\xi_{\varepsilon,2})$ such that

$$\xi_{\varepsilon,1} := \begin{cases} 1 & \text{in } \overline{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0,0)}, \\ 0 & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0,0), \end{cases}$$

and

$$\xi_{\varepsilon,2} := \begin{cases} 0 & \text{in } \overline{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+3)k_\varepsilon}(0,0)}, \\ 1 & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{\frac{2\beta}{\varepsilon}} + (5j_\varepsilon+4)k_\varepsilon(0,0), \end{cases}$$

and $0 \leq \xi_{\varepsilon,1}, \xi_{\varepsilon,2} \leq 1$, $|\nabla \xi_{\varepsilon,1}|, |\nabla \xi_{\varepsilon,2}| \leq \frac{C}{k_\varepsilon}$, and we set

$$\tilde{w}_{\varepsilon,i} := \xi_{\varepsilon,i}\tilde{w}_\varepsilon \text{ and } w_{\varepsilon,i}(x,y) := \tilde{w}_{\varepsilon,i}\left(x - \frac{x_\varepsilon}{\varepsilon}, y\right) \text{ for } i = 1, 2.$$

Since $w_\varepsilon \in X_\varepsilon$, we see that $w_{\varepsilon,i} \in X_{\varepsilon_i}$ for $i = 1, 2$. Hence, (i)-(iii) hold true. Now, direct calculations show that

$$\begin{aligned}
\|w_\varepsilon - w_{\varepsilon,1} - w_{\varepsilon,2}\|^2_\varepsilon &\leq C \iint_{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+4)k_\varepsilon}(0,0)\setminus B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0,0)} y^{1-2s}|\nabla \tilde{w}_\varepsilon|^2\, dxdy \\
&+ C\int_{\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+4)k_\varepsilon}(0)\setminus\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0)} V(\varepsilon x + x_\varepsilon)|\tilde{w}_\varepsilon|^2\, dx \\
&+ C\iint_{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0,0)\setminus B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0,0)} y^{1-2s}|\nabla \xi_{\varepsilon,1}|^2|\tilde{w}_\varepsilon|^2\, dxdy \\
&+ C\iint_{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+4)k_\varepsilon}(0,0)\setminus B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+3)k_\varepsilon}(0,0)} y^{1-2s}|\nabla \xi_{\varepsilon,2}|^2|\tilde{w}_\varepsilon|^2\, dxdy \\
&=: (I)_\varepsilon + (II)_\varepsilon + (III)_\varepsilon + (IV)_\varepsilon.
\end{aligned}$$

Using (5.14) we deduce that $(I)_\varepsilon, (II)_\varepsilon = o(1)$. Moreover, arguing as in (5.5), it follows from (5.14) that

$$(III)_\varepsilon \leq C\left(\iint_{B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0,0)\setminus B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0,0)} y^{1-2s}|\tilde{w}_\varepsilon|^{2\gamma}\, dxdy\right)^{\frac{1}{\gamma}} = o(1).$$

In a similar fashion we can prove that $(IV)_\varepsilon = o(1)$. In conclusion, (iv) holds true. Moreover, by (5.14), we see that (v) is satisfied. Taking into account (i)-(v), (f_1)-(f_2) and the boundedness of (w_ε) in X_ε we get

$$\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \|w_{\varepsilon,1}\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \|w_{\varepsilon,2}\|^2_{X^s(\mathbb{R}^{N+1}_+)} + o(1), \tag{5.15}$$

$$\int_{\mathbb{R}^N} V_\varepsilon(x)w^2_\varepsilon(x,0)\, dx = \int_{\mathbb{R}^N} V_\varepsilon(x)w^2_{\varepsilon,1}(x,0)\, dx + \int_{\mathbb{R}^N} V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\, dx + o(1), \tag{5.16}$$

$$\int_{\mathbb{R}^N} F(w_\varepsilon(x,0))\, dx = \int_{\mathbb{R}^N} F(w_{\varepsilon,1}(x,0))\, dx + \int_{\mathbb{R}^N} F(w_{\varepsilon,2}(x,0))\, dx + o(1). \tag{5.17}$$

By $(M1)$, we know that

$$\widehat{M}(t_1 + t_2) = \widehat{M}(t_1) + \int_{t_1}^{t_1+t_2} M(\tau)\, d\tau \geq \widehat{M}(t_1) + m_0 t_2,$$

which together with (5.15)-(5.17), the boundedness of (w_ε) in X_ε and $G(x,t)\leq F(t)$ implies that

$$J_\varepsilon(w_\varepsilon)\geq I_\varepsilon(w_{\varepsilon,1})+\frac{m_0}{2}\|w_{\varepsilon,2}\|^2_{X^s(\mathbb{R}^{N+1}_+)}+\frac{1}{2}\int_{\mathbb{R}^N}V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\,dx-\int_{\mathbb{R}^N}F(w_{\varepsilon,2}(x,0))\,dx+o(1). \tag{5.18}$$

Now, we prove that $\|w_{\varepsilon,2}\|_\varepsilon\to 0$ as $\varepsilon\to 0$. By (5.12), (iv) and the definition of $w_{\varepsilon,2}$, we see that

$$\begin{aligned}
\|w_{\varepsilon,2}\|_\varepsilon &\leq \left\|w_{\varepsilon,1}-\phi_0\left(\frac{\varepsilon}{\beta}\sqrt{\left|x-\frac{x_\varepsilon}{\varepsilon}\right|^2+y^2}\right)W_0\left(x-\frac{x_\varepsilon}{\varepsilon},y\right)\right\|_\varepsilon+2d_0+o(1)\\
&=\left\|w_{\varepsilon,1}-\phi_0\left(\frac{\varepsilon}{\beta}\sqrt{\left|x-\frac{x_\varepsilon}{\varepsilon}\right|^2+y^2}\right)W_0\left(x-\frac{x_\varepsilon}{\varepsilon},y\right)\right\|_{X_\varepsilon\left(B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0,0)\right)}+2d_0+o(1)\\
&\leq \|w_{\varepsilon,2}\|_{X_\varepsilon\left(B^+_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0,0)\right)}+2d_0+o(1)\\
&=4d_0+o(1),
\end{aligned}$$

which yields

$$\limsup_{\varepsilon\to 0}\|w_{\varepsilon,2}\|_\varepsilon\leq 4d_0. \tag{5.19}$$

On the other hand, using $\langle J'_\varepsilon(w_\varepsilon),w_{\varepsilon,1}\rangle=o(1)$, $\langle Q'_\varepsilon(w_\varepsilon),w_{\varepsilon,2}\rangle=\langle Q'_\varepsilon(w_{\varepsilon,2}),w_{\varepsilon,2}\rangle\geq 0$, $(M1)$, (V_1), (f_1)-(f_2), (iii), (iv), (5.19), the boundedness of (w_ε) in X_ε, we get

$$\begin{aligned}
&m_0\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla w_{\varepsilon,2}|^2\,dxdy+\int_{\mathbb{R}^N}V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\,dx\\
&\leq M(\|w_\varepsilon\|^2_\varepsilon)\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla w_{\varepsilon,2}|^2\,dxdy+\int_{\mathbb{R}^N}V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\,dx\\
&\leq M(\|w_\varepsilon\|^2_\varepsilon)\iint_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla w_{\varepsilon,2}|^2\,dxdy+\int_{\mathbb{R}^N}V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\,dx+\langle Q'(w_{\varepsilon,2}),w_{\varepsilon,2}\rangle\\
&=\int_{\mathbb{R}^N}g_\varepsilon(x,w_{\varepsilon,2}(x,0))w_{\varepsilon,2}(x,0)\,dx+o(1)\\
&\leq\delta\int_{\mathbb{R}^N}w^2_{\varepsilon,2}(x,0)\,dx+C_\delta\int_{\mathbb{R}^N}|w_{\varepsilon,2}(x,0)|^{2^*_s}+o(1)\\
&\leq\frac{\delta}{V_1}\int_{\mathbb{R}^N}V_\varepsilon(x)w^2_{\varepsilon,2}(x,0)\,dx+C_\delta|w_{\varepsilon,2}(x,0)|^{2^*_s}_{2^*_s}+o(1).
\end{aligned}$$

Then, choosing $\delta>0$ sufficiently small and using Lemma 2.1 we deduce that $\|w_{\varepsilon,2}\|^2_\varepsilon\leq C\|w_{\varepsilon,2}\|^{2^*_s}_\varepsilon+o(1)$. Taking $d_0>0$ small enough, we deduce that $\|w_{\varepsilon,2}\|_\varepsilon=o(1)$. Hence, in view of (5.18), we have

$$J_\varepsilon(w_\varepsilon)\geq I_\varepsilon(w_{\varepsilon,1})+o(1). \tag{5.20}$$

Up to a subsequence, we can find $\tilde{w}\in X^{1,s}(\mathbb{R}^{N+1}_+)$ such that

$$\tilde{w}_{\varepsilon,1}\rightharpoonup\tilde{w}\text{ in }X^{1,s}(\mathbb{R}^{N+1}_+)\text{ and }\tilde{w}_{\varepsilon,1}(\cdot,0)\rightharpoonup\tilde{w}(\cdot,0)\text{ in }L^q_{loc}(\mathbb{R}^N)\quad\forall q\in[1,2^*_s). \tag{5.21}$$

In what follows we show that

$$\tilde{w}_{\varepsilon,1}(\cdot,0)\to\tilde{w}(\cdot,0)\text{ in }L^q(\mathbb{R}^N)\quad\forall q\in(2,2^*_s). \tag{5.22}$$

Indeed, by vanishing Lions-type lemma (see Lemma 3.3 in [36]), we assume by contradiction that there exists $r>0$ such that

$$\liminf_{\varepsilon\to 0}\sup_{z\in\mathbb{R}^N}\int_{\Gamma^0_1(z)}|\tilde{w}_{\varepsilon,1}(x,0)-\tilde{w}(x,0)|^2\,dx=2r>0.$$

Then, for $\varepsilon>0$ small, there exists $z_\varepsilon\in\mathbb{R}^N$ such that

$$\int_{\Gamma^0_1(z_\varepsilon)}|\tilde{w}_{\varepsilon,1}(x,0)-\tilde{w}(x,0)|^2\,dx\geq r>0. \tag{5.23}$$

By (5.21) we see that (z_ε) is unbounded, so, up to a subsequence, $|z_\varepsilon| \to \infty$. Then, by (5.23),

$$\liminf_{\varepsilon\to 0} \int_{\Gamma_1^0(z_\varepsilon)} |\tilde{w}_{\varepsilon,1}(x,0)|^2 \, dx \geq r > 0. \tag{5.24}$$

Since $\xi_{\varepsilon,1}(x,0) = 0$ for $|x| \geq (\frac{2\beta}{\varepsilon}) + (5j_\varepsilon + 2)k_\varepsilon$, we deduce that $|z_\varepsilon| < (\frac{2\beta}{\varepsilon}) + (5j_\varepsilon + 3)k_\varepsilon$ for $\varepsilon > 0$ small enough. Therefore, we may assume that

$$\varepsilon\, z_\varepsilon \to z_0 \in \overline{\Gamma_{3\beta}^0}(0) \quad \text{ and } \quad \bar{w}_\varepsilon(x,y) := \tilde{w}_{\varepsilon,1}(x+z_\varepsilon, y) \rightharpoonup \bar{w}(x,y) \text{ in } X^{1,s}(\mathbb{R}^{N+1}_+). \tag{5.25}$$

Now, we show that $\bar{w}$ satisfies

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla \bar{w}) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{\alpha_0}\frac{\partial \bar{w}}{\partial \nu^{1-2s}} = -V(x_0+z_0)\bar{w}(\cdot,0) + f(\bar{w}(\cdot,0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{5.26}$$

where

$$\alpha_0 := \lim_{\varepsilon\to 0} M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}).$$

Fix $k \geq 1$. Since $x_0 + z_0 \in \mathcal{M}^{4\beta} \subset \Lambda$, there exists $n_0 = n_0(k) \in \mathbb{N}$ such that $\varepsilon\, x + x_\varepsilon + \varepsilon\, z_\varepsilon \in \Lambda$ for all $x \in \Gamma_k^0(0)$ and $n \geq n_0$. By the definition of χ_ε and $g(x,t)$ it follows that

$$\left\langle Q'(w_\varepsilon), \phi\left(\cdot - \frac{x_\varepsilon}{\varepsilon} - z_\varepsilon\right)\right\rangle = 0 \text{ and } g(\varepsilon\, x + x_\varepsilon + \varepsilon\, z_\varepsilon, t)\phi = f(t)\phi,$$

for all $n \geq n_0$ and $\phi \in C_c^\infty(B_k^+(0,0) \cup \Gamma_k^0(0))$. From $\langle J'_\varepsilon(w_\varepsilon), \phi(\cdot - \frac{x_\varepsilon}{\varepsilon} - z_\varepsilon)\rangle = o(1)$, (iv) and $\|w_{\varepsilon,2}\|_\varepsilon = o(1)$ we can deduce that

$$\begin{aligned} o(1) = M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}) &\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla \bar{w}_\varepsilon \nabla \phi \, dxdy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon\, x + x_\varepsilon + \varepsilon\, z_\varepsilon)\bar{w}_\varepsilon(x,0)\phi(x,0)\, dx - \int_{\mathbb{R}^N} f(\bar{w}_\varepsilon(x,0))\phi(x,0)\, dx. \end{aligned}$$

Note that by $(M1)$ and the boundedness of (w_ε) in X_ε it holds $m_0 \leq \alpha_0 \leq C$. Then, by (5.25) and the arbitrariness of k we get

$$0 = \alpha_0 \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla \bar{w} \nabla \phi \, dxdy + \int_{\mathbb{R}^N} V(x_0+z_0)\bar{w}(x,0)\phi(x,0)\, dx - \int_{\mathbb{R}^N} f(\bar{w}(x,0))\phi(x,0)\, dx,$$

for all $\phi \in C_c^\infty(\mathbb{R}^{N+1}_+)$, which proves the claim.

Since $\bar{w} \neq 0$ by (5.24), it follows from the Pohozaev identity that

$$d_{V(x_0+z_0)} \leq \frac{s}{N}\alpha_0 \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \bar{w}|^2 \, dxdy, \tag{5.27}$$

where

$$d_{V(x_0+z_0)} := \inf\left\{L_{\alpha_0, V(x_0+z_0)}(u) : u \in X^{1,s}(\mathbb{R}^{N+1}_+)\setminus\{0\} : L'_{\alpha_0,V(x_0+z_0)}(u) = 0\right\}$$

and

$$L_{\alpha_0,V(x_0+z_0)}(u) := \frac{\alpha_0}{2}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \frac{V(x_0+z_0)}{2}\int_{\mathbb{R}^N} u^2(x,0)\, dx - \int_{\mathbb{R}^N} F(u(x,0))\, dx.$$

We observe that, by the results in [7], it turns out that $d_{V(x_0+z_0)} > 0$. Then, for $R > 0$ large enough we get

$$\begin{aligned} \liminf_{\varepsilon\to 0} \frac{s}{N}\alpha_0 \iint_{B_R^+(z_\varepsilon + (\frac{x_\varepsilon}{\varepsilon}),0)} y^{1-2s}|\nabla w_\varepsilon|^2 \, dxdy &= \liminf_{\varepsilon\to 0} \frac{s}{N}\alpha_0 \iint_{B_R^+(z_\varepsilon + (\frac{x_\varepsilon}{\varepsilon}),0)} y^{1-2s}|\nabla w_{\varepsilon,1}|^2 \, dxdy \\ &= \liminf_{\varepsilon\to 0} \frac{s}{N}\alpha_0 \iint_{B_R^+(0,0)} y^{1-2s}|\nabla \bar{w}_\varepsilon|^2 \, dxdy \\ &\geq \frac{s}{N}\alpha_0 \iint_{B_R^+(0,0)} y^{1-2s}|\nabla \bar{w}|^2 \, dxdy \\ &\geq \frac{1}{2}\frac{s}{N}\alpha_0 \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \bar{w}|^2 \, dxdy \\ &\geq \frac{1}{2} d_{V(x_0+z_0)} > 0. \end{aligned}$$

On the other hand, arguing as in (5.5), it follows from (5.12) and $|z_\varepsilon|\to\infty$ that

$$
\begin{aligned}
&\alpha_0 \iint_{B_R^+(z_\varepsilon+(\frac{x_\varepsilon}{\varepsilon}),0)} y^{1-2s}|\nabla w_\varepsilon|^2\,dxdy\\
&\leq C\iint_{B_R^+(z_\varepsilon+(\frac{x_\varepsilon}{\varepsilon}),0)} y^{1-2s}\left|\nabla\left(\phi_0\left(\frac{\varepsilon}{\beta}\sqrt{\left|x-\frac{x_\varepsilon}{\varepsilon}\right|^2+y^2}\right)W_0\left(x-\frac{x_\varepsilon}{\varepsilon},y\right)\right)\right|^2 dxdy+Cd_0\\
&\leq C\iint_{B_R^+(z_\varepsilon,0)} y^{1-2s}|\nabla W_0|^2\,dxdy+C\,\varepsilon^2\iint_{B_R^+(z_\varepsilon,0)} y^{1-2s}|W_0|^2\,dxdy+Cd_0\\
&\leq C\iint_{B_R^+(z_\varepsilon,0)} y^{1-2s}|\nabla W_0|^2\,dxdy+C\,\varepsilon^2R^2\left(\iint_{B_R^+(z_\varepsilon,0)} y^{1-2s}|W_0|^{2\gamma}\,dxdy\right)^{\frac{1}{2\gamma}}+Cd_0\\
&=Cd_0+o(1)
\end{aligned}
$$

which leads to a contradiction for $d_0>0$ small enough. Consequently, (5.22) holds true.
Then, by (f_1)-(f_2) and (5.22), we have as $\varepsilon\to 0$

$$
\int_{\mathbb{R}^N} F(\tilde{w}_{\varepsilon,1}(x,0))\,dx\to\int_{\mathbb{R}^N}F(\tilde{w}(x,0))\,dx \text{ and } \int_{\mathbb{R}^N} f(\tilde{w}_{\varepsilon,1}(x,0))\tilde{w}_{\varepsilon,1}(x,0)\,dx\to\int_{\mathbb{R}^N} f(\tilde{w}(x,0))\tilde{w}(x,0)\,dx. \tag{5.28}
$$

Moreover, we can see that as $\varepsilon\to 0$

$$
\int_{\mathbb{R}^N} g(\varepsilon\,x+x_\varepsilon,\tilde{w}_\varepsilon(x,0))\tilde{w}_{\varepsilon,1}(x,0)\,dx\to\int_{\mathbb{R}^N} f(\tilde{w}(x,0))\tilde{w}(x,0)\,dx. \tag{5.29}
$$

Indeed, using $x_\varepsilon\to x_0\in\mathcal{M}^\beta\subset\Lambda$ and the definition of $\tilde{w}_{\varepsilon,1}$, for all $x\in\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0)$ we have

$$
g(\varepsilon\,x+x_\varepsilon,\tilde{w}_\varepsilon(x,0))\tilde{w}_{\varepsilon,1}(x,0)=f(\tilde{w}_\varepsilon(x,0))\tilde{w}_{\varepsilon,1}(x,0), \tag{5.30}
$$

since $\varepsilon\,x+x_\varepsilon\in\mathcal{M}^{4\beta}\subset\Lambda$ for all $x\in\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0)$ and $\varepsilon>0$ small. Furthermore, as $\varepsilon\to 0$

$$
\int_{\mathbb{R}^N} f(\tilde{w}_\varepsilon(x,0))\tilde{w}_{\varepsilon,1}(x,0)\,dx=\int_{\mathbb{R}^N} f(\tilde{w}_{\varepsilon,1}(x,0))\tilde{w}_{\varepsilon,1}(x,0)\,dx+o(1), \tag{5.31}
$$

because (f_1),(f_2) and (5.22) yield

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int_{\mathbb{R}^N}[f(\tilde{w}_\varepsilon(x,0))-f(\tilde{w}_{\varepsilon,1}(x,0))]\tilde{w}_{\varepsilon,1}(x,0)\,dx\right|\\
&\quad=\limsup_{\varepsilon\to 0}\left|\int_{\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0)\setminus\Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+1)k_\varepsilon}(0)}[f(\tilde{w}_\varepsilon(x,0))-f(\tilde{w}_{\varepsilon,1}(x,0))]\tilde{w}_{\varepsilon,1}(x,0)\,dx\right|\\
&\quad\leq\delta C+C_\delta\limsup_{\varepsilon\to 0}|\tilde{w}_{\varepsilon,1}(\cdot,0)|_{L^{p+1}(\mathbb{R}^N\setminus\Gamma^0_{2\beta/\varepsilon}(0))}\\
&\quad\leq\delta C+C_\delta\left[\limsup_{\varepsilon\to 0}|\tilde{w}_{\varepsilon,1}(\cdot,0)-\tilde{w}(\cdot,0)|_{p+1}+\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus\Gamma^0_{2\beta/\varepsilon}(0)}|\tilde{w}(x,0)|^{p+1}\,dx\right]\\
&\quad=\delta C\quad\forall\delta>0.
\end{aligned}
$$

Gathering (5.28), (5.30) and (5.31) we get (5.29).
Now, we note that, arguing as before, $\tilde{w}$ satisfies

$$
\begin{cases}
-\operatorname{div}(y^{1-2s}\nabla\tilde{w})=0 & \text{in } \mathbb{R}^{N+1}_+,\\
\frac{1}{\alpha_0}\frac{\partial\tilde{w}}{\partial\nu^{1-2s}}=-V(x_0)\tilde{w}(\cdot,0)+f(\tilde{w}(\cdot,0)) & \text{in } \mathbb{R}^N,
\end{cases} \tag{5.32}
$$

with

$$
\alpha_0:=\lim_{\varepsilon\to 0}M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)})=\lim_{\varepsilon\to 0}M(\|w_{\varepsilon,1}\|^2_{X^s(\mathbb{R}^{N+1}_+)})=\lim_{\varepsilon\to 0}M(\|\tilde{w}_{\varepsilon,1}\|^2_{X^s(\mathbb{R}^{N+1}_+)}),
$$

where in the second identity we used that $\|w_\varepsilon-w_{\varepsilon,1}\|_\varepsilon=o(1)$ thanks to (iv) and $\|w_{\varepsilon,2}\|_\varepsilon=o(1)$, and in the third one that $\tilde{w}_{\varepsilon,1}(x,y)=w_{\varepsilon,1}(x+\frac{x_\varepsilon}{\varepsilon},y)$.

Taking into account (5.21), (5.29), (5.32), (iv) and $\langle J'_\varepsilon(w_\varepsilon), w_{\varepsilon,1}\rangle = o(1)$, $\|w_{\varepsilon,2}\|_\varepsilon = o(1)$, $\langle Q'_\varepsilon(w_\varepsilon), w_{\varepsilon,1}\rangle = 0$ and $\tilde{w}_{\varepsilon,1}(x,y) = w_{\varepsilon,1}(x+\frac{x_\varepsilon}{\varepsilon}, y)$, we have

$$\begin{aligned}
&\alpha_0 \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}|^2\, dxdy + \int_{\mathbb{R}^N} V(x_0)\tilde{w}^2(x,0)\, dx \\
&\le \liminf_{\varepsilon\to 0}\left[M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}_{\varepsilon,1}|^2\, dxdy + \int_{\mathbb{R}^N} V(\varepsilon\, x + x_\varepsilon)\tilde{w}^2_{\varepsilon,1}(x,0)\, dx\right] \\
&\le \limsup_{\varepsilon\to 0}\left[M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}_{\varepsilon,1}|^2\, dxdy + \int_{\mathbb{R}^N} V(\varepsilon\, x + x_\varepsilon)\tilde{w}^2_{\varepsilon,1}(x,0)\, dx\right] \\
&= \limsup_{\varepsilon\to 0}\left[M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}\nabla w_\varepsilon \nabla w_{\varepsilon,1}\, dxdy + \int_{\mathbb{R}^N} V_\varepsilon(x) w_\varepsilon(x,0) w_{\varepsilon,1}(x,0)\, dx\right] \\
&= \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N} g_\varepsilon(x, w_\varepsilon(x,0)) w_{\varepsilon,1}(x,0)\, dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} g(\varepsilon\, x + x_\varepsilon, \tilde{w}_\varepsilon(x,0))\tilde{w}_{\varepsilon,1}(x,0)\, dx \\
&= \int_{\mathbb{R}^N} f(\tilde{w}(x,0))\tilde{w}(x,0)\, dx \\
&= \alpha_0 \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}|^2\, dxdy + \int_{\mathbb{R}^N} V(x_0)\tilde{w}^2(x,0)\, dx
\end{aligned}$$

which yields

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_{\varepsilon,1}|^2\, dxdy = \lim_{\varepsilon\to 0}\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}_{\varepsilon,1}|^2\, dxdy = \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{w}|^2\, dxdy \tag{5.33}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} V(\varepsilon\, x) w^2_{\varepsilon,1}(x,0)\, dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} V(\varepsilon\, x + x_\varepsilon)\tilde{w}^2_{\varepsilon,1}(x,0)\, dx = \int_{\mathbb{R}^N} V(x_0)\tilde{w}^2(x,0)\, dx. \tag{5.34}$$

In particular,

$$\alpha_0 = M(\|\tilde{w}\|^2_{X^s(\mathbb{R}^{N+1}_+)}).$$

Putting together (5.20), (5.28), (5.33), (5.34) we deduce that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(w_\varepsilon) \ge \liminf_{\varepsilon\to 0} I_\varepsilon(w_{\varepsilon,1}) \ge L_{V(x_0)}(\tilde{w})$$

which combined with (5.12) gives

$$L_{V(x_0)}(\tilde{w}) \le c_{V_0}.$$

Since $\tilde{w} \ne 0$, it follows from (5.2) that

$$L_{V(x_0)}(\tilde{w}) \ge c_{V(x_0)}.$$

Then, using the fact that $x_0 \in \mathcal{M}^\beta \subset \Lambda$, the above inequalities and the monotonicity of $m \mapsto c_m$ (see Remark 3.1), we have that $V(x_0) = V_0$ and thus $x_0 \in \mathcal{M}$. At this point, it is clear that there exist $W \in \mathcal{S}_{V_0}$ and $z_0 \in \mathbb{R}^N$ such that $\tilde{w}(x,y) = W(x - z_0, y)$.
On the other hand, observing that

$$V(x_0) = V_0 \le V(\varepsilon\, x + x_\varepsilon) \text{ on } \Gamma^0_{\frac{2\beta}{\varepsilon}+(5j_\varepsilon+2)k_\varepsilon}(0),$$

we combine (5.33) with (5.34) to infer that $\tilde{w}_{\varepsilon,1} \to \tilde{w}$ in $X^{1,s}(\mathbb{R}^{N+1}_+)$ as $\varepsilon \to 0$, which implies that

$$\lim_{\varepsilon\to 0}\left\| w_\varepsilon - \phi_0\left(\frac{\varepsilon}{\beta}\sqrt{\left|x - \left(\frac{x_\varepsilon}{\varepsilon} + z_0\right)\right|^2 + y^2}\right) W\left(x - \left(\frac{x_\varepsilon}{\varepsilon} + z_0\right), y\right)\right\|_\varepsilon = 0.$$

This ends the proof of lemma. □

Corollary 5.1. *For any $d \in (0, d_0)$ there exist constants $\omega > 0$ and $\varepsilon_d > 0$ such that $\|J'_\varepsilon(w)\|_{(X_\varepsilon)^{-1}} \geq \omega$ for $w \in J_\varepsilon^{d_\varepsilon} \cap (E_\varepsilon^{d_0} \setminus E_\varepsilon^d)$ and $\varepsilon \in (0, \varepsilon_d)$. Here d_ε is defined as in (5.11).*

Proof. Assume by contradiction that there exist $d \in (0, d_0)$, (ε_n) and (w_n) such that

$$\varepsilon_n \in \left(0, \frac{1}{n}\right), \quad w_n \in J_{\varepsilon_n}^{d_{\varepsilon_n}} \cap (E_{\varepsilon_n}^{d_0} \setminus E_{\varepsilon_n}^d), \quad \|J'_{\varepsilon_n}(w_n)\|_{(X_{\varepsilon_n})^{-1}} < \frac{1}{n}.$$

By Lemma 5.3, we can find $(z_n) \subset \mathbb{R}^N$, $x_0 \in \mathcal{M}$ and $W \in \mathcal{S}_{V_0}$ such that

$$\lim_{n\to\infty} |\,\varepsilon_n z_n - x_0| = 0 \text{ and } \lim_{n\to\infty} \|w_n - \phi_0(\varepsilon_n \sqrt{|x - z_n|^2 + y^2}/\beta) W(x - z_n, y)\|_{\varepsilon_n} = 0,$$

which imply that $w_n \in E_{\varepsilon_n}^d$ for n sufficiently large. This is impossible because $w_n \in E_{\varepsilon_n}^{d_0} \setminus E_{\varepsilon_n}^d$. □

Lemma 5.4. *Given $\lambda > 0$ there exist $\varepsilon_0 > 0$ and $d_0 > 0$ small enough such that*

$$J_\varepsilon(w) > c_{V_0} - \lambda \quad \text{for all } w \in E_\varepsilon^{d_0} \quad \text{and } \varepsilon \in (0, \varepsilon_0).$$

Proof. If $w \in E_\varepsilon$ then there exist $W \in \mathcal{S}_{V_0}$ and $x' \in \mathcal{M}^\beta$ such that

$$w(x, y) = \phi_0(\sqrt{|\,\varepsilon\, x - x'|^2 + \varepsilon^2\, y^2}/\beta) W(x - (x'/\,\varepsilon), y).$$

Using $L_{V_0}(W) = c_{V_0}$, (V_2) and $G(x, t) \leq F(t)$ we get

$$\begin{aligned} J_\varepsilon(w) - c_{V_0} \geq \frac{1}{2}\left[\widehat{M}(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \widehat{M}(\|W\|^2_{X^s(\mathbb{R}^{N+1}_+)})\right] + \frac{V_0}{2}\int_{\mathbb{R}^N} (\phi_0^2(\varepsilon\,|x|/\beta) - 1) W^2(x, 0)\, dx \\ - \int_{\mathbb{R}^N} F(\phi_0^2(\varepsilon\,|x|/\beta) W(x, 0)) - F(W(x, 0))\, dx \end{aligned}$$

independently of $x' \in \mathcal{M}^\beta$. Arguing as in the proof of Lemma 5.1, we can see that there exists $\varepsilon_0 > 0$ such that

$$J_\varepsilon(w) - c_{V_0} > -\frac{\lambda}{2} \quad \text{for all } w \in E_\varepsilon \quad \text{and } \varepsilon \in (0, \varepsilon_0).$$

Now, if $v \in E_\varepsilon^d$, then there exists $w \in E_\varepsilon$ such that $\|w - v\|_\varepsilon \leq d$. Hence, $v = w + z$ with $\|z\|_\varepsilon \leq d$. Observing that $Q_\varepsilon(w) = 0$, we have

$$\begin{aligned} J_\varepsilon(v) - J_\varepsilon(w) \geq \frac{1}{2}[\widehat{M}(\|w + z\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \widehat{M}(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})] + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)[(w(x, 0) + z(x, 0))^2 - w^2(x, 0)]\, dx \\ - \int_{\mathbb{R}^N} G_\varepsilon(x, w(x, 0) + z(x, 0)) - G_\varepsilon(x, w(x, 0))\, dx. \end{aligned}$$

Since E_ε is uniformly bounded for $\varepsilon \in (0, \varepsilon_0)$ (see the estimates in the proof of Lemma 5.1), we obtain that for $\varepsilon \in (0, \varepsilon_0)$

$$|\|w + z\|^2_\varepsilon - \|w\|^2_\varepsilon| \leq \|z\|^2_\varepsilon + 2\|w\|_\varepsilon \|z\|_\varepsilon \leq d^2 + Cd \to 0 \text{ as } d \to 0.$$

Moreover, noting that $\widehat{M}(t_2) - \widehat{M}(t_1) = \int_{t_1}^{t_2} M(\tau)\, d\tau$ and $(M5)$ yield

$$|\widehat{M}(\|w + z\|^2_{X^s(\mathbb{R}^{N+1}_+)}) - \widehat{M}(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})| \leq M(C)|\|w + z\|^2_{X^s(\mathbb{R}^{N+1}_+)} - \|z\|^2_{X^s(\mathbb{R}^{N+1}_+)}| \to 0 \text{ as } d \to 0,$$

we can find $d_0 > 0$ small enough such that

$$J_\varepsilon(v) > J_\varepsilon(w) - \frac{\lambda}{2} > c_{V_0} - \lambda \quad \forall v \in E_\varepsilon^{d_0} \quad \forall\, \varepsilon \in (0, \varepsilon_0).$$

This ends the proof of lemma. □

By Corollary 5.1 and Lemma 5.4, we fix $d_1 \in (0, \frac{d_0}{3})$ and corresponding $\omega > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} &\|J'_\varepsilon(w)\|_{(X_\varepsilon)^{-1}} \geq \omega \quad \text{for all } w \in J_\varepsilon^{d_\varepsilon} \cap (E_\varepsilon^{d_0} \setminus \mathcal{E}_\varepsilon^{d_1}) \\ &J_\varepsilon(w) > \frac{c_{V_0}}{2} \quad \text{for all } w \in E_\varepsilon^{d_0}. \end{aligned}$$

Lemma 5.5. *There exists $\alpha > 0$ such that*

$$|t - 1/t_0| \le \alpha \text{ implies that } \gamma_\varepsilon(t) \in E_\varepsilon^{d_1} \text{ for all } \varepsilon \in (0, \varepsilon_0),$$

where γ_ε is given by (5.8) and t_0 was chosen in (5.3).

Proof. Firstly, we note that there exists $C_0 > 0$ such that

$$\left\| \phi_0 \left(\frac{\varepsilon}{\beta} \sqrt{|x|^2 + y^2} \right) v \right\|_\varepsilon \le C_0 \|v\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} \quad \forall \varepsilon \in (0, \varepsilon_0) \quad \forall v \in X^{1,s}(\mathbb{R}^{N+1}_+).$$

Since the map $\psi : [0, t_0] \to X^{1,s}(\mathbb{R}^{N+1}_+)$ defined as $\psi(t) := W_t^*$ is continuous, we can find $\sigma > 0$ such that $\|W_t^* - W^*\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} < \frac{d_1}{C_0}$ whenever $|t - 1| \le \sigma$. Hence, if $|tt_0 - 1| \le \sigma$, then $|t - \frac{1}{t_0}| \le \frac{\sigma}{t_0} =: \alpha$ and this yields

$$\|\gamma_\varepsilon(t) - W_{\varepsilon,1}\|_\varepsilon = \left\| \phi_0 \left(\frac{\varepsilon}{\beta} \sqrt{|x|^2 + y^2} \right) (W_{tt_0}^* - W^*) \right\|_\varepsilon \le C_0 \|W_{tt_0}^* - W^*\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} < d_1.$$

Since $W_{\varepsilon,1} \in E_\varepsilon$ (recall that $0 \in \mathcal{M}$ and $W^* \in \mathcal{S}_{V_0}$), we deduce that $\gamma_\varepsilon(t) \in E_\varepsilon^{d_1}$. □

Lemma 5.6. *For α given in Lemma 5.5 there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that*

$$J_\varepsilon(\gamma_\varepsilon(t)) < c_{V_0} - \rho, \quad \text{for any } \varepsilon \in (0, \varepsilon_0) \quad \text{and } |t - 1/t_0| \ge \alpha.$$

Proof. By $(M5)$ and (5.3), we know that $t = 1$ is a maximum point of $L_{V_0}(W_t^*)$ in $[0, t_0]$ (see the proof of Lemma 3.5). Then, we find $\rho > 0$ such that

$$L_{V_0}(W_t^*) < c_{V_0} - 2\rho \text{ for } |t - 1| \ge t_0\alpha.$$

On the other hand, by Lemma 5.1, there exists $\varepsilon_0 > 0$ such that

$$\sup_{t \in [0,t_0]} |J_\varepsilon(W_{\varepsilon,t}) - L_{V_0}(W_t^*)| < \rho \text{ for } \varepsilon \in (0, \varepsilon_0).$$

Consequently, for $|t - 1| \ge t_0\alpha$ and $\varepsilon \in (0, \varepsilon_0)$, we have

$$J_\varepsilon(W_{\varepsilon,t}) \le L_{V_0}(W_t^*) + |J_\varepsilon(W_{\varepsilon,t}) - L_{V_0}(W_t^*)| < c_{V_0} - 2\rho + \rho = c_{V_0} - \rho.$$

□

In the light of Lemma 5.5 and Lemma 5.6, we can argue as in the proof of Proposition 5.2 in [32] (see also [13,28,35]), to obtain the following result that we state without giving the details.

Lemma 5.7. *There exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ there exists a sequence $(w_{n,\varepsilon}) \subset J_\varepsilon^{d_\varepsilon + \varepsilon} \cap E_\varepsilon^{d_0}$ such that $J'_\varepsilon(w_{n,\varepsilon}) \to 0$ in $(X_\varepsilon)^{-1}$ as $n \to \infty$.*

Now we are ready to give the proof of the main result of this section.

Proof of Theorem 1.1. By Lemma 5.7, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$ there exists a sequence $(w_{n,\varepsilon}) \subset J_\varepsilon^{d_\varepsilon + \varepsilon} \cap E_\varepsilon^{d_0}$ such that $J'_\varepsilon(w_{n,\varepsilon}) \to 0$ in $(X_\varepsilon)^{-1}$ as $n \to \infty$. Since $(w_{n,\varepsilon})$ is bounded in X_ε, up to a subsequence, as $n \to \infty$, we have

$$w_{n,\varepsilon} \rightharpoonup w_\varepsilon \text{ in } X_\varepsilon, \tag{5.35}$$

and

$$\lambda_{n,\varepsilon} := \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) w_{n,\varepsilon}^2(x, 0)\, dx - 1 \right)_+ \to \lambda_\varepsilon. \tag{5.36}$$

Then, it is easy to verify that

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w_\varepsilon) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{1}{\alpha_\varepsilon} \frac{\partial w_\varepsilon}{\partial \nu^{1-2s}} = -V_\varepsilon w_\varepsilon(\cdot, 0) - 4\lambda_\varepsilon \chi_\varepsilon w_\varepsilon(\cdot, 0) + g_\varepsilon(x, w_\varepsilon(\cdot, 0)) & \text{in } \mathbb{R}^N, \end{cases} \tag{5.37}$$

where

$$\alpha_\varepsilon := \lim_{n \to \infty} M(\|w_{n,\varepsilon}\|^2_{X^s(\mathbb{R}^{N+1}_+)}).$$

By $(M1)$, $(M4)$ and the boundedness of $(w_{n,\varepsilon})$ in X_ε we know that

$$m_0 \le \alpha_\varepsilon \le C \quad \forall \varepsilon \in (0, \bar{\varepsilon}]. \tag{5.38}$$

Next, we show that $(w_{n,\varepsilon})$ is tight in $X^s(\mathbb{R}^{N+1}_+)$ (see definition 3.2.1 in [25]). To prove this, for all fixed $\varepsilon \in (0,\bar{\varepsilon}]$, take $R>0$ such that $\Lambda_\varepsilon \subset \Gamma^0_R(0)$, and set $\phi_R(x,y) := \bar{\phi}(\sqrt{|x|^2+y^2}/R)$ where $\bar{\phi} \in C^\infty(\mathbb{R}_+)$ is such that $\bar{\phi}=0$ in $[0,1]$, $\bar{\phi}=1$ in $[2,\infty)$, $0\le\bar{\phi}\le 1$ and $|\bar{\phi}'|_\infty \le C$. Since $(\phi_R w_{n,\varepsilon})$ is bounded in X_ε for each $\varepsilon\in(0,\bar{\varepsilon}]$, we deduce that $\langle J'_\varepsilon(w_{n,\varepsilon}), \phi_R w_{n,\varepsilon}\rangle \to 0$ as $n\to\infty$, and so, by the definition of g_ε, we get

$$\alpha_\varepsilon \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_{n,\varepsilon}|^2\phi_R\,dxdy + \int_{\mathbb{R}^N} V_\varepsilon(x) w^2_{n,\varepsilon}(x,0)\phi_R(x,0)\,dx \\ \le \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x) w^2_{n,\varepsilon}(x,0)\phi_R(x,0)\,dx - \alpha_\varepsilon \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} w_{n,\varepsilon}\nabla w_{n,\varepsilon}\nabla\phi_R\,dxdy. \tag{5.39}$$

Arguing as in (5.5), and using Hölder's inequality, (5.38), (5.35) and Lemma 2.2-(ii), we get

$$\limsup_{n\to\infty}\left|\alpha_\varepsilon \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} w_{n,\varepsilon}\nabla w_{n,\varepsilon}\nabla\phi_R\,dxdy\right| \\ \le \frac{C}{R}\limsup_{n\to\infty}\left[\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_{n,\varepsilon}|^2\,dxdy\right)^{\frac{1}{2}}\left(\iint_{B^+_{2R}(0,0)\setminus B^+_R(0,0)} y^{1-2s}|w_{n,\varepsilon}|^2\,dxdy\right)^{\frac{1}{2}}\right] \\ \le \frac{C}{R}\left(\iint_{B^+_{2R}(0,0)\setminus B^+_R(0,0)} y^{1-2s}|w_\varepsilon|^2\,dxdy\right)^{\frac{1}{2}} \\ \le C\left(\iint_{B^+_{2R}(0,0)\setminus B^+_R(0,0)} y^{1-2s}|w_\varepsilon|^{2\gamma}\,dxdy\right)^{\frac{1}{2\gamma}} \to 0 \text{ as } R\to\infty. \tag{5.40}$$

Putting together (5.38), (5.39) and (5.40) we obtain

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{N+1}_+\setminus B^+_{2R}(0,0)} y^{1-2s}|\nabla w_{n,\varepsilon}|^2\,dxdy + \int_{\mathbb{R}^N\setminus\Gamma^0_{2R}(0)} V_\varepsilon(x) w^2_{n,\varepsilon}(x,0)\,dx = 0, \tag{5.41}$$

which implies that $(w_{n,\varepsilon})$ is tight in X_ε. In particular, by (5.41) and the compactness of $H^s(\mathbb{R}^N)\subset L^2_{loc}(\mathbb{R}^N)$, we deduce that $w_{n,\varepsilon}(\cdot,0)\to w_\varepsilon(\cdot,0)$ in $L^2(\mathbb{R}^N)$ as $n\to\infty$. Hence, by interpolation, $w_{n,\varepsilon}(\cdot,0)\to w_\varepsilon(\cdot,0)$ in $L^q(\mathbb{R}^N)$ for all $q\in[2,2^*_s)$. By the definition of g_ε, (f_1)-(f_2), we have as $n\to\infty$

$$\int_{\mathbb{R}^N} g_\varepsilon(x, w_{n,\varepsilon}(x,0))w_{n,\varepsilon}(x,0)\,dx \to \int_{\mathbb{R}^N} g_\varepsilon(x,w_\varepsilon(x,0))w_\varepsilon(x,0)\,dx. \tag{5.42}$$

In the light of (5.35), (5.37), (5.42), $\langle J'_\varepsilon(w_{n,\varepsilon}), w_{n,\varepsilon}\rangle\to 0$ and arguing as at the end of the proof of Lemma 5.3, we deduce that

$$w_{n,\varepsilon}\to w_\varepsilon \text{ in } X_\varepsilon \text{ as } n\to\infty,\ \alpha_\varepsilon = M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)}) \text{ and } \lambda_\varepsilon = \left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)w^2_\varepsilon(x,0)\,dx - 1\right)_+. \tag{5.43}$$

Since $\mathcal{S}_{V_0}$ is compact in $X^{1,s}(\mathbb{R}^{N+1}_+)$, it is easy to check that $0\notin E^{d_0}_\varepsilon$ for $\varepsilon>0$, $d_0>0$ small. Hence, $w_\varepsilon\in E^{d_0}_\varepsilon\cap J^{d_\varepsilon+\varepsilon}$ is a nontrivial solution to (5.37).

Now, for any sequence (ε_n) such that $\varepsilon_n\to 0$ as $n\to\infty$, by Lemma 5.3 there exist, up to a subsequence, $(z_n)\subset\mathbb{R}^N$, $x_0\in\mathcal{M}$ and $W\in\mathcal{S}_{V_0}$ such that

$$\lim_{n\to\infty}|\varepsilon_n z_n - x_0| = 0 \tag{5.44}$$

and

$$\lim_{n\to\infty}\|w_{\varepsilon_n} - \phi_0(\varepsilon_n\sqrt{|x-z_n|^2+y^2}/\beta)W(x-z_n,y)\|_{\varepsilon_n} = 0,$$

which implies that

$$\lim_{n\to\infty}\|\bar{w}_{\varepsilon_n} - W\|_{X^{1,s}(\mathbb{R}^{N+1}_+)} = 0, \tag{5.45}$$

where $\bar{w}_{\varepsilon_n}(x,y) := w_{\varepsilon_n}(x+z_n,y)$. In view of (5.37), (5.38), (5.43) and (5.45), we can use a Moser iteration scheme (see for instance [6,11,25]) and repeat the same arguments in [3,9,11,37] to deduce that

$$\lim_{|x|\to\infty} \bar{w}_{\varepsilon_n}(x,0)=0 \text{ uniformly for } \varepsilon_n \text{ small,} \tag{5.46}$$

which guarantees the existence of a constant $\rho>0$ such that $f(\tilde{w}_{\varepsilon_n}(x,0)) \le \frac{V_0}{2}\tilde{w}_{\varepsilon_n}(x,0)$ for all $|x|\ge\rho$ and ε_n small. When $|x|\le\rho$, it follows from (5.44) that $\Gamma^0_{\varepsilon_n\rho}(\varepsilon_n z_n)\subset\Lambda$ for ε_n small enough, and so

$$g_{\varepsilon_n}(x+z_n,\bar{w}_{\varepsilon_n}(x,0)) = f(\bar{w}_{\varepsilon_n}(x,0)) \text{ for } \varepsilon_n \text{ small.} \tag{5.47}$$

From (5.46) and (f_1), we can find $R>0$ big enough such that

$$f(\bar{w}_{\varepsilon_n}(x,0)) \le \frac{1}{2}V(\varepsilon_n x+\varepsilon_n z_n)\bar{w}_{\varepsilon_n}(x,0) \text{ for } x\in\mathbb{R}^N\setminus\Gamma^0_R(0).$$

On the other hand, arguing as in [3,8,9], we see that

$$|\bar{w}_{\varepsilon_n}(x,0)| \le \frac{C}{1+|x|^{N+2s}} \text{ for } \varepsilon_n \text{ small,}$$

for some $C>0$ independent of ε_n. Then, noting that $\mathbb{R}^N\setminus(\Lambda_{\varepsilon_n}-z_n)\subset\mathbb{R}^N\setminus\Gamma^0_{\frac{\beta}{\varepsilon_n}}(0)$, we obtain

$$\begin{aligned}\varepsilon_n^{-1}\int_{\mathbb{R}^N\setminus\Lambda_{\varepsilon_n}} w^2_{\varepsilon_n}(x,0)\,dx &= \varepsilon_n^{-1}\int_{\mathbb{R}^N\setminus(\Lambda_{\varepsilon_n}-z_n)} \bar{w}^2_{\varepsilon_n}(x,0)\,dx\\ &\le C\,\varepsilon_n^{-1}\int_{\mathbb{R}^N\setminus\Gamma^0_{\frac{\beta}{\varepsilon_n}}(0)} \frac{1}{(1+|x|^{N+2s})^2}\,dx \to 0 \text{ as } n\to\infty,\end{aligned}$$

which implies that $Q_{\varepsilon_n}(w_{\varepsilon_n})=0$ for ε_n small enough. This together with (5.47) implies that w_{ε_n} is a solution to (2.2). Hence, $u_{\varepsilon_n}(x):=w_{\varepsilon_n}(\frac{x}{\varepsilon_n},0)$ is a solution to (1.1). Since $u_\varepsilon\in L^\infty(\mathbb{R}^N)$, $u_\varepsilon\ge0$ in $\mathbb{R}^N$, V and f are continuous functions, and using $(M1)$, from the Harnack inequality [19,33] we have that $u_\varepsilon>0$ in $\mathbb{R}^N$.

Now, let P_n be a global maximum point of $\bar{w}_{\varepsilon_n}(\cdot,0)$. Since $\bar{w}_{\varepsilon_n}$ solves (5.1) with V_{ε_n} replaced by $V_{\varepsilon_n}(\cdot+z_n)$, it follows from (V_1), (f_1)-(f_2) that

$$V_1|\bar{w}_{\varepsilon_n}(\cdot,0)|_2^2 \le \frac{V_1}{2}|\bar{w}_{\varepsilon_n}(\cdot,0)|_2^2 + C|\bar{w}_{\varepsilon_n}(\cdot,0)|_\infty^{2^*_s-2}|\bar{w}_{\varepsilon_n}(\cdot,0)|_2^2$$

which implies that $|\bar{w}_{\varepsilon_n}(\cdot,0)|_\infty\ge\delta>0$ for all $n\in\mathbb{N}$. Then, $\bar{w}_{\varepsilon_n}(P_n,0)\ge\delta>0$ for all $n\in\mathbb{N}$, and (P_n) is bounded by (5.46). Noting that $u_{\varepsilon_n}(x)=\bar{w}_{\varepsilon_n}(\frac{x}{\varepsilon_n}-z_n,0)$, we deduce that $x_n:=\varepsilon_n P_n+\varepsilon_n z_n$ is a global maximum point of u_{ε_n}. From (5.44) we get $x_n\to x_0\in\mathcal{M}$ as $n\to\infty$. Finally, we can argue as in [8,9,37] to deduce the polynomial decay of u_ε.

□

6. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We borrow some arguments used in [50].
In view of Proposition 4.1 there exists $\kappa>0$ such that

$$\sup_{u\in\mathcal{S}_{V_0}}|u(\cdot,0)|_\infty = \sup_{u\in\tilde{\mathcal{S}}_{V_0}}|u(\cdot,0)|_\infty<\kappa. \tag{6.1}$$

For any $k>\max_{t\in[0,\kappa]}f(t)$, define $f_k(t):=\min\{f(t),k\}$. Now, we consider the truncated problem

$$\begin{cases}\varepsilon^{2s}M(\varepsilon^{2s-N}[u]_s^2)(-\Delta)^s u+V(x)u=f_k(u) & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N),\quad u>0 & \text{in } \mathbb{R}^N.\end{cases} \tag{6.2}$$

In what follows, we prove that, for small $\varepsilon>0$, there exists a positive solution v_ε to (6.2) satisfying the properties of Theorem 1.2. Clearly, v_ε is a solution to (1.1) if $|v_\varepsilon|_\infty<\kappa$. We consider the limiting problem

$$\begin{cases}M([u]_s^2)(-\Delta)^s u+V_0u=f_k(u) & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N),\quad u>0 & \text{in } \mathbb{R}^N,\end{cases} \tag{6.3}$$

and the corresponding extended problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w)=0 & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{1}{M(\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)})}\frac{\partial w}{\partial \nu^{1-2s}}=-V_0 w(\cdot,0)+f_k(w(\cdot,0)) & \text{in } \mathbb{R}^N,\end{cases} \tag{6.4}$$

whose associated energy functional is given by

$$L^k_{V_0}(u)=\frac{1}{2}\widehat{M}(\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)})+\frac{V_0}{2}|u(\cdot,0)|_2^2-\int_{\mathbb{R}^N}F_k(u(x,0))\,dx.$$

Lemma 6.1. *Under the same assumptions of Theorem 1.2, (6.4) admits a positive ground state solution.*

Proof. Firstly we show that f_k satisfies (f_1)-(f_3). It is clear that (f_1)-(f_2) are true. Now, for any $u\in\widetilde{\mathcal{S}}_{V_0}$, we know that u fulfills the Pohozaev identity

$$\frac{N-2s}{N}\|u\|^2_{X^s(\mathbb{R}^{N+1}_+)}=N\int_{\mathbb{R}^N}F(u(x,0))-\frac{V_0}{2}u^2(x,0)\,dx,$$

which yields

$$\int_{\mathbb{R}^N}F(u(x,0))-\frac{V_0}{2}u^2(x,0)\,dx\geq 0.$$

If $F(u(x,0))-\frac{V_0}{2}u^2(x,0)\leq 0$ for all $x\in\mathbb{R}^N$, then $\frac{F(u(x,0))}{u^2(x,0)}=V_0>0$ for all $x\in\mathbb{R}^N$. Using (f_1') and that $u(x,0)\to 0$ as $|x|\to\infty$, we get $\frac{F(u(x,0))}{u^2(x,0)}\to 0$ as $|x|\to\infty$, that is a contradiction. Then, we can find $x_0\in\mathbb{R}^N$ such that $F(u(x_0,0))>\frac{V_0}{2}u^2(x_0,0)$. Since $|u(x_0,0)|<\kappa$, it follows that $F_k(u(x,0))=F(u(x,0))$ for all $x\in\mathbb{R}^N$. Hence, letting $T=u(x_0,0)>0$, we obtain that $F_k(T)>\frac{V_0}{2}T^2$, that is (f_3) is satisfied. From [7, 16, 51] we know that

$$(-\Delta)^s u+V_0u=f_k(u) \text{ in } \mathbb{R}^N$$

admits a radially symmetric ground state solution. At this point, we apply Lemma 3.7 to deduce the assertion. □

Let $\mathcal{S}^k_{V_0}$ be the set of ground state solutions u to (6.3) such that $u(0,0)=\max_{x\in\mathbb{R}^N}u(x,0)$. Then, by Lemma 6.1 we deduce that $\mathcal{S}^k_{V_0}\neq\emptyset$.

Lemma 6.2. *For $k>\max_{t\in[0,\kappa]}f(t)$, we have*

$$\mathcal{S}^k_{V_0}=\mathcal{S}_{V_0}.$$

Proof. In the light of Lemma 4.1 and Lemma 4.2 it is enough to prove that $\widetilde{S}^k_{V_0}=\widetilde{S}_{V_0}$. This is proved in Corollary 4.3 in [39]. □

Now we provide the proof of the main result of this section.

Proof of Theorem 1.2. Since f_k satisfies (f_1)-(f_3), we can invoke Theorem 1.1 to deduce that, fixed $k>\max_{t\in[0,\kappa]}f(t)$, there exists $\varepsilon_0>0$ such that (6.2) admits a positive solution v_ε for $\varepsilon\in(0,\varepsilon_0)$. Moreover, there exists $U\in\mathcal{S}^k_{V_0}$ and a maximum point x_ε of v_ε such that $\lim_{\varepsilon\to 0}dist(x_\varepsilon,\mathcal{M})=0$ and $v_\varepsilon(\varepsilon\cdot+x_\varepsilon)\to U(\cdot+z_0)$ as $\varepsilon\to 0$ in $H^s(\mathbb{R}^N)$, for some $z_0\in\mathbb{R}^N$. Letting $w_\varepsilon=v_\varepsilon(\varepsilon\cdot+x_\varepsilon)$ we see that w_ε satisfies

$$M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)})(-\Delta)^s w_\varepsilon+V_\varepsilon\left(x+\frac{x_\varepsilon}{\varepsilon}\right)w_\varepsilon=f(w_\varepsilon) \text{ in } \mathbb{R}^N.$$

Clearly,

$$m_0\leq\inf_{\varepsilon<\varepsilon_0}M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)})\leq\sup_{\varepsilon<\varepsilon_0}M(\|w_\varepsilon\|^2_{X^s(\mathbb{R}^{N+1}_+)})<\infty.$$

Then, we can argue as in Step 2 of the proof of Theorem 1.1 in [39] and use Lemma 6.2 to infer that there exists $\varepsilon^*>0$ such that $|v_\varepsilon|_\infty<\kappa$ for all $\varepsilon\in(0,\varepsilon^*)$, which implies that $f_k(v_\varepsilon)=f(v_\varepsilon)$ in $\mathbb{R}^N$. In conclusion, v_ε is a positive solution to (1.1). □

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