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28 June 2026

CONCENTRATION PHENOMENA FOR THE FRACTIONAL RELATIVISTIC SCHRÖDINGER-CHOQUARD EQUATION

VINCENZO AMBROSIO

Abstract. We consider the fractional relativistic Schrödinger-Choquard equation

$$\begin{cases} (-\Delta+m^2)^s u+V(\varepsilon\, x)u=\left(\frac{1}{|x|^{\mu}}*F(u)\right)f(u) & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon>0$ is a small parameter, $s\in(0,1)$, $m>0$, $N>2s$, $\mu\in(0,2s)$, $(-\Delta+m^2)^s$ is the fractional relativistic Schrödinger operator, $V:\mathbb{R}^N\to\mathbb{R}$ is a continuous potential having a local minimum, $f:\mathbb{R}\to\mathbb{R}$ is a continuous nonlinearity with subcritical growth at infinity and $F(t)=\int_0^t f(\tau)d\tau$. Exploiting appropriate variational arguments, we construct a family of solutions concentrating around the local minimum of V as $\varepsilon\to 0$.

1. Introduction

In this paper, we focus on the nonlinear fractional elliptic problem

$$\begin{cases} (-\Delta+m^2)^s u+V(\varepsilon\, x)u=\left(\frac{1}{|x|^{\mu}}*F(u)\right)f(u) & \text{in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\varepsilon>0$ is a small parameter, $m>0$, $s\in(0,1)$, $N>2s$ and $\mu\in(0,2s)$. The potential $V:\mathbb{R}^N\to\mathbb{R}$ is continuous function verifying the following conditions:

(V_1) there exists $V_1\in(0,m^{2s})$ such that $-V_1:=\inf_{x\in\mathbb{R}^N}V(x)$,

(V_2) there exists a bounded open set $\Lambda\subset\mathbb{R}^N$ such that

$$-V_0:=\inf_{x\in\Lambda}V(x)<\min_{x\in\partial\Lambda}V(x),$$

with $V_0>0$ and $0\in M:=\{x\in\Lambda: V(x)=-V_0\}$,

and the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $f(t)=0$ for $t\leq 0$ and it satisfies the following assumptions:

(f_1) $\lim_{t\to 0}\frac{f(t)}{t}=0$,

(f_2) there exists $q\in\left(2,\frac{2(N-\mu)}{N-2s}\right)$ such that

$$\lim_{t\to\infty}\frac{f(t)}{t^{q-1}}=0,$$

(f_3) $0<4F(t)\leq 2tf(t)$ for all $t>0$, where $F(t):=\int_0^t f(\tau)\,d\tau$,

(f_4) the function $t\mapsto\frac{f(t)}{t}$ is increasing in $(0,\infty)$.

The fractional relativistic Schrödinger operator $(-\Delta+m^2)^s$ is defined for all $u\in\mathcal{S}(\mathbb{R}^N)$ by setting

$$\mathcal{F}((-\Delta+m^2)^s u)(k):=(|k|^2+m^2)^s\mathcal{F}u(k), \quad k\in\mathbb{R}^N,$$

where

$$\mathcal{F}u(k):=(2\pi)^{-\frac{N}{2}}\int_{\mathbb{R}^N}e^{-\imath k\cdot x}u(x)\,dx, \quad k\in\mathbb{R}^N,$$

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denotes the Fourier transform of u and $\mathcal{S}(\mathbb{R}^N)$ stands for the Schwartz space of rapidly decreasing functions. Alternatively, $(-\Delta+m^2)^s$ can be expressed in terms of singular integrals by setting

$$(-\Delta+m^2)^s u(x) := m^{2s}u(x) + C_{N,s}\, m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\, dy, \quad x\in\mathbb{R}^N,$$

where $P.V.$ is the Cauchy principal value, $C_{N,s}>0$ is a suitable constant, and K_ν denotes the modified Bessel function of the third kind of index ν (see [51]). We recall that when $s=\frac{1}{2}$, the fractional relativistic Schrödinger operator is closely related to $\sqrt{-\Delta+m^2}-m$ which has a relevant role in relativistic quantum mechanics because it corresponds to the kinetic energy of a relativistic particle with mass m. The study of $\sqrt{-\Delta+m^2}-m$ has been heavily influenced by several works on the stability of relativistic matter (see [32]). On the other hand, the operator $(-\Delta+m^2)^s$ is very important in the theory of Bessel potentials and finds application in harmonic analysis and partial differential equations (see [13,28,47]). There exists also a deep connection between the fractional relativistic Schrödinger operator and the theory of Lévy processes due to the fact that $m^{2s}-[(-\Delta+m^2)^s]$ is the infinitesimal generator of a Lévy process called the $2s$-stable relativistic process (see [18,43]). For a more detailed discussion on $(-\Delta+m^2)^s$ and its applications, we refer the interested reader to [12].

As $m\to 0$, $(-\Delta+m^2)^s$ becomes the fractional Laplacian $(-\Delta)^s$ defined for all $u\in\mathcal{S}(\mathbb{R}^N)$ via Fourier transform by

$$\mathcal{F}((-\Delta)^s u)(k) := |k|^{2s}\mathcal{F}u(k), \quad k\in\mathbb{R}^N,$$

or via singular integrals by

$$(-\Delta)^s u(x) := C'_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x\in\mathbb{R}^N, \tag{1.2}$$

where $C'_{N,s}>0$ is an appropriate constant. In recent years, a great interest has been devoted to the fractional Laplacian, both for its interesting theoretical structure and in view of real world concrete applications; see [15] for more details. We point out that the main difference between the fractional operators $(-\Delta)^s$ and $(-\Delta+m^2)^s$ consists in the fact that the first operator is homogeneous in scaling whereas this property is not true for the latter one. When $(-\Delta+m^2)^s$ is replaced by $(-\Delta)^s$, after rescaling, equation (1.1) is related to the singularly perturbed nonlinear fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which has been widely investigated by several authors. For instance, when $\varepsilon=1$, V is constant, and $f(u)=|u|^{p-2}u$ with $p\in(2-\frac{\mu}{N}, \frac{2N-\mu}{N-2s})$, d'Avenia et al. [22] examined regularity, existence, multiplicity, non existence, symmetry and decay properties of solutions to (1.3). Shen et al. [45] proved the existence of ground state solutions for (1.3) when $\varepsilon=1$, V is constant and f is a general subcritical nonlinearity. Assuming a local condition on the potential V and that f is a continuous function satisfying (f_1)-(f_4), the author in [6] combined a penalization method with Ljusternik-Schnirelmann theory to show that, for $\varepsilon>0$ sufficiently small, the number of solutions to (1.3) depends on the topology of the set where the potential V attains its minimum value. We also mention [7,8,14,27,40,49] for another interesting results to nonlinear fractional Choquard equations. Note that in absence of the convolution term $\frac{1}{|x|^\mu}*F(u)$, (1.3) reduces to the singularly perturbed nonlinear fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

for which many existence and multiplicity results appeared in the literature; see [10] and the references therein for more details. Equation (1.4) stems from the study of standing wave solutions $\Psi(x,t)=u(x)e^{-\imath\frac{E}{\varepsilon}t}$, with $E\in\mathbb{R}$, for the time-dependent fractional Schrödinger equation

$$\imath\,\varepsilon\frac{\partial\Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + (V(x)+E)\Psi - f(\Psi) \quad \text{in } \mathbb{R}^N\times\mathbb{R},$$

where $f(\rho e^{\imath\xi})=f(\rho)e^{\imath\xi}$ for all $\rho,\xi\in\mathbb{R}$, proposed by Laskin [29] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Furthermore, (1.4) can be understood as the fractional analog of the well-known nonlinear Schrödinger equation (see [23,25,42,50,52]).

We stress that if $s \to 1$ then (1.3) boils down to the following semilinear elliptic equation with a nonlocal nonlinearity

$$-\varepsilon^2\,\Delta u + V(x)u = \varepsilon^{\mu-N}\left(\frac{1}{|x|^{\mu}} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Indeed, when $N=3$, $V\equiv 1$, $\mu=\varepsilon=1$ and $F(t)=\frac{t^2}{\sqrt{2}}$, (1.5) is the so-called Choquard-Pekar equation

$$-\Delta u + u = \left(\frac{1}{|x|} * u^2\right) u \ \text{ in } \mathbb{R}^3, \tag{1.6}$$

which was elaborated by Pekar in 1954 [41] to study the quantum mechanics of a polaron at rest. In 1976 Choquard used (1.6) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one-component plasma [30]. Equation (1.6) was also proposed by Penrose in 1996 as a model of self-gravitating matter and is known in that context as the Schrödinger-Newton equation [35]. From a mathematical point of view, the early existence and symmetry results are due to Lieb [30] and Lions [33]. For more details on the Choquard equation and some of its variants and generalizations, one can see [1–3, 19, 26, 34, 36–38, 46] and the references therein.

On the other hand, a great attention has been devoted to the fractional relativistic Schrödinger equation

$$(-\Delta+m^2)^s u + V(\varepsilon\,x)u = f(x,u) \text{ in } \mathbb{R}^N, \tag{1.7}$$

which arises when we look for standing wave solutions of the fractional Schrödinger-Klein-Gordon equation

$$\imath\frac{\partial\Psi}{\partial t} = (-\Delta+m^2)^s\Psi - \varphi(x,|\Psi|)\Psi \quad \text{in } \mathbb{R}^N\times\mathbb{R},$$

describing the behavior of bosonic systems; see [24, 32]. Some existence and multiplicity results for (1.7) with local nonlinearities f can be found in [5, 9–11, 44], whereas if f is a nonlocal Choquard term then we refer to [16, 20, 21]. In particular, in [9] the author investigated the existence and multiplicity of concentrating solutions to (1.7) for $\varepsilon>0$ small enough, when V satisfies (V_1)-(V_2) and $f(x,u)=f(u)$ is a superlinear continuous nonlinearity with subcritical growth at infinity. These results have been recently extended for critical nonlinearities in [11]. In [20] the authors performed a semiclassical analysis for (1.7) with $s=1/2$, $f(x,u)=\left(\frac{1}{|x|^{N-\alpha}} * |u|^p\right)|u|^{p-2}u$, $p\in[2,\frac{2N}{N-1})$ and $\alpha\in((N-1)p-N,N)$.

Strongly motivated by the above papers, in this work we analyze the existence of solutions to (1.1) which concentrate around local minimum points of the potential V as $\varepsilon\to 0$. More precisely, we prove the following main result.

Theorem 1.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) hold. Then, for every small $\varepsilon>0$, there exists a solution u_ε to (1.1) such that u_ε has a maximum point x_ε satisfying*

$$\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon\,x_\varepsilon, M)=0,$$

and for which

$$0<u_\varepsilon(x)\le C_1 e^{-C_2|x-x_\varepsilon|} \quad \text{for all } x\in\mathbb{R}^N,$$

for suitable constants $C_1, C_2>0$. Moreover, for each sequence (ε_n), with $\varepsilon_n\to 0$, there exists a subsequence, still denoted by itself, such that there exist a point $x_0\in M$, with $\varepsilon_n\,x_{\varepsilon_n}\to x_0$, and a positive ground state solution $u\in H^s(\mathbb{R}^N)$ of the limiting problem

$$(-\Delta+m^2)^s u - V_0 u = \left(\frac{1}{|x|^{\mu}} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N,$$

for which we have

$$u_{\varepsilon_n}(x) = u(x-x_{\varepsilon_n}) + \mathcal{R}_n(x),$$

where

$$\lim_{n\to\infty}\|\mathcal{R}_n\|_{H^s(\mathbb{R}^N)}=0.$$

The proof of Theorem 1.1 is based on appropriate variational arguments. The principal difficulties lie in the presence of two non-local terms, namely, the fractional relativistic Schrödinger operator $(-\Delta+m^2)^s$ and the convolution $\frac{1}{|x|^{\mu}} * F(u)$, and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^N$. To overcome these obstacles, we first use the extension method [17, 48] to transform (1.1) into a degenerate elliptic equation in the upper half-space $\mathbb{R}^{N+1}_+$ with a nonlinear and nonlocal Neumann boundary condition on $\partial\mathbb{R}^{N+1}_+$. Inspired by [9], we study the extended problem by adapting the penalization approach in [23]. Roughly speaking, we modify properly the nonlinearity f outside Λ in such a way that the mountain pass theorem [4] is directly applicable to the corresponding extended modified energy functional J_ε. Then, taking advantage of the energy-minimality of the mountain pass solution v_ε, we show that the trace $v_\varepsilon(\cdot,0)$ of v_ε turns out to be a solution to the original problem with the desired properties when $\varepsilon>0$ is sufficiently small. However, since we have to handle a Choquard term, we will carry out a more careful analysis. We start by recalling the following Hardy-Littlewood-Sobolev inequality which will be often used throughout the paper.

Theorem 1.2. *[31, Theorem 4.3] Let $p,r\in(1,\infty)$ and $\mu\in(0,N)$ with $\frac{1}{p}+\frac{\mu}{N}+\frac{1}{r}=2$. Let $g\in L^p(\mathbb{R}^N)$ and $h\in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(N,\mu,p)>0$, independent of g and h, such that*

$$\left|\iint_{\mathbb{R}^{2N}}\frac{g(x)h(y)}{|x-y|^\mu}\,dxdy\right|\leq C(N,\mu,p)|g|_p|h|_r.$$

We note that if $F(t)=|t|^\sigma$ for some $\sigma>0$ and $u\in H^s(\mathbb{R}^N)$, then Theorem 1.2 guarantees that the integral

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(u(x))\right)F(u(x))\,dx$$

is well-defined whenever $F(u)\in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. In light of the fact that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for all $r\in[2,2^*_s]$, we have to require that $\frac{2N\sigma}{2N-\mu}\in[2,2^*_s]$, which holds when

$$\sigma\in\left[\frac{2N-\mu}{N},\frac{2N-\mu}{N-2s}\right].$$

Our assumptions on μ and q ensure us that $\frac{2N-\mu}{N}<2<q<\frac{2(N-\mu)}{N-2s}<\frac{2N-\mu}{N-2s}$, and thus every continuous function f fulfilling (f_1)-(f_2) is such that $F(u)\in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ for all $u\in H^s(\mathbb{R}^N)$. The restrictions on μ and q also imply that the convolution $\frac{1}{|x|^\mu}*F(u)$ is a bounded term, and thanks to assumptions (f_3) and (f_4) we will be able to implement the penalization scheme. To recover compactness, we will verify that J_ε satisfies a local Palais-Smale condition and some additional efforts with respect to [9] will be needed due to the presence of the Choquard term; see Lemmas 3.2 and 3.5. Moreover, some convenient estimates on the quadratic part of J_ε will be established to deduce the strong convergence of the Palais-Smale sequences of J_ε; see Lemma 3.4. Finally, in order to prove that the trace of the mountain pass solution of the extended modified problem is indeed a solution to (1.1) for $\varepsilon>0$ small enough, we will exploit a Moser iteration technique [39], a comparison argument and some fundamental properties of the Bessel kernel; see Lemma 5.3 and Theorem 1.1.

To the best of our knowledge, this is the first existence result of concentrating solutions to (1.1), under conditions (V_1)-(V_2) on the potential V and (f_1)-(f_4) on the nonlinearity f, respectively.

The paper is organized as follows. In section 2 we introduce the notations and some function spaces. In section 3 we consider the modified problem. In section 4 we examine the limiting fractional relativistic Schrödinger-Choquard equation related to the extended modified problem. In the last section, we give the proof of Theorem 1.1.

2. PRELIMINARIES

We begin by fixing the notations used along the paper. We denote the upper half-space in $\mathbb{R}^N$ by

$$\mathbb{R}^{N+1}_+:=\left\{(x,y)\in\mathbb{R}^{N+1}:y>0\right\}.$$

For $(x,y)\in\mathbb{R}^{N+1}_+$, we set $|(x,y)|:=\sqrt{|x|^2+y^2}$. The letters C, C', C'' and C_i will be frequently employed to denote various positive constants whose exact values are irrelevant. For $x\in\mathbb{R}^N$ and $R>0$, we will denote by $B_R(x)$ the ball in $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ with radius $r>0$. When $x=0$, we put $B_R:=B_R(0)$. For $(x_0,y_0)\in\mathbb{R}^{N+1}_+$ and $R>0$, we set

$$B^+_R(x_0,y_0):=\{(x,y)\in\mathbb{R}^{N+1}_+:|(x,y)-(x_0,y_0)|<R\},$$

and $B^+_R:=B^+_R(0,0)$. Let $p\in[1,\infty]$ and $A\subset\mathbb{R}^N$ be a measurable set. With $A^c:=\mathbb{R}^N\setminus A$ we denote the complement of A in $\mathbb{R}^N$. We will use $|u|_{L^p(A)}$ for the L^p-norm of $u:\mathbb{R}^N\to\mathbb{R}$. If $A=\mathbb{R}^N$, we only write

$|u|_p$ instead of $|u|_{L^p(\mathbb{R}^N)}$. We denote by $\|v\|_{L^p(\mathbb{R}^{N+1}_+)}$ the norm of $v \in L^p(\mathbb{R}^{N+1}_+)$. For a generic real-valued function w, we set $w^+ := \max\{w, 0\}$ and $w^- := \min\{w, 0\}$.

Next we introduce some fractional Sobolev spaces and we collect some results found in [9]. With $H^s(\mathbb{R}^N)$ we denote the fractional Sobolev space given by the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$|u|_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|k|^2 + m^2)^s |\mathcal{F}u(k)|^2 dk \right)^{\frac{1}{2}}.$$

It is well-known that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*_s]$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^*_s)$. Let $X^s(\mathbb{R}^{N+1}_+)$ be the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|v\|_{X^s(\mathbb{R}^{N+1}_+)} := \left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (|\nabla v|^2 + m^2 v^2)\, dxdy \right)^{\frac{1}{2}}.$$

We recall that $X^s(\mathbb{R}^{N+1}_+)$ is continuously embedded into $L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})$, where $\gamma := 1 + \frac{2}{N-2s}$, that is there exists $\hat{S} > 0$ such that

$$\|v\|_{L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})} \leq \hat{S} \|v\|_{X^s(\mathbb{R}^{N+1}_+)} \quad \text{for all } v \in X^s(\mathbb{R}^{N+1}_+).$$

Here $L^r(\mathbb{R}^{N+1}_+, y^{1-2s})$ stands for the weighted Lebesgue space, with $r \in (1, \infty)$, endowed with the norm

$$\|v\|_{L^r(\mathbb{R}^{N+1}_+, y^{1-2s})} := \left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} |v|^r\, dxdy \right)^{\frac{1}{r}}.$$

Moreover, $X^s(\mathbb{R}^{N+1}_+)$ is compactly embedded into $L^2(B^+_R, y^{1-2s})$ for all $R > 0$.
Let $\mathrm{Tr} : X^s(\mathbb{R}^{N+1}_+) \to H^s(\mathbb{R}^N)$ be the unique linear trace operator such that

$$\sqrt{\sigma_s} |\mathrm{Tr}(v)|_{H^s(\mathbb{R}^N)} \leq \|v\|_{X^s(\mathbb{R}^{N+1}_+)} \quad \text{for all } v \in X^s(\mathbb{R}^{N+1}_+). \tag{2.1}$$

where $\sigma_s := 2^{1-2s}\Gamma(1-s)/\Gamma(s)$. Hereafter, we will denote $\mathrm{Tr}(v)$ by $v(\cdot, 0)$. We observe that (2.1) implies that

$$\sigma_s m^{2s} \int_{\mathbb{R}^N} v^2(x, 0)\, dx \leq \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (|\nabla v|^2 + m^2 v^2)\, dxdy, \tag{2.2}$$

for all $v \in X^s(\mathbb{R}^{N+1}_+)$, which can be also written as

$$\sigma_s \int_{\mathbb{R}^N} v^2(x, 0)\, dx \leq m^{-2s} \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla v|^2\, dxdy + m^{2-2s} \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v^2\, dxdy. \tag{2.3}$$

The following embeddings are valid.

Theorem 2.1. $\mathrm{Tr}(X^s(\mathbb{R}^{N+1}_+))$ *is continuously embedded into* $L^r(\mathbb{R}^N)$ *for all* $r \in [2, 2^*_s]$ *and compactly embedded into* $L^r_{loc}(\mathbb{R}^N)$ *for all* $r \in [1, 2^*_s)$.

Now we introduce the extension problem for $(-\Delta + m^2)^s$. More precisely, given $u \in H^s(\mathbb{R}^N)$ we can find a unique function $U \in X^s(\mathbb{R}^{N+1}_+)$ that satisfies the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) + m^2 y^{1-2s} U = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U = u & \text{on } \partial\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \{0\}. \end{cases}$$

The function U is called the extension of u and enjoys the following properties:

(*i*)

$$\frac{\partial U}{\partial \nu^{1-2s}} := -\lim_{y \to 0} y^{1-2s} \frac{\partial U}{\partial y}(x, y) = \sigma_s (-\Delta + m^2)^s u(x) \quad \text{in } H^{-s}(\mathbb{R}^N),$$

where $H^{-s}(\mathbb{R}^N)$ denotes the dual of $H^s(\mathbb{R}^N)$,

(*ii*) $\sqrt{\sigma_s}|u|_{H^s(\mathbb{R}^N)} = \|U\|_{X^s(\mathbb{R}^{N+1}_+)} \leq \|W\|_{X^s(\mathbb{R}^{N+1}_+)}$ for all $W \in X^s(\mathbb{R}^{N+1}_+)$ such that $W(\cdot, 0) = u$,

(iii) if $u \in \mathcal{S}(\mathbb{R}^N)$ then $U \in C^\infty(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$ and it can be expressed as

$$U(x,y) = \int_{\mathbb{R}^N} P_{s,m}(x-z,y)u(z)\,dz,$$

where

$$P_{s,m}(x,y) := c_{N,s}\, y^{2s} m^{\frac{N+2s}{2}} |(x,y)|^{-\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(m|(x,y)|),$$

and $c_{N,s} > 0$ is an appropriate constant.

From the previous facts, instead of (1.1), we will consider the following local degenerate equation in $\mathbb{R}^{N+1}_+$ together with a nonlinear and nonlocal Neumann boundary condition on $\partial\mathbb{R}^{N+1}_+$

$$\left\{\begin{array}{ll} -\operatorname{div}(y^{1-2s}\nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial v}{\partial \nu^{1-2s}} = \sigma_s \left[-V_\varepsilon(x) v + \left(\frac{1}{|x|^\mu} * F(v)\right) f(v)\right] & \text{on } \partial\mathbb{R}^{N+1}_+, \end{array}\right. \tag{2.4}$$

where $V_\varepsilon(x) := V(\varepsilon x)$. In order to lighten the notation, we omit the constant σ_s from the second equation in (2.4). To study (2.4) via suitable variational methods, for every $\varepsilon > 0$, we introduce the space

$$X_\varepsilon := \left\{ v \in X^s(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} V_\varepsilon(x) v^2(x,0)\,dx < \infty \right\}$$

equipped with the norm

$$\|v\|_\varepsilon := \left(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \int_{\mathbb{R}^N} V_\varepsilon(x) v^2(x,0)\,dx \right)^{\frac{1}{2}}.$$

Obviously, $X_\varepsilon \subset X^s(\mathbb{R}^{N+1}_+)$, and thanks to (2.2) and (V_1) we have that

$$\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)} \leq \left(\frac{m^{2s}}{m^{2s} - V_1} \right) \|v\|^2_\varepsilon \quad \text{for all } v \in X_\varepsilon. \tag{2.5}$$

Note that X_ε is a Hilbert space endowed with the inner product

$$\langle v, w \rangle_\varepsilon := \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla v \cdot \nabla w + m^2 v w)\,dxdy + \int_{\mathbb{R}^N} V_\varepsilon(x) v(x,0) w(x,0)\,dx \quad \text{for all } v, w \in X_\varepsilon.$$

Finally, we denote by X^*_ε the dual space of X_ε.

3. THE PENALIZED PROBLEM

Inspired by [9], we adapt the penalization method in [23] to examine (2.4). Fix

$$\theta \in \left(0, \frac{m^{2s} - V_1}{V_1}\right). \tag{3.1}$$

Take $\ell > 0$ such that $\frac{V_1}{\ell} < \lim_{t\to\infty} \frac{f(t)}{t} \in (0,\infty]$, and let $a > 0$ be the unique number such that $f(a) = \frac{V_1}{\ell} a$. Define

$$\tilde{f}(t) := \left\{ \begin{array}{ll} f(t) & \text{for } t < a, \\ \frac{V_1}{\ell} t & \text{for } t \geq a, \end{array} \right.$$

and

$$g(x,t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x)) \tilde{f}(t) \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where χ_Λ stands for the characteristic function of Λ. Put $G(x,t) := \int_0^t g(x,\tau)\,d\tau$. On account of (f_1)-(f_4), we can verify that g is a Carathéodory function satisfying the following conditions:

(g_1) $\lim_{t\to 0} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$,
(g_2) $\lim_{t\to\infty} \frac{g(x,t)}{t^{q-1}} = 0$ uniformly in $x \in \mathbb{R}^N$,
(g_3) (i) $0 < 4G(x,t) \leq 2tg(x,t)$ for all $x \in \Lambda$ and $t > 0$,
 (ii) $0 < 2G(x,t) \leq tg(x,t) \leq \frac{V_1}{\ell} t^2$ for all $x \in \Lambda^c$ and $t > 0$,
(g_4) the functions $t \mapsto g(x,t)$ and $t \mapsto \frac{G(x,t)}{t}$ are increasing for all $x \in \mathbb{R}^N$ and $t > 0$.

Let us consider the auxiliary problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v)+m^2y^{1-2s}v=0 & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial v}{\partial\nu^{1-2s}}=-V_\varepsilon(x)v+\left(\frac{1}{|x|^\mu}*G_\varepsilon(x,v)\right)g_\varepsilon(x,v) & \text{on } \partial\mathbb{R}^{N+1}_+,\end{cases} \tag{3.2}$$

where $g_\varepsilon(x,t):=g(\varepsilon x,t)$. Clearly, if v_ε is a positive solution of (3.2) such that $v_\varepsilon(x,0)<a$ for all $x\in\Lambda_\varepsilon^c$, where $\Lambda_\varepsilon:=\{x\in\mathbb{R}^N:\varepsilon x\in\Lambda\}$, then v_ε is a positive solution of (2.4). The energy functional associated with (3.2) is defined as

$$J_\varepsilon(v):=\frac{1}{2}\|v\|_\varepsilon^2-\Sigma_\varepsilon(v)\quad \text{for all } v\in X_\varepsilon,$$

where

$$\Sigma_\varepsilon(v):=\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G_\varepsilon(x,v(x,0))\right)G_\varepsilon(x,v(x,0))\,dx\quad \text{for all } v\in X_\varepsilon.$$

It is easy to verify that $J_\varepsilon\in C^1(X_\varepsilon,\mathbb{R})$ and that its differential is given by

$$\langle J_\varepsilon'(v),w\rangle=\langle v,w\rangle_\varepsilon-\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G_\varepsilon(x,v(x,0))\right)g_\varepsilon(x,v(x,0))w(x,0)\,dx\quad \text{for all } v,w\in X_\varepsilon.$$

Next we prove that J_ε has the geometric structure required by the mountain pass theorem [4].

Lemma 3.1. *The functional J_ε possesses the following properties:*

(i) $J_\varepsilon(0)=0$,

(ii) there exist $\alpha,\rho>0$ such that $J_\varepsilon(v)\geq\alpha$ for all $v\in X_\varepsilon$ such that $\|v\|_\varepsilon=\rho$,

(iii) there exists $e\in X_\varepsilon$ such that $\|e\|_\varepsilon>\rho$ and $J_\varepsilon(e)<0$.

Proof. Evidently, (i) holds. Exploiting the growth conditions (g_1)-(g_2), given $\eta>0$ we can find $C_\eta>0$ such that

$$|g_\varepsilon(x,t)|\leq\eta|t|+C_\eta|t|^{q-1}\quad \text{for } (x,t)\in\mathbb{R}^N\times\mathbb{R}, \tag{3.3}$$

and

$$|G_\varepsilon(x,t)|\leq\frac{\eta}{2}|t|^2+\frac{C_\eta}{q}|t|^q\quad \text{for } (x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{3.4}$$

In view of Theorem 1.2 and (3.4), we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G_\varepsilon(x,v(x,0))\right)G_\varepsilon(x,v(x,0))\,dx\right| &\leq C|G_\varepsilon(\cdot,v(\cdot,0))|_\tau|G_\varepsilon(\cdot,v(\cdot,0))|_\tau\\ &\leq C'\left(\int_{\mathbb{R}^N}(\eta|v(x,0)|^2+C_\eta|v(x,0)|^q)^\tau\,dx\right)^{\frac{2}{\tau}},\end{aligned} \tag{3.5}$$

where $\tau:=\frac{2N}{2N-\mu}$. Since $\mu\in(0,2s)$ and $q\in\left(2,\frac{2(N-\mu)}{N-2s}\right)$, we have that $2\tau\in(2,2_s^*)$ and $q\tau\in(2,2_s^*)$. Then Theorem 2.1 ensures that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*G_\varepsilon(x,v(x,0))\right)G_\varepsilon(x,v(x,0))\,dx\right| &\leq C''(\eta\|v\|_\varepsilon^2+C_\eta\|v\|_\varepsilon^q)^2\\ &\leq C_0(\eta^2\|v\|_\varepsilon^4+C_\eta^2\|v\|_\varepsilon^{2q}).\end{aligned} \tag{3.6}$$

Consequently,

$$J_\varepsilon(v)\geq\frac{1}{2}\|v\|_\varepsilon^2-\frac{C_0}{2}(\eta^2\|v\|_\varepsilon^4+C_\eta^2\|v\|_\varepsilon^{2q}).$$

Let $v\in X_\varepsilon$ be such that $\|v\|_\varepsilon=\rho\in(0,1)$. Because $\|v\|_\varepsilon^4\leq\|v\|_\varepsilon^2$ and taking

$$\eta\in\left(0,\frac{1}{\sqrt{C_0}}\right),$$

we arrive at

$$J_\varepsilon(v)\geq C_1\|v\|_\varepsilon^2-C_2\|v\|_\varepsilon^{2q}.$$

By virtue of $q > 2$, we can select

$$\rho \in \left(0, \min\left\{1, \left(\frac{C_1}{C_2}\right)^{\frac{1}{2(q-1)}}\right\}\right)$$

so that

$$\inf_{v\in X_\varepsilon : \|v\|_\varepsilon=\rho} J_\varepsilon(v) \geq \rho^2(C_1 - C_2\rho^{2q-2}) =: \alpha > 0.$$

Hence, (ii) is true. Finally, let $v_0 \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ be such that $v_0 \geq 0$, $v_0 \not\equiv 0$ and $\mathrm{supp}(v_0(\cdot, 0)) \subset \Lambda_\varepsilon$. Define

$$\varsigma(t) := \Sigma_\varepsilon\left(\frac{tv_0}{\|v_0\|_\varepsilon}\right) \quad \text{for } t > 0.$$

In light of $G_\varepsilon(\cdot, v_0(\cdot, 0)) = F(v_0(\cdot, 0))$ and (f_3), we see that, for all $t > 0$,

$$\begin{aligned}
\varsigma'(t) &= \langle \Sigma'_\varepsilon\left(\frac{tv_0}{\|v_0\|_\varepsilon}\right), \frac{v_0}{\|v_0\|_\varepsilon}\rangle \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F\left(\frac{tv_0(x,0)}{\|v_0\|_\varepsilon}\right)\right) f\left(\frac{tv_0(x,0)}{\|v_0\|_\varepsilon}\right) \frac{v_0(x,0)}{\|v_0\|_\varepsilon}\, dx \\
&= \frac{4}{t}\int_{\mathbb{R}^N} \frac{1}{2}\left(\frac{1}{|x|^\mu} * F\left(\frac{tv_0(x,0)}{\|v_0\|_\varepsilon}\right)\right) \frac{1}{2} f\left(\frac{tv_0(x,0)}{\|v_0\|_\varepsilon}\right) \frac{tv_0(x,0)}{\|v_0\|_\varepsilon}\, dx \\
&\geq \frac{4}{t}\varsigma(t).
\end{aligned} \tag{3.7}$$

Integrating (3.7) over $[1, t\|v_0\|_\varepsilon]$ with $t > \frac{1}{\|v_0\|_\varepsilon}$, we have

$$\varsigma(t\|v_0\|_\varepsilon) \geq \varsigma(1)(t\|v_0\|_\varepsilon)^4$$

which gives

$$\Sigma_\varepsilon(tv_0) \geq \Sigma_\varepsilon\left(\frac{v_0}{\|v_0\|_\varepsilon}\right)\|v_0\|^4_\varepsilon t^4.$$

Therefore,

$$J_\varepsilon(tv_0) = \frac{t^2}{2}\|v_0\|^2_\varepsilon - \Sigma_\varepsilon(tv_0) \leq C_3t^2 - C_4t^4 \quad \text{for all } t > \frac{1}{\|v_0\|_\varepsilon}.$$

Then (iii) follows by taking $e := tv_0$ with t sufficiently large. □

Combining Lemma 3.1 with a variant of the mountain pass theorem without the Palais-Smale condition (see [52, Theorem 2.9]), we can find a Palais-Smale sequence $(v_n) \subset X_\varepsilon$ at the mountain pass level c_ε, that is

$$J_\varepsilon(v_n) \to c_\varepsilon \quad \text{and} \quad J'_\varepsilon(v_n) \to 0 \text{ in } X^*_\varepsilon,$$

as $n \to \infty$, where

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)),$$

and

$$\Gamma_\varepsilon := \{\gamma \in C([0, 1], X_\varepsilon) : \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\}.$$

Using (f_4), it is standard to verify that c_ε can be characterized as

$$c_\varepsilon = \inf_{v\in\mathcal{N}_\varepsilon} J_\varepsilon(v) = \inf_{v\in X_\varepsilon\setminus\{0\}} \max_{t\geq0} J_\varepsilon(tv),$$

where

$$\mathcal{N}_\varepsilon := \{v \in X_\varepsilon : \langle J'_\varepsilon(v), v\rangle = 0\}$$

is the Nehari manifold associated with J_ε. Since $supp(v_0(\cdot, 0)) \subset \Lambda_\varepsilon$, there exists $\kappa > 0$, independent of ε, ℓ, and a, such that

$$c_\varepsilon < \kappa.$$

Let us define

$$\mathcal{B} := \left\{v \in X_\varepsilon : \|v\|^2_\varepsilon \leq 4(\kappa + 1)\right\},$$

and we set

$$\tilde{K}_\varepsilon(v)(x) := \frac{1}{|x|^\mu} * G_\varepsilon(x, v(x, 0)).$$

Lemma 3.2. *There exists* $\ell_0 > 0$ *such that*

$$\frac{\sup_{v\in\mathcal{B}} |\tilde{K}_\varepsilon(v)|_\infty}{\ell_0} < \theta \quad \text{for all } \varepsilon > 0,$$

where θ *is given in* (3.1).

Proof. We begin by proving that there exists $\tilde{C}_0 > 0$ such that

$$\sup_{v\in\mathcal{B}} |\tilde{K}_\varepsilon(v)|_\infty \leq \tilde{C}_0. \tag{3.8}$$

Fix $v \in \mathcal{B}$. Utilizing (3.4), we can see that

$$\begin{aligned} |\tilde{K}_\varepsilon(v)(x)| &= \left| \int_{\mathbb{R}^N} \frac{G_\varepsilon(\xi, v(\xi,0))}{|x-\xi|^\mu}\, d\xi \right| \\ &\leq \left| \int_{|x-y|\leq 1} \frac{G_\varepsilon(\xi, v(\xi,0))}{|x-\xi|^\mu}\, d\xi \right| + \left| \int_{|x-\xi|>1} \frac{G_\varepsilon(\xi, v(\xi,0))}{|x-\xi|^\mu}\, d\xi \right| \\ &\leq C \int_{|x-\xi|\leq 1} \frac{|v(\xi,0)|^2 + |v(\xi,0)|^q}{|x-\xi|^\mu}\, d\xi + C \int_{\mathbb{R}^N} (|v(\xi,0)|^2 + |v(\xi,0)|^q)\, d\xi \\ &\leq C \int_{|x-\xi|\leq 1} \frac{|v(\xi,0)|^2 + |v(\xi,0)|^q}{|x-\xi|^\mu}\, d\xi + C' \quad \text{for a.e. } x \in \mathbb{R}^N, \end{aligned} \tag{3.9}$$

where we have used Theorem 2.1 and the fact that $v \in \mathcal{B}$. Since $\mu \in (0, 2s)$ and $q \in \left(2, \frac{2(N-\mu)}{N-2s}\right)$, we can select

$$t \in \left(\frac{N}{N-\mu}, \frac{2^*_s}{2}\right] \quad \text{and} \quad r \in \left(\frac{N}{N-\mu}, \frac{2^*_s}{q}\right].$$

Exploiting Hölder's inequality, Theorem 2.1, $v \in \mathcal{B}$ and $N - 1 - \frac{t\mu}{t-1} > -1$, we get

$$\begin{aligned} \int_{|x-\xi|\leq 1} \frac{|v(\xi,0)|^2}{|x-\xi|^\mu}\, d\xi &\leq \left(\int_{|x-\xi|\leq 1} |v(\xi,0)|^{2t}\, d\xi \right)^{\frac{1}{t}} \left(\int_{|x-\xi|\leq 1} \frac{1}{|x-\xi|^{\frac{t\mu}{t-1}}}\, d\xi \right)^{\frac{t-1}{t}} \\ &\leq C_1 \left(\int_0^1 \rho^{N-1-\frac{t\mu}{t-1}}\, d\rho \right)^{\frac{t-1}{t}} < \infty. \end{aligned} \tag{3.10}$$

In a similar fashion, due to $N - 1 - \frac{r\mu}{r-1} > -1$, we obtain

$$\begin{aligned} \int_{|x-\xi|\leq 1} \frac{|v(\xi,0)|^q}{|x-\xi|^\mu}\, d\xi &\leq \left(\int_{|x-\xi|\leq 1} |v(\xi,0)|^{rq}\, d\xi \right)^{\frac{1}{r}} \left(\int_{|x-\xi|\leq 1} \frac{1}{|x-\xi|^{\frac{r\mu}{r-1}}}\, d\xi \right)^{\frac{r-1}{r}} \\ &\leq C_2 \left(\int_0^1 \rho^{N-1-\frac{r\mu}{r-1}}\, d\rho \right)^{\frac{r-1}{r}} < \infty. \end{aligned} \tag{3.11}$$

In light of (3.10) and (3.11), we arrive at

$$\int_{|x-\xi|\leq 1} \frac{|v(\xi,0)|^2 + |v(\xi,0)|^q}{|x-\xi|^\mu}\, d\xi \leq C_3 \quad \text{for a.e. } x \in \mathbb{R}^N,$$

which combined with (3.9) yields $|\tilde{K}_\varepsilon(v)|_\infty \leq \tilde{C}_0$ for all $v \in \mathcal{B}$, and so (3.8) holds. Thus, we can find $\ell_0 > 0$ such that

$$\frac{\sup_{v\in\mathcal{B}} |\tilde{K}_\varepsilon(v)|_\infty}{\ell_0} \leq \frac{\tilde{C}_0}{\ell_0} < \theta.$$

The proof is now complete. □

Henceforth, we assume that $\ell > \ell_0$ and consider the penalized nonlinearity g with this choice of ℓ.

In the next lemma, we establish the boundedness of Palais-Smale sequences of J_ε.

Lemma 3.3. *Let $c \in [c_\varepsilon, \kappa]$ and $(v_n) \subset X_\varepsilon$ be a Palais-Smale sequence of J_ε at the level c. Then, (v_n) is bounded in X_ε. Furthermore, there exists $n_0 \in \mathbb{N}$ such that*

$$\|v_n\|_\varepsilon^2 \leq 4(\kappa+1) \quad \text{for all } n \geq n_0.$$

Proof. By assumptions, we know that

$$J_\varepsilon(v_n) \to c \quad \text{and} \quad J'_\varepsilon(v_n) \to 0 \text{ in } X_\varepsilon^*, \tag{3.12}$$

as $n \to \infty$. Using (3.12), (g_3), (2.2) and (2.5), we can see that

$$\begin{aligned} c + o_n(1)\|v_n\|_\varepsilon &\geq J_\varepsilon(v_n) - \frac{1}{4}\langle J'_\varepsilon(v_n), v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{4}\right)\|v_n\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G_\varepsilon(x, v_n(x,0))\right)[2g_\varepsilon(x, v_n(x,0))v_n(x,0) - 4G_\varepsilon(x, v_n(x,0))]\,dx \\ &\geq \frac{1}{4}\|v_n\|_\varepsilon^2. \end{aligned}$$

Since $c \in [c_\varepsilon, \kappa]$, there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n\|_\varepsilon^2 \leq 4(\kappa+1) \quad \text{for all } n \geq n_0.$$

This concludes the proof of the lemma. □

Now, we prove the following useful result.

Lemma 3.4. *Let $c \in \mathbb{R}$ and $(v_n) \subset X_\varepsilon$ be a Palais-Smale sequence of J_ε at the level c. Then, for all $\xi > 0$ there exists $R = R(\xi) > 0$ such that*

$$\limsup_{n\to\infty}\left[\iint_{\mathbb{R}^{N+1}_+ \setminus B_R^+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\,dxdy + \int_{\mathbb{R}^N \setminus B_R}(V_\varepsilon(x) + V_1)v_n^2(x,0)\,dx\right] < \xi. \tag{3.13}$$

Proof. For $R > 0$, take $\eta_R \in C^\infty(\overline{\mathbb{R}^{N+1}_+})$ such that

$$\eta_R(x,y) := \begin{cases} 0 & \text{if } (x,y) \in B^+_{R/2}, \\ 1 & \text{if } (x,y) \in \overline{\mathbb{R}^{N+1}_+} \setminus B_R^+, \end{cases}$$

with $0 \leq \eta_R \leq 1$ and $\|\nabla \eta_R\|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq C/R$, for some $C > 0$ independent of $R > 0$. By Lemma 3.3, (v_n) is a bounded Palais-Smale sequence in X_ε and there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n\|_\varepsilon^2 \leq 4(\kappa+1) \quad \text{for all } n \geq n_0.$$

Thus, by Lemma 3.2, we know that

$$\frac{\sup_{n\geq n_0}|\tilde{K}_\varepsilon(v_n)|_\infty}{\ell} \leq \theta < \frac{m^{2s} - V_1}{V_1}. \tag{3.14}$$

Because $\langle J'_\varepsilon(v_n), v_n\eta_R^2\rangle = o_n(1)$, we can write

$$\begin{aligned} &\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2\,dxdy + \int_{\mathbb{R}^N}(V_\varepsilon(x) + V_1)v_n^2(x,0)\eta_R^2(x,0)\,dx \\ &= \int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G_\varepsilon(x, v_n(x,0))\right)g_\varepsilon(x, v_n(x,0))v_n(x,0)\eta_R^2(x,0)\,dx + V_1\int_{\mathbb{R}^N} v_n^2(x,0)\eta_R^2(x,0)\,dx \\ &\quad - 2\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}v_n\eta_R\nabla v_n \cdot \nabla\eta_R\,dxdy + o_n(1). \end{aligned} \tag{3.15}$$

Choose $R>0$ such that $\Lambda_\varepsilon \subset B_{R/2}$. Then, thanks to (g_3)-(ii) and (3.14), we can see that, for all $n \geq n_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G_\varepsilon(x, v_n(x,0))\right) g_\varepsilon(x, v_n(x,0)) v_n(x,0)\eta_R^2(x,0)\,dx + V_1 \int_{\mathbb{R}^N} v_n^2(x,0)\eta_R^2(x,0)\,dx \\
&\leq V_1 \int_{\mathbb{R}^N\setminus B_{R/2}} \frac{\sup_{n\geq n_0} |\tilde{K}_\varepsilon(v_n)|_\infty}{\ell} v_n^2(x,0)\eta_R^2(x,0)\,dx + V_1 \int_{\mathbb{R}^N} v_n^2(x,0)\eta_R^2(x,0)\,dx \qquad (3.16)\\
&\leq V_1(1+\theta)\int_{\mathbb{R}^N} v_n^2(x,0)\eta_R^2(x,0)\,dx.
\end{aligned}$$

Now, exploiting Hölder's inequality, $0 \leq \eta_R \leq 1$, $\|\nabla \eta_R\|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq C/R$, and the boundedness of (v_n) in X_ε, we obtain

$$\begin{aligned}
\left|2\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n \eta_R \nabla v_n \cdot \nabla \eta_R\,dxdy\right| &\leq \frac{C_1}{R}\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|v_n||\nabla v_n|\,dxdy \\
&\leq \frac{C_1}{R}\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n^2\,dxdy\right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v_n|^2\,dxdy\right)^{\frac{1}{2}} \\
&\leq \frac{C_2}{R}. \qquad (3.17)
\end{aligned}$$

On the other hand, applying (2.3) to $v_n\eta_R$, and utilizing $\|\nabla \eta_R\|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq C/R$ and (3.17), we have

$$\begin{aligned}
&V_1(1+\theta)\int_{\mathbb{R}^N} v_n^2(x,0)\eta_R^2(x,0)\,dx \\
&\leq V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla(v_n\eta_R)|^2\,dxdy + V_1 m^{2-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n^2\eta_R^2\,dxdy \\
&= V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla v_n|^2\eta_R^2\,dxdy + V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \eta_R|^2 v_n^2\,dxdy \\
&\quad + 2V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n\eta_R \nabla v_n\cdot\nabla\eta_R\,dxdy + V_1 m^{2-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n^2\eta_R^2\,dxdy \\
&\leq V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2\,dxdy + \frac{C_3}{R^2}\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} v_n^2\,dxdy + \frac{C_4}{R} \\
&\leq V_1 m^{-2s}(1+\theta)\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2\,dxdy + \frac{C_5}{R^2} + \frac{C_4}{R}. \qquad (3.18)
\end{aligned}$$

From (3.15)-(3.18), we derive that

$$\begin{aligned}
&\left[1 - V_1 m^{-2s}(1+\theta)\right]\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2\,dxdy + \int_{\mathbb{R}^N}(V_\varepsilon(x) + V_1)v_n^2(x,0)\eta_R^2(x,0)\,dx \\
&\leq \frac{C_6}{R} + \frac{C_5}{R^2} + o_n(1). \qquad (3.19)
\end{aligned}$$

In view of (3.1), (V_1) and the definition of η_R, we deduce that (3.19) implies (3.13). □

Remark 3.1. *Let* $r \in [2, 2^*_s]$. *Using* $\eta_R(\cdot,0) = 1$ *in* B_R^c, *Theorem* 2.1, $|x+y|^2 \leq 2(|x|^2+|y|^2)$ *for* $x, y \in \mathbb{R}^N$, *and* $\|\nabla \eta_R\|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq C/R$, *we can see that*

$$\begin{aligned}
&\left(\int_{B_R^c} |v_n(x,0)|^r\,dx\right)^{\frac{2}{r}} \leq \left(\int_{\mathbb{R}^N} |(v_n\eta_R)(x,0)|^r\,dx\right)^{\frac{2}{r}} \\
&\leq C_1\left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla(v_n\eta_R)|^2 + m^2 v_n^2\eta_R^2)\,dxdy\right)
\end{aligned}$$

$$\leq C_2 \left[\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2 \, dxdy + \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \eta_R|^2 v_n^2 \, dxdy \right]$$
$$\leq C_2 \left[\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v_n|^2 + m^2 v_n^2)\eta_R^2 \, dxdy + \frac{C_3}{R^2} \right],$$

which combined with (3.19) *yields*

$$\left(\int_{B_R^c} |v_n(x,0)|^r \, dx \right)^{\frac{2}{r}} \leq \frac{C'}{R} + \frac{C''}{R^2} + o_n(1).$$

Consequently,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{B_R^c} |v_n(x,0)|^r \, dx = 0 \quad \textit{for all } r \in [2, 2^*_s]. \tag{3.20}$$

Let us now prove that J_ε satisfies the following compactness condition.

Lemma 3.5. *Let $c \in [c_\varepsilon, \kappa]$. Then J_ε satisfies the Palais-Smale condition at the level c.*

Proof. Let $c \in [c_\varepsilon, \kappa]$ and $(v_n) \subset X_\varepsilon$ be a Palais-Smale sequence at the level c, namely

$$J_\varepsilon(v_n) \to c \quad \text{ and } \quad J'_\varepsilon(v_n) \to 0 \text{ in } X^*_\varepsilon,$$

as $n \to \infty$. From Lemma 3.3, it follows that (v_n) is bounded in X_ε. Hence, thanks to the reflexivity of X_ε and Theorem 2.1, we may assume, up to a subsequence, that $v_n \rightharpoonup v$ in X_ε, $v_n(\cdot,0) \to v(\cdot,0)$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, 2^*_s)$, and $v_n(\cdot,0) \to v(\cdot,0)$ a.e. in $\mathbb{R}^N$. Moreover, there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n\|^2_\varepsilon \leq 4(\kappa+1) \quad \text{ for all } n \geq n_0.$$

By Lemma 3.2, we also have

$$\frac{\sup_{n\geq n_0} |\tilde{K}_\varepsilon(v_n)|_\infty}{\ell} \leq \theta. \tag{3.21}$$

Let us first check that v is a critical point of J_ε. Owing to $v_n \rightharpoonup v$ in X_ε, we know that, for all $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$,

$$\langle v_n, \phi \rangle_\varepsilon \to \langle v, \phi \rangle_\varepsilon. \tag{3.22}$$

On the other hand, since $(G_\varepsilon(\cdot, v_n(\cdot,0)))$ is bounded in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, $v_n(\cdot,0) \to v(\cdot,0)$ a.e. in $\mathbb{R}^N$, and $t \mapsto G(x,t)$ is continuous for a.e. $x \in \mathbb{R}^N$, we see that

$$G_\varepsilon(\cdot, v_n(\cdot,0)) \rightharpoonup G_\varepsilon(\cdot, v(\cdot,0)) \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

According to Theorem 1.2, the map

$$h \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N) \mapsto \frac{1}{|\cdot|^\mu} * h \in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$$

is a linear bounded operator, and so

$$\tilde{K}_\varepsilon(v_n) = \frac{1}{|\cdot|^\mu} * G_\varepsilon(\cdot, v_n(\cdot,0)) \rightharpoonup \frac{1}{|\cdot|^\mu} * G_\varepsilon(\cdot, v(\cdot,0)) = \tilde{K}_\varepsilon(v) \text{ in } L^{\frac{2N}{\mu}}(\mathbb{R}^N).$$

Because $g_\varepsilon(\cdot, v_n(\cdot,0)) \to g_\varepsilon(\cdot, v(\cdot,0))$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, \frac{2^*_s}{q-1})$, we deduce that, for all $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$,

$$\int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))\phi(x,0)\, dx \to \int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v)(x) g_\varepsilon(x, v(x,0))\phi(x,0)\, dx. \tag{3.23}$$

Thus, due to $\langle J'_\varepsilon(v_n), \phi \rangle = o_n(1)$ for all $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$, it follows from (3.22) and (3.23) that $\langle J'_\varepsilon(v), \phi \rangle = 0$ for all $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$. By virtue of the density of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ in X_ε, we can infer that v is a critical point of J_ε. Next we prove that $v_n \to v$ in X_ε. Note that $v_n \rightharpoonup v$ in X_ε and $\langle J'_\varepsilon(v_n), v_n - v \rangle = o_n(1)$ yield

$$\|v_n - v\|^2_\varepsilon = \langle J'_\varepsilon(v_n), v_n - v \rangle + \int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))(v_n(x,0) - v(x,0))\, dx + o_n(1)$$

$$= \int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))(v_n(x,0) - v(x,0))\, dx + o_n(1).$$

Therefore, it suffices to show that

$$\int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))(v_n(x,0) - v(x,0))\, dx = o_n(1) \tag{3.24}$$

to conclude that $v_n \to v$ in X_ε. Exploiting (3.3), (3.21), $v_n(\cdot,0) \to v(\cdot,0)$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, 2^*_s)$, and the dominated convergence theorem, we obtain that, for all fixed $R > 0$,

$$\int_{B_R} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))(v_n(x,0) - v(x,0))\, dx = o_n(1). \tag{3.25}$$

Pick $\xi > 0$. Using (3.3), (3.20), (3.21), we have that, for $R = R(\xi) > 0$ sufficiently large,

$$\begin{aligned} \limsup_{n\to\infty} \left| \int_{B_R^c} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0)) v_n(x,0)\, dx \right| &\leq C \limsup_{n\to\infty} \int_{B_R^c} (|v_n(x,0)|^2 + |v_n(x,0)|^q)\, dx \\ &\leq C_1 \xi. \end{aligned} \tag{3.26}$$

In a similar fashion, applying Hölder's inequality, we get

$$\limsup_{n\to\infty} \left| \int_{B_R^c} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0)) v(x,0)\, dx \right| \leq C_2 \xi. \tag{3.27}$$

Combining (3.25), (3.26) and (3.27), we deduce that

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} \tilde{K}_\varepsilon(v_n)(x) g_\varepsilon(x, v_n(x,0))(v_n(x,0) - v(x,0))\, dx \right| \leq C_3 \xi,$$

and due to the arbitrariness of $\xi > 0$ we arrive at (3.24). The proof of the lemma is now complete. □

We can now establish an existence result for (3.2).

Theorem 3.1. *For all $\varepsilon > 0$ there exists $v_\varepsilon \in X_\varepsilon \setminus \{0\}$ such that*

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon \quad \text{and} \quad J'_\varepsilon(v_\varepsilon) = 0. \tag{3.28}$$

Proof. Owing to Lemmas 3.1 and 3.5, we can invoke the mountain pass theorem [4] to infer that for every $\varepsilon > 0$ there exists a mountain pass solution $v_\varepsilon \in X_\varepsilon \setminus \{0\}$ to (3.2). □

Remark 3.2. *As $g(\cdot, t) = 0$ for $t \leq 0$, it is easy to see that $\langle J'_\varepsilon(v_\varepsilon), v_\varepsilon^- \rangle = 0$ implies $v_\varepsilon \geq 0$ in $\mathbb{R}^{N+1}_+$.*

4. The limiting fractional relativistic Schrödinger-Choquard problem

Let $\omega > -m^{2s}$ and introduce the following limiting problem related to (1.1)

$$\begin{cases} (-\Delta + m^2)^s u + \omega u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

The extended problem associated with (4.1) is given by

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla v) + m^2 y^{1-2s} v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial v}{\partial \nu^{1-2s}} = -\omega v + \left(\frac{1}{|x|^\mu} * F(v)\right) f(v) & \text{on } \partial\mathbb{R}^{N+1}_+, \end{cases} \tag{4.2}$$

and its corresponding energy functional is defined as

$$L_\omega(v) := \frac{1}{2}\|v\|^2_{Y_\omega} - \Sigma_0(v),$$

where

$$\Sigma_0(v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(v(x,0))\right) F(v(x,0))\, dx,$$

and $Y_\omega := X^s(\mathbb{R}^{N+1}_+)$ is equipped with the norm

$$\|v\|_{Y_\omega} := \left(\|v\|^2_{X^s(\mathbb{R}^{N+1}_+)} + \omega|v(\cdot,0)|_2^2\right)^{\frac{1}{2}}.$$

Reasoning as in [9, pag. 5671], we can verify that $\|\cdot\|_{Y_\omega}$ is a norm equivalent to $\|\cdot\|_{X^s(\mathbb{R}^{N+1}_+)}$. Clearly, Y_ω is a Hilbert space with the inner product

$$\langle v,w\rangle_{Y_\omega} := \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla v\cdot\nabla w + m^2 vw)\,dxdy + \omega\int_{\mathbb{R}^N} v(x,0)w(x,0)\,dx \quad \text{for all } v,w\in Y_\omega.$$

Let $\mathcal{M}_\omega$ be the Nehari manifold associated with L_ω, that is

$$\mathcal{M}_\omega := \{v\in Y_\omega : \langle L'_\omega(v), v\rangle = 0\}.$$

As in section 3, we can check that L_ω has a mountain pass geometry [4]. Thus, invoking a variant of the mountain pass theorem without the Palais-Smale condition (see [52, Theorem 2.9]), we can find a Palais-Smale sequence $(v_n)\subset Y_\omega$ at the mountain pass level d_ω of L_ω given by

$$d_\omega := \inf_{\gamma\in\Gamma_\omega}\max_{t\in[0,1]} L_\omega(\gamma(t)),$$

where

$$\Gamma_\omega := \{\gamma\in C([0,1],Y_\omega) : \gamma(0)=0,\, L_\omega(\gamma(1))<0\}.$$

Exploiting (f_3), we can show that (v_n) is bounded in Y_ω. Using (f_4), we can also see that

$$0 < d_\omega = \inf_{v\in\mathcal{M}_\omega} L_\omega(v) = \inf_{v\in Y_\omega\setminus\{0\}}\max_{t\geq 0} L_\omega(tv).$$

The aim of this section is to obtain the existence of a ground state solution for (4.2). We first recall a vanishing Lions type result.

Lemma 4.1. *[9, Lemma 3.3] Let $t\in[2,2^*_s)$ and $R>0$. If $(v_n)\subset X^s(\mathbb{R}^{N+1}_+)$ is a bounded sequence such that*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)} |v_n(x,0)|^t\,dx = 0,$$

then $v_n(\cdot,0)\to 0$ in $L^r(\mathbb{R}^N)$ for all $r\in(2,2^*_s)$.

The next lemma will be very helpful.

Lemma 4.2. *Let $(v_n)\subset Y_\omega$ be a Palais-Smale sequence for L_ω at the level d_ω and such that $v_n\rightharpoonup 0$ in Y_ω. Then we have either*
(a) $v_n\to 0$ in Y_ω, or
(b) there exist $(z_n)\subset\mathbb{R}^N$ and $R,\beta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(z_n)} v_n^2(x,0)\,dx \geq \beta.$$

Proof. Suppose that (*b*) does not hold. Hence, for all $R>0$,

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)} v_n^2(x,0)\,dx = 0.$$

Taking Lemma 4.1 into account, we know that $v_n(\cdot,0)\to 0$ in $L^r(\mathbb{R}^N)$ for all $r\in(2,2^*_s)$. Therefore, in view of the growth assumptions on f and Theorem 1.2, we deduce that

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*F(v_n(x,0))\right)f(v_n(x,0))v_n(x,0)\,dx = o_n(1). \tag{4.3}$$

In light of $\langle L'_\omega(v_n),v_n\rangle = o_n(1)$ and (4.3), we can conclude that $v_n\to 0$ in Y_ω. $\square$

Remark 4.1. *If* v *is the weak limit of a Palais-Smale sequence* (v_n) *for* L_ω *at the level* d_ω*, then we may assume* $v \not\equiv 0$*. Indeed, if* $v_n \rightharpoonup 0$ *and* $v_n \not\to 0$ *in* Y_ω*, then we can apply Lemma 4.2 to find* $(z_n) \subset \mathbb{R}^N$ *and* $R, \beta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(z_n)} v_n^2(x,0)\,dx \geq \beta.$$

Set $\tilde{v}_n(x,y) := v_n(x+z_n, y)$*. Due to the invariance of* $\mathbb{R}^N$ *by translation, it turns out that* $(\tilde{v}_n)$ *is a bounded Palais-Smale sequence for* L_ω *at the level* d_ω *such that* $\tilde{v}_n \rightharpoonup \tilde{v}$ *in* Y_ω *with* $\tilde{v} \not\equiv 0$*.*

Now we can prove the main result of this section.

Theorem 4.1. *Let* $\omega > -m^{2s}$*. Then* (4.1) *has a ground state solution.*

Proof. Since L_ω has a mountain pass geometry [4], we can produce a Palais-Smale sequence $(v_n) \subset Y_\omega$ at the level d_ω. Thus, (v_n) is bounded in Y_ω. Thanks to the reflexivity of Y_ω and Theorem 2.1, by passing to a subsequence if necessary, we may assume that, up to a subsequence, there exists $v \in Y_\omega$ such that $v_n \rightharpoonup v$ in Y_ω, $v_n(\cdot,0) \to v(\cdot,0)$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, 2^*_s)$, and $v_n(\cdot,0) \to v(\cdot,0)$ a.e. in $\mathbb{R}^N$. Arguing as in the proof of Lemma 3.5, we can verify that $\langle L'_\omega(v), \varphi\rangle = 0$ for all $\varphi \in Y_\omega$. From Remark 4.1, we may suppose that $v \not\equiv 0$. Using $v_n \rightharpoonup v$ in Y_ω and the following nonlocal Brezis-Lieb type results for Σ_ω and Σ'_ω (see [8, Lemma 2.5] and also [1, Lemma 3.5]):

$$\Sigma_0(v_n) - \Sigma_0(v_n - v) - \Sigma_0(v) = o_n(1),$$

and

$$\langle \Sigma'_0(v_n) - \Sigma'_0(v_n - v) - \Sigma'_0(v), \varphi\rangle = o_n(1) \quad \text{for all } \varphi \in Y_\omega : \ \|\varphi\|_{Y_\omega} \leq 1,$$

we have that

$$L_\omega(v_n - v) = d_\omega - L_\omega(v) + o_n(1) \ \text{ and } \ L'_\omega(v_n - v) = o_n(1).$$

On the other hand, by (f_3) and Fatou's lemma, we see that

$$d_\omega \leq L_\omega(v) - \frac{1}{2}\langle L'_\omega(v), v\rangle \leq \liminf_{n\to\infty}\left(L_\omega(v_n) - \frac{1}{2}\langle L'_\omega(v_n), v_n\rangle\right) = d_\omega,$$

which yields $d_\omega = L_\omega(v)$. Therefore,

$$L_\omega(v_n - v) = o_n(1) \ \text{ and } \ L'_\omega(v_n - v) = o_n(1),$$

and so

$$\limsup_{n\to\infty} \frac{1}{4}\|v_n - v\|^2_{Y_\omega} \leq \limsup_{n\to\infty}\left(L_\omega(v_n - v) - \frac{1}{4}\langle L'_\omega(v_n - v), v_n - v\rangle\right) = 0.$$

Consequently, $v_n \to v$ in Y_ω. Finally, we show that $v(\cdot,0)$ is positive. As $f(t) = 0$ for $t \leq 0$, it follows from $\langle L'_\omega(v), v^-\rangle = 0$ that $v \geq 0$ in $\mathbb{R}^{N+1}_+$ and $v \not\equiv 0$. Arguing as in the proof of Lemma 3.2, we can prove that $\frac{1}{|\cdot|^\mu} * F(v(\cdot,0)) \in L^\infty(\mathbb{R}^N)$. Then, a Moser iteration argument (see [12, Lemma 4.1] or Lemma 5.3 below) ensures that $v(\cdot,0) \in L^r(\mathbb{R}^N)$ for all $r \in [2,\infty]$. From [9, Corollary 3], we deduce that $v(\cdot,0) \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Invoking the weak Harnack inequality [10, Proposition 4.2.11], we conclude that $v(\cdot,0) > 0$ in $\mathbb{R}^N$. □

Finally, we establish the following relation between c_ε and $d_{V(0)} = d_{-V_0}$ (note that $V(0) = -V_0 > -m^{2s}$ due to $0 \in M$).

Lemma 4.3. *The following inequality holds:*

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leq d_{V(0)}.$$

Proof. By virtue of Theorem 4.1, we can find a ground state solution w to (4.2) with $\omega = V(0)$. Let $\phi \in C^\infty_c(\mathbb{R})$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ in $[-1,1]$ and $\phi = 0$ in $\mathbb{R} \setminus (-2,2)$. Suppose that $B_2 \subset \Lambda$. Put $w_\varepsilon(x,y) := \phi(\varepsilon\,|(x,y)|)w(x,y)$ and note that $\text{supp}(w_\varepsilon(\cdot,0)) \subset \Lambda_\varepsilon$. It follows from the dominated convergence theorem that

$$w_\varepsilon \to w \text{ in } Y_{V(0)} \ \text{ and } \ L_{V(0)}(w_\varepsilon) \to L_{V(0)}(w) \text{ as } \varepsilon \to 0. \tag{4.4}$$

Now, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$J_\varepsilon(t_\varepsilon w_\varepsilon) = \max_{t\geq 0} J_\varepsilon(t w_\varepsilon).$$

From $\langle J'_\varepsilon(t_\varepsilon w_\varepsilon), w_\varepsilon\rangle = 0$, we derive that

$$t_\varepsilon \|w_\varepsilon\|^2_\varepsilon = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(t_\varepsilon w_\varepsilon(x,0))\right) f(t_\varepsilon w_\varepsilon(x,0)) w_\varepsilon(x,0)\, dx.$$

Let us prove that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. Assume by contradiction that, up to a subsequence, $t_\varepsilon \to \infty$. Because

$$\|w_\varepsilon\|^2_\varepsilon = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \frac{F(t_\varepsilon w_\varepsilon(x,0))}{t_\varepsilon}\right) f(t_\varepsilon w_\varepsilon(x,0)) w_\varepsilon(x,0)\, dx, \tag{4.5}$$

and utilizing (4.4) and $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{F(t)}{t} = \infty$ (thanks to (f_3)), we can apply Fatou's lemma to infer that $\|w\|_{Y_{V(0)}} = \infty$, which is impossible. Thus, up to a subsequence, $t_\varepsilon \to t_0 \in [0,\infty)$. If $t_0 = 0$, then the growth assumptions on f and Theorem 1.2 imply

$$\|w_\varepsilon\|^2_\varepsilon \leq C t_\varepsilon^2 \|w_\varepsilon\|^4_\varepsilon + C t_\varepsilon^{2q-2} \|w_\varepsilon\|^{2q}_\varepsilon,$$

from which $\|w\|_{Y_{V(0)}} = 0$, a contradiction. Hence, $t_\varepsilon \to t_0 \in (0,\infty)$. Letting $\varepsilon \to 0$ in (4.5), we find

$$\|w\|^2_{Y_{V(0)}} = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * \frac{F(t_0 w(x,0))}{t_0}\right) f(t_0 w(x,0)) w(x,0)\, dx.$$

Since $w \in \mathcal{M}_{V(0)}$, $t \mapsto f(t)$ and $t \mapsto \frac{F(t)}{t}$ are increasing in $(0,\infty)$ (by (f_3) and (f_4)), we can deduce that $t_0 = 1$. Now, using the definition of c_ε, we can see that

$$c_\varepsilon \leq \max_{t\geq 0} J_\varepsilon(t w_\varepsilon) = J_\varepsilon(t_\varepsilon w_\varepsilon) = L_{V(0)}(t_\varepsilon w_\varepsilon) + \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} (V_\varepsilon(x) - V(0)) w_\varepsilon^2(x,0)\, dx.$$

Because $V_\varepsilon(x)$ is bounded on the support of $w_\varepsilon(\cdot,0)$ and $V_\varepsilon(x) \to V(0)$ as $\varepsilon \to 0$, we can use the dominated convergence theorem, (4.4), $t_\varepsilon \to 1$ as $\varepsilon \to 0$, and the above inequality to obtain the assertion. □

5. THE PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1. In light of Theorem 3.1, we know that for all $\varepsilon > 0$ there exists a non-negative mountain pass solution v_ε to (3.2). Now we prove the following result.

Lemma 5.1. *There exist $r, \beta, \varepsilon^* > 0$ and $(y_\varepsilon) \subset \mathbb{R}^N$ such that*

$$\int_{B_r(y_\varepsilon)} v_\varepsilon^2(x,0)\, dx \geq \beta \quad \text{for all } \varepsilon \in (0,\varepsilon^*).$$

Proof. Let us first show that there exists $\alpha > 0$, independent of $\varepsilon > 0$, such that

$$\|v_\varepsilon\|^2_\varepsilon \geq \alpha \quad \text{for all } \varepsilon > 0. \tag{5.1}$$

Indeed, from (3.28), (3.3) and Theorem 1.2, given $\eta > 0$ we can find $C_\eta > 0$ such that

$$\begin{aligned}\|v_\varepsilon\|^2_\varepsilon &= \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G_\varepsilon(x, v_\varepsilon(x,0))\right) g_\varepsilon(x, v_\varepsilon(x,0)) v_\varepsilon(x,0)\, dx \\ &\leq C_0(\eta^2 \|v_\varepsilon\|^4_\varepsilon + C_\eta \|v_\varepsilon\|^{2q}_\varepsilon).\end{aligned}$$

If $\|v_\varepsilon\|_\varepsilon \geq 1$ then we are done. Assume that $\|v_\varepsilon\|_\varepsilon < 1$. Therefore, $\|v_\varepsilon\|^4_\varepsilon \leq \|v_\varepsilon\|^2_\varepsilon$. Fixed $\eta \in \left(0, \frac{1}{\sqrt{C_0}}\right)$, we get

$$(1 - C_0\eta^2)\|v_\varepsilon\|^2_\varepsilon \leq C_0 C_\eta \|v_\varepsilon\|^{2q}_\varepsilon,$$

and so the assertion follows from $q > 2$. Let now $(\varepsilon_n) \subset (0,\infty)$ be such that $\varepsilon_n \to 0$. Suppose by contradiction that there exists $r > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} v_{\varepsilon_n}^2(x,0)\, dx = 0.$$

From Lemma 4.1, we derive that $v_{\varepsilon_n}(\cdot,0) \to 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (2, 2^*_s)$. Thus, in view of the growth assumptions on g and Theorem 1.2, we obtain

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G_{\varepsilon_n}(x, v_{\varepsilon_n}(x,0)) \right) g_{\varepsilon_n}(x, v_{\varepsilon_n}(x,0)) v_{\varepsilon_n}(x,0)\, dx = o_n(1).$$

This fact and $\langle J'_{\varepsilon_n}(v_{\varepsilon_n}), v_{\varepsilon_n}\rangle = 0$ imply that $\|v_{\varepsilon_n}\|_{\varepsilon_n} \to 0$ which contradicts (5.1). ☐

Lemma 5.2. *For each sequence* (ε_n) *such that* $\varepsilon_n \to 0$*, consider the sequence* $(y_{\varepsilon_n}) \subset \mathbb{R}^N$ *given in Lemma 5.1. Set* $w_n(x,y) := v_{\varepsilon_n}(x + y_{\varepsilon_n}, y)$*. Then there exist a subsequence of* (w_n)*, still denoted by itself, and* $w \in X^s(\mathbb{R}^{N+1}_+) \setminus \{0\}$ *such that*

$$w_n \to w \quad \text{in } X^s(\mathbb{R}^{N+1}_+).$$

Moreover, there exists $x_0 \in \Lambda$ *such that*

$$\varepsilon_n\, y_{\varepsilon_n} \to x_0 \quad \text{and} \quad V(x_0) = -V_0.$$

Proof. Hereafter, we denote by (y_n) and (v_n), the sequences (y_{ε_n}) and (v_{ε_n}), respectively. Using (3.28) and Lemma 4.3, we can argue as in the proof of Lemma 3.3 to deduce that (v_n) is bounded in X_{ε_n}. Furthermore, there exists $n_0 \in \mathbb{N}$ such that

$$\|v_n\|^2_{\varepsilon_n} \le 4(\kappa + 1) \quad \text{for all } n \ge n_0,$$

and so, by Lemma 3.2, we have

$$\frac{\sup_{n\ge n_0} |\tilde{K}_\varepsilon(v_n)|_\infty}{\ell} \le \theta < \frac{m^{2s} - V_1}{V_1}. \tag{5.2}$$

Exploiting (2.5), we know that (w_n) is bounded in $X^s(\mathbb{R}^{N+1}_+)$. Hence, there are a subsequence of (w_n), still denoted by itself, and $w \in X^s(\mathbb{R}^{N+1}_+)$ such that, as $n \to \infty$,

$$\begin{aligned} &w_n \rightharpoonup w \text{ in } X^s(\mathbb{R}^{N+1}_+),\\ &w_n(\cdot,0) \to w(\cdot,0) \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, 2^*_s),\\ &w_n(\cdot,0) \to w(\cdot,0) \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{5.3}$$

In addition, by Lemma 5.1 and (5.3),

$$\int_{B_r} w^2(x,0)\, dx \ge \beta > 0, \tag{5.4}$$

and so $w \not\equiv 0$. In what follows, we prove that $(\varepsilon_n\, y_n)$ is bounded in $\mathbb{R}^N$. For this purpose, we verify that

$$\mathrm{dist}(\varepsilon_n\, y_n, \overline{\Lambda}) \to 0 \quad \text{as } n \to \infty. \tag{5.5}$$

Assume by contradiction that (5.5) is not true. Then we can find $\delta > 0$ and a subsequence of $(\varepsilon_n\, y_n)$, still denoted by itself, such that

$$\mathrm{dist}(\varepsilon_n\, y_n, \overline{\Lambda}) \ge \delta \quad \text{for all } n \in \mathbb{N}.$$

Accordingly, there is $R > 0$ such that $B_R(\varepsilon_n\, y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Since $w \ge 0$ and $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ is dense in $X^s(\mathbb{R}^{N+1}_+)$, we can select $(\psi_j) \subset C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ such that $\psi_j \ge 0$ in $\overline{\mathbb{R}^{N+1}_+}$ and $\psi_j \to w$ in $X^s(\mathbb{R}^{N+1}_+)$ as $j \to \infty$. Fix $j \in \mathbb{N}$. Taking $\phi = \psi_j$ into the relation $\langle J'_{\varepsilon_n}(v_n), \phi\rangle = 0$, we have

$$\begin{aligned} &\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla w_n \cdot \nabla \psi_j + m^2 w_n \psi_j)\, dxdy + \int_{\mathbb{R}^N} V_{\varepsilon_n}(x + y_n) w_n(x,0)\psi_j(x,0)\, dx\\ &= \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G_{\varepsilon_n}(x + y_n, w_n(x,0)) \right) g_{\varepsilon_n}(x + y_n, w_n(x,0)) \psi_j(x,0)\, dx. \end{aligned} \tag{5.6}$$

Let us observe that $\tilde{f}(t) \leq \frac{V_1}{\ell} t$ for all $t > 0$ and (5.2) yield

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G_{\varepsilon_n}(x + y_n, w_n(x,0)) \right) g_{\varepsilon_n}(x + y_n, w_n(x,0)) \psi_j(x,0)\, dx \\
&\leq \ell\theta \int_{\mathbb{R}^N} g_{\varepsilon_n}(x + y_n, w_n(x,0)) \psi_j(x,0)\, dx \\
&= \ell\theta \int_{B_{\frac{R}{\varepsilon_n}}} g_{\varepsilon_n}(x + y_n, w_n(x,0)) \psi_j(x,0)\, dx + \ell\theta \int_{B^c_{\frac{R}{\varepsilon_n}}} g_{\varepsilon_n}(x + y_n, w_n(x,0)) \psi_j(x,0)\, dx \\
&\leq \theta V_1 \int_{B_{\frac{R}{\varepsilon_n}}} w_n(x,0) \psi_j(x,0)\, dx + \ell\theta \int_{B^c_{\frac{R}{\varepsilon_n}}} f(w_n(x,0)) \psi_j(x,0)\, dx.
\end{aligned} \tag{5.7}$$

Putting together (V_1), (5.6) and (5.7), we arrive at

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (\nabla w_n \cdot \nabla \psi_j + m^2 w_n \psi_j)\, dxdy - V_1 (1+\theta) \int_{\mathbb{R}^N} w_n(x,0) \psi_j(x,0)\, dx \\
&\leq \ell\theta \int_{B^c_{\frac{R}{\varepsilon_n}}} f(w_n(x,0)) \psi_j(x,0)\, dx.
\end{aligned}$$

Using (5.3), $(\psi_j) \subset C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$, $\varepsilon_n \to 0$ and the growth assumptions on f, we can see that, as $n \to \infty$,

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (\nabla w_n \cdot \nabla \psi_j + m^2 w_n \psi_j)\, dxdy - V_1 (1+\theta) \int_{\mathbb{R}^N} w_n(x,0) \psi_j(x,0)\, dx \\
&\to \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (\nabla w \cdot \nabla \psi_j + m^2 w \psi_j)\, dxdy - V_1 (1+\theta) \int_{\mathbb{R}^N} w(x,0) \psi_j(x,0)\, dx,
\end{aligned}$$

and

$$\int_{B^c_{\frac{R}{\varepsilon_n}}} f(w_n(x,0)) \psi_j(x,0)\, dx \to 0.$$

Consequently, for every $j \in \mathbb{N}$,

$$\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (\nabla w \cdot \nabla \psi_j + m^2 w \psi_j)\, dxdy - V_1 (1+\theta) \int_{\mathbb{R}^N} w(x,0) \psi_j(x,0)\, dx \leq 0.$$

Passing to the limit as $j \to \infty$ in the above inequality, we find

$$\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1 (1+\theta) |w(\cdot,0)|_2^2 dx \leq 0,$$

which combined with (2.2) and (3.1) ensures that

$$0 \leq \left(1 - \frac{V_1}{m^{2s}} (1+\theta) \right) \|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} \leq 0,$$

but this contradicts (5.4). Hence, (5.5) is valid. Then we can extract a subsequence of $(\varepsilon_n\, y_n)$, still denoted by itself, such that $\varepsilon_n\, y_n \to x_0$ as $n \to \infty$, for some $x_0 \in \overline{\Lambda}$. Next, we prove that

$$x_0 \in \Lambda. \tag{5.8}$$

From (5.6), $\psi_j \geq 0$ in $\overline{\mathbb{R}^{N+1}_+}$ and $0 \leq g(x,t) \leq f(t)$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, we derive that

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} (\nabla w_n \cdot \nabla \psi_j + m^2 w_n \psi_j)\, dxdy + \int_{\mathbb{R}^N} V_{\varepsilon_n}(x + y_n) w_n(x,0) \psi_j(x,0)\, dx \\
&\leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(w_n(x,0)) \right) f(w_n(x,0)) \psi_j(x,0)\, dx.
\end{aligned}$$

Taking the limit as $n \to \infty$ and exploiting (5.3), the continuity of V, $\frac{1}{|\cdot|^{\mu}} * F(w_n(\cdot,0)) \rightharpoonup \frac{1}{|\cdot|^{\mu}} * F(w(\cdot,0))$ in $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, $f(w_n(\cdot,0)) \to f(w(\cdot,0))$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, \frac{2^*_s}{q-1})$, and that ψ_j has compact support in $\overline{\mathbb{R}^{N+1}_+}$, we get

$$
\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(\nabla w \cdot \nabla \psi_j + m^2 w \psi_j)\, dxdy + \int_{\mathbb{R}^N} V(x_0) w(x,0)\psi_j(x,0)\, dx \\
&\leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\mu}} * F(w(x,0)) \right) f(w(x,0))\psi_j(x,0)\, dx.
\end{aligned}
$$

Letting $j \to \infty$, we obtain that

$$
\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla w|^2 + m^2 w^2)\, dxdy + \int_{\mathbb{R}^N} V(x_0) w^2(x,0)\, dx \leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{\mu}} * F(w(x,0)) \right) f(w(x,0)) w(x,0)\, dx.
$$

Therefore, there exists $t_1 \in (0,1)$ such that $t_1 w \in \mathcal{M}_{V(x_0)}$. Thus, by Lemma 4.3, we have

$$
d_{V(x_0)} \leq L_{V(x_0)}(t_1 w) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(v_n) = \liminf_{n\to\infty} c_{\varepsilon_n} \leq d_{V(0)},
$$

that is, $d_{V(x_0)} \leq d_{V(0)}$, whence $V(x_0) \leq V(0) = -V_0$. Because $-V_0 = \inf_{x\in\overline{\Lambda}} V(x)$, we can infer that $V(x_0) = -V_0$. This fact and (V_2) imply that $x_0 \notin \partial\Lambda$. As a result, (5.8) is true. In order to complete the proof of the lemma, it remains to show that, as $n \to \infty$,

$$
w_n \to w \text{ in } X^s(\mathbb{R}^{N+1}_+). \tag{5.9}
$$

Set $\tilde{\Lambda}_n := \frac{\Lambda - \varepsilon_n \tilde{y}_n}{\varepsilon_n}$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$, we define the following functions:

$$
\begin{aligned}
&\tilde{\chi}^1_n(x) := \begin{cases} 1 & \text{if } x \in \tilde{\Lambda}_n, \\ 0 & \text{if } x \in \tilde{\Lambda}^c_n, \end{cases} \\
&\tilde{\chi}^2_n(x) := 1 - \tilde{\chi}^1_n(x), \\
&h^1_n(x) := \frac{1}{4}(V_{\varepsilon_n}(x+y_n) + V_1) w^2_n(x,0)\tilde{\chi}^1_n(x), \\
&h^1(x) := \frac{1}{4}(V(x_0) + V_1) w^2(x,0), \\
&h^2_n(x) := \Big[\frac{1}{4}(V_{\varepsilon_n}(x+y_n) + V_1) w^2_n(x,0) \\
&\quad + \left(\frac{1}{|x|^{\mu}} * G_{\varepsilon_n}(x+y_n, w_n(x,0)) \right) \left(\frac{1}{4} g_{\varepsilon_n}(x+y_n, w_n(x,0)) w_n(x,0) - \frac{1}{2} G_{\varepsilon_n}(x+y_n, w_n(x,0)) \right) \Big] \tilde{\chi}^2_n(x), \\
&h^3_n(x) := \left(\frac{1}{|x|^{\mu}} * G_{\varepsilon_n}(x+y_n, w_n(x,0)) \right) \left(\frac{1}{4} g_{\varepsilon_n}(x+y_n, w_n(x,0)) w_n(x,0) - \frac{1}{2} G_{\varepsilon_n}(x+y_n, w_n(x,0)) \right) \tilde{\chi}^1_n(x), \\
&h^3(x) := \left(\frac{1}{|x|^{\mu}} * F(w(x,0)) \right) \left(\frac{1}{4} f(w(x,0)) w(x,0) - \frac{1}{2} F(w(x,0)) \right).
\end{aligned}
$$

From (f_3), (g_3) and (V_1), it follows that $h^j_n, h^j \geq 0$ in $\mathbb{R}^N$ for all $j = 1,2,3$ and $n \in \mathbb{N}$. Moreover, recalling that, as $n \to \infty$, $w_n(x,0) \to w(x,0)$ for a.e. $x \in \mathbb{R}^N$ and $\varepsilon_n y_n \to x_0 \in \Lambda$, we can see that, as $n \to \infty$,

$$
\tilde{\chi}^1_n(x) \to 1,\ h^1_n(x) \to h^1(x),\ h^2_n(x) \to 0 \ \text{ and } \ h^3_n(x) \to h^3(x) \ \text{ for a.e. } x \in \mathbb{R}^N.
$$

Then, using Lemma 4.3, the weak lower semicontinuity of $\|\cdot\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1|\cdot|^2_2$, Fatou's lemma, and the invariance of $\mathbb{R}^N$ by translation, we deduce that

$$
\begin{aligned}
d_{V(0)} &\geq \limsup_{n\to\infty} c_{\varepsilon_n} = \limsup_{n\to\infty} \left(J_{\varepsilon_n}(v_n) - \frac{1}{4}\langle J'_{\varepsilon_n}(v_n), v_n \rangle \right) \\
&\geq \limsup_{n\to\infty} \left\{ \frac{1}{4}\left[\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1 |w_n(\cdot,0)|^2_2 \right] + \int_{\mathbb{R}^N} (h^1_n + h^2_n + h^3_n)\, dx \right\} \\
&\geq \liminf_{n\to\infty} \left\{ \frac{1}{4}\left[\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1 |w_n(\cdot,0)|^2_2 \right] + \int_{\mathbb{R}^N} (h^1_n + h^2_n + h^3_n)\, dx \right\}
\end{aligned}
$$

$$\geq \frac{1}{4}\left[\|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1|w(\cdot,0)|_2^2\right] + \int_{\mathbb{R}^N}(h^1+h^3)\,dx \geq d_{V(0)}.$$

Hence,

$$\lim_{n\to\infty}\|w_n\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1|w_n(\cdot,0)|_2^2 = \|w\|^2_{X^s(\mathbb{R}^{N+1}_+)} - V_1|w(\cdot,0)|_2^2, \tag{5.10}$$

and

$$h^1_n \to h^1,\ h^2_n \to 0 \text{ and } h^3_n \to h^3 \text{ in } L^1(\mathbb{R}^N).$$

Consequently,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(V_{\varepsilon_n}(x+y_n)+V_1)w_n^2(x,0)\,dx = \int_{\mathbb{R}^N}(V(x_0)+V_1)w^2(x,0)\,dx,$$

and so

$$\lim_{n\to\infty}|w_n(\cdot,0)|_2^2 = |w(\cdot,0)|_2^2. \tag{5.11}$$

From (5.10), (5.11), and the fact that $X^s(\mathbb{R}^{N+1}_+)$ is a Hilbert space, we infer that (5.9) holds. □

Now we establish a crucial result.

Lemma 5.3. *Let (w_n) be the sequence defined as in Lemma 5.2. Then, $(w_n(\cdot,0)) \subset L^\infty(\mathbb{R}^N)$ and there exists $C>0$ such that*

$$|w_n(\cdot,0)|_\infty \leq C \quad \text{for all } n\in\mathbb{N}. \tag{5.12}$$

Furthermore,

$$w_n(\cdot,0)\to 0 \text{ as } |x|\to\infty \text{ uniformly in } n\in\mathbb{N}, \tag{5.13}$$

and there exists $\delta>0$ such that

$$|w_n(\cdot,0)|_\infty \geq \delta \quad \text{for all } n\in\mathbb{N}. \tag{5.14}$$

Proof. Let us observe that, for each $n\in\mathbb{N}$, w_n is a weak solution to

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w_n) + m^2y^{1-2s}w_n = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial w_n}{\partial \nu^{1-2s}} = -V_n(x)w_n + K_n(x)g_n(x,w_n) & \text{on } \partial\mathbb{R}^{N+1}_+, \end{cases} \tag{5.15}$$

where

$$\begin{aligned} V_n(x) &:= V_{\varepsilon_n}(x+y_n), \\ g_n(x,w_n(x,0)) &:= g_{\varepsilon_n}(x+y_n,w_n(x,0)), \\ G_n(x,t) &:= \int_0^t g_n(x,\tau)\,d\tau, \\ K_n(x) &:= \frac{1}{|x|^\mu} * G_{\varepsilon_n}(x+y_n,w_n(x,0)). \end{aligned}$$

Since $(w_n(\cdot,0))$ is bounded in $H^s(\mathbb{R}^N)$, we can proceed as in the proof of Lemma 3.2 to find $C_0>0$ such that

$$\sup_{n\in\mathbb{N}}|K_n|_\infty \leq C_0. \tag{5.16}$$

Now we develop a Moser iteration argument [39] to prove (5.12). For each $n\in\mathbb{N}$ and $L>0$, we define $z_{n,L} := w_n w_{n,L}^{2\beta}$, where $w_{n,L} := \min\{w_n, L\}$ and $\beta>0$ will be chosen later. Inserting $z_{L,n}$ in the weak formulation of (5.15), we get

$$\begin{aligned} &\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}w_{n,L}^{2\beta}(|\nabla w_n|^2 + m^2w_n^2)\,dxdy + \iint_{D_{n,L}} 2\beta y^{1-2s}w_{n,L}^{2\beta}|\nabla w_n|^2\,dxdy \\ &\quad = -\int_{\mathbb{R}^N} V_n(x)w_n^2(x,0)w_{n,L}^{2\beta}(x,0)\,dx + \int_{\mathbb{R}^N} K_n(x)g_n(x,w_n(x,0))w_n(x,0)w_{n,L}^{2\beta}(x,0)\,dx, \end{aligned} \tag{5.17}$$

where $D_{n,L} := \{(x,y) \in \mathbb{R}^{N+1}_+ : w_n(x,y) \le L\}$. A direct calculation shows that

$$\begin{aligned}\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla(w_n w_{n,L}^{\beta})|^2\,dxdy &= \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} w_{n,L}^{2\beta}|\nabla w_n|^2\,dxdy \\ &+ \iint_{D_{n,L}} (2\beta+\beta^2) y^{1-2s} w_{n,L}^{2\beta}|\nabla w_n|^2\,dxdy. \end{aligned} \tag{5.18}$$

On the other hand, using (V_1), (3.3) and (5.16), we see that

$$\begin{aligned}&-\int_{\mathbb{R}^N} V_n(x) w_n^2(x,0) w_{n,L}^{2\beta}(x,0)\,dx + \int_{\mathbb{R}^N} K_n(x) g_n(x, w_n(x,0)) w_n(x,0) w_{n,L}^{2\beta}(x,0)\,dx \\ &\le \int_{\mathbb{R}^N} \left[(V_1+C_0) w_n^2(x,0) w_{n,L}^{2\beta}(x,0) + C_0 C_1 w_n^q(x,0) w_{n,L}^{2\beta}(x,0)\right]\,dx.\end{aligned} \tag{5.19}$$

Combining (5.17), (5.18), (5.19), we have

$$\begin{aligned}&\|w_n w_{n,L}^{\beta-1}\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla(w_n w_{n,L}^{\beta})|^2 + m^2 w_n^2 w_{n,L}^{2\beta})\,dxdy \\ &= \iint_{\mathbb{R}^{N+1}_+} y^{1-2s} w_{n,L}^{2\beta}(|\nabla w_n|^2 + m^2 w_n^2)\,dxdy + \iint_{D_{n,L}} 2\beta\left(1+\frac{\beta}{2}\right) y^{1-2s} w_{n,L}^{2\beta}|\nabla w_n|^2\,dxdy \\ &\le C_\beta\left[\iint_{\mathbb{R}^{N+1}_+} y^{1-2s} w_{n,L}^{2\beta}(|\nabla w_n|^2 + m^2 w_n^2)\,dxdy + \iint_{D_{n,L}} 2\beta y^{1-2s} w_{n,L}^{2\beta}|\nabla w_n|^2\,dxdy\right] \\ &= C_\beta\left[-\int_{\mathbb{R}^N} V_n(x) w_n^2(x,0) w_{n,L}^{2\beta}(x,0)\,dx + \int_{\mathbb{R}^N} K_n(x) g_n(x,w_n(x,0)) w_n(x,0) w_{n,L}^{2\beta}(x,0)\,dx\right] \\ &\le C_\beta\left[\int_{\mathbb{R}^N} (V_1+C_0) w_n^2(x,0) w_{n,L}^{2\beta}(x,0) + C_0 C_1 w_n^q(x,0) w_{n,L}^{2\beta}(x,0)\,dx\right],\end{aligned} \tag{5.20}$$

where we set $C_\beta := 1 + \frac{\beta}{2}$. Thanks to the boundedness of (w_n) in $X^s(\mathbb{R}^{N+1}_+)$ and Theorem 2.1, we can verify that there exist $C_2 > 0$ independent of n, L, β, and $h_n \in L^{N/2s}(\mathbb{R}^N)$ with $h_n \ge 0$ and independent of L and β, such that

$$(V_1+C_0) w_n^2(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) + C_0 C_1 w_n^q(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) \le (C_2 + h_n) w_n^2(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) \quad \text{on } \mathbb{R}^N. \tag{5.21}$$

To prove (5.21), we first notice that

$$\begin{aligned}&(V_1+C_0) w_n^2(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) + C_0 C_1 w_n^q(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) \\ &\le (V_1+C_0) w_n^2(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) + C_0 C_1 w_n^{q-2}(\cdot,0) w_n^2(\cdot,0) w_{n,L}^{2\beta}(\cdot,0) \quad \text{on } \mathbb{R}^N.\end{aligned}$$

Moreover,

$$w_n^{q-2}(\cdot,0) \le 1 + h_n \quad \text{on } \mathbb{R}^N, \tag{5.22}$$

where $h_n := \chi_{\{w_n(\cdot,0)>1\}} w_n^{q-2}(\cdot,0) \in L^{N/2s}(\mathbb{R}^N)$. Indeed, we can observe that

$$w_n^{q-2}(\cdot,0) = \chi_{\{w_n(\cdot,0)\le 1\}} w_n^{q-2}(\cdot,0) + \chi_{\{w_n(\cdot,0)>1\}} w_n^{q-1}(\cdot,0) \le 1 + \chi_{\{w_n(\cdot,0)>1\}} w_n^{q-2}(\cdot,0) \quad \text{on } \mathbb{R}^N.$$

If $(q-2)\frac{N}{2s} < 2$ then, recalling that $(w_n(\cdot,0))$ is bounded in $L^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \chi_{\{w_n(\cdot,0)>1\}} (w_n(x,0))^{(q-2)\frac{N}{2s}}\,dx \le \int_{\mathbb{R}^N} \chi_{\{w_n(\cdot,0)>1\}} w_n^2(x,0)\,dx \le C \quad \text{for all } n \in \mathbb{N}.$$

If $2 \le (q-2)\frac{N}{2s}$ then, due to $q \in \left(2, \frac{2(N-\mu)}{N-2s}\right)$, we get $(q-2)\frac{N}{2s} \in [2, 2^*_s]$, and using Theorem 2.1 and the boundedness of (w_n) in $X^s(\mathbb{R}^{N+1}_+)$, we see that

$$\int_{\mathbb{R}^N} \chi_{\{w_n(\cdot,0)>1\}} (w_n(x,0))^{(q-2)\frac{N}{2s}}\,dx \le C' \|w_n\|^{(q-2)\frac{N}{2s}}_{X^s(\mathbb{R}^{N+1}_+)} \le C'' \quad \text{for all } n \in \mathbb{N},$$

where $C', C'' > 0$ depend only on N, s and q. Hence, (5.22) holds, and thus (5.21) is valid. In light of (5.20) and (5.21), we obtain

$$\|w_n w_{n,L}^{\beta}\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le C_\beta \int_{\mathbb{R}^N} (C_2 + h_n(x)) w_n^2(x,0) w_{n,L}^{2\beta}(x,0)\, dx. \tag{5.23}$$

Letting $L \to \infty$ in (5.23) and invoking Fatou's lemma and the monotone convergence theorem, we find

$$\|w_n^{\beta+1}\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le C_2 C_\beta \int_{\mathbb{R}^N} w_n^{2(\beta+1)}(x,0)\, dx + C_\beta \int_{\mathbb{R}^N} h_n(x) w_n^{2(\beta+1)}(x,0)\, dx. \tag{5.24}$$

Fix $M > 1$ and put $\Omega_{1,n} := \{h_n \le M\}$ and $\Omega_{2,n} := \{h_n > M\}$. From Hölder's inequality, we derive that

$$\begin{aligned}\int_{\mathbb{R}^N} h_n(x) w_n^{2(\beta+1)}(x,0)\, dx &= \int_{\Omega_{1,n}} h_n(x) w_n^{2(\beta+1)}(x,0)\, dx + \int_{\Omega_{2,n}} h_n(x) w_n^{2(\beta+1)}(x,0)\, dx \\ &\le M |w_n^{\beta+1}(\cdot,0)|_2^2 + \epsilon(M) |w_n^{\beta+1}(\cdot,0)|^2_{2^*_s},\end{aligned} \tag{5.25}$$

where

$$\epsilon(M) := \sup_{n\in\mathbb{N}} \left(\int_{\Omega_{2,n}} h_n^{N/2s} dx \right)^{\frac{2s}{N}} \to 0 \quad \text{as } M \to \infty,$$

because $w_n(\cdot,0) \to w(\cdot,0)$ in $H^s(\mathbb{R}^N)$. Combining (5.24) with (5.25) gives

$$\|w_n^{\beta+1}\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le C_\beta (C_2 + M) |w_n^{\beta+1}(\cdot,0)|_2^2 + C_\beta \epsilon(M) |w_n^{\beta+1}(\cdot,0)|^2_{2^*_s}. \tag{5.26}$$

On the other hand, Theorem 2.1 implies

$$|w_n^{\beta+1}(\cdot,0)|^2_{2^*_s} \le C_*^2 \|w_n^{\beta+1}\|^2_{X^s(\mathbb{R}^{N+1}_+)}. \tag{5.27}$$

Therefore, taking $M > 0$ big enough such that

$$\epsilon(M) C_\beta C_*^2 < \frac{1}{2},$$

and exploiting (5.26) and (5.27), we arrive at

$$|w_n^{\beta+1}(\cdot,0)|^2_{2^*_s} \le 2 C_*^2 C_\beta (C_2 + M) |w_n^{\beta+1}(\cdot,0)|_2^2. \tag{5.28}$$

Now we start a bootstrap argument. Since $w_n(\cdot,0) \in L^{2^*_s}(\mathbb{R}^N)$ and $|w_n(\cdot,0)|_{2^*_s} \le C$ for all $n \in \mathbb{N}$, we can use (5.28) with $\beta_1 + 1 = \frac{N}{N-2s}$ to infer that $w_n(\cdot,0) \in L^{(\beta_1+1)2^*_s}(\mathbb{R}^N) = L^{\frac{2N^2}{(N-2s)^2}}(\mathbb{R}^N)$. Applying again (5.28), after k iterations, we get $w_n(\cdot,0) \in L^{\frac{2N^k}{(N-2s)^k}}(\mathbb{R}^N)$, and so $w_n(\cdot,0) \in L^r(\mathbb{R}^N)$ for all $r \in [2,\infty)$ with

$$|w_n(\cdot,0)|_r \le C_r \quad \text{for all } n \in \mathbb{N}. \tag{5.29}$$

Next we show the boundedness of $(w_n(\cdot,0))$ in $L^\infty(\mathbb{R}^N)$. We first observe that (5.29) and the definition of h_n yield $(h_n) \subset L^{\frac{N}{s}}(\mathbb{R}^N)$ and $|h_n|_{\frac{N}{s}} \le D$ for all $n \in \mathbb{N}$. Then, by virtue of the generalized Hölder inequality and Young's inequality with $\lambda > 0$, we can see that for all $\lambda > 0$

$$\begin{aligned}\int_{\mathbb{R}^N} h_n(x) w_n^{2(\beta+1)}(x,0)\, dx &\le |h_n|_{\frac{N}{s}} |w_n^{\beta+1}(\cdot,0)|_2 |w_n^{\beta+1}(\cdot,0)|_{2^*_s} \\ &\le D \left(\lambda |w_n^{\beta+1}(\cdot,0)|_2^2 + \frac{1}{\lambda} |w_n^{\beta+1}(\cdot,0)|^2_{2^*_s} \right).\end{aligned}$$

Using (5.24), (5.27) and the above inequality, we have

$$|w_n^{\beta+1}(\cdot,0)|^2_{2^*_s} \le C_*^2 \|w_n^{\beta+1}\|^2_{X^s(\mathbb{R}^{N+1}_+)} \le C_\beta C_*^2 (C_2 + D\lambda) |w_n^{\beta+1}(\cdot,0)|_2^2 + \frac{C_\beta D C_*^2}{\lambda} |w_n^{\beta+1}(\cdot,0)|^2_{2^*_s}. \tag{5.30}$$

Taking $\lambda > 0$ such that

$$\frac{C_\beta D C_*^2}{\lambda} = \frac{1}{2},$$

we obtain that

$$|w_n^{\beta+1}(\cdot,0)|^2_{2^*_s} \le 2 C_\beta (C_2 + D\lambda) C_*^2 |w_n^{\beta+1}(\cdot,0)|_2^2 = M_\beta |w_n^{\beta+1}(\cdot,0)|_2^2,$$

where $M_\beta := 2C_\beta(C_2 + D\lambda)C_*^2$. Now the advantage is that we are able to control the dependence on β of M_β as follows:

$$M_\beta \le C_3 C_\beta^2 \le C_4(1+\beta)^2 \le M_0^2 e^{2\sqrt{\beta+1}},$$

for some $M_0 > 0$ independent of β. Therefore,

$$|w_n(\cdot,0)|_{2_s^*(\beta+1)} \le M_0^{\frac{1}{\beta+1}} e^{\frac{1}{\sqrt{\beta+1}}} |w_n(\cdot,0)|_{2(\beta+1)}.$$

Choosing $\beta_0 = 0$ and $2(\beta_{j+1}+1) = 2_s^*(\beta_j+1)$ for $j \in \mathbb{N}$, an iteration argument reveals that for all $k \in \mathbb{N}\cup\{0\}$

$$|w_n(\cdot,0)|_{2_s^*(\beta_k+1)} \le M_0^{\sum_{j=0}^k \frac{1}{\beta_j+1}} e^{\sum_{j=0}^k \frac{1}{\sqrt{\beta_j+1}}} |w_n(\cdot,0)|_{2(\beta_0+1)}.$$

Because $\beta_j + 1 = \left(\frac{N}{N-2s}\right)^j$ for all $j \in \mathbb{N}\cup\{0\}$, we have

$$\upsilon_1 := \sum_{j=0}^{\infty} \frac{1}{\beta_j+1} < \infty \quad \text{and} \quad \upsilon_2 := \sum_{j=0}^{\infty} \frac{1}{\sqrt{\beta_j+1}} < \infty,$$

and recalling that $|w_n(\cdot,0)|_2 \le C$ for all $n \in \mathbb{N}$, we infer that

$$|w_n(\cdot,0)|_\infty = \lim_{k\to\infty} |w_n(\cdot,0)|_{2_s^*(\beta_k+1)} \le M_0^{\upsilon_1} e^{\upsilon_2} C =: \bar{C} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, (5.12) is valid. Now, we notice that $w_n(\cdot,0)$ is a weak solution to

$$(-\Delta+m^2)^s w_n(\cdot,0) = -V_n(x) w_n(\cdot,0) + K_n(x) g_n(x, w_n(\cdot,0)) \quad \text{in } \mathbb{R}^N.$$

Pick $\eta \in (0, m^{2s} - V_1)$. Using (V_1), (3.3) and (5.16), we see that $w_n(\cdot,0)$ is a weak subsolution to

$$(-\Delta+m^2)^s w_n(\cdot,0) = (V_1 + C_0\eta) w_n(\cdot,0) + C_\eta C_0 w_n^{q-1}(\cdot,0) =: \zeta_n \quad \text{in } \mathbb{R}^N, \tag{5.31}$$

for some $C_\eta > 0$. Exploiting $w_n \to w$ in $X^s(\mathbb{R}^{N+1}_+)$, (5.12) and the interpolation inequality for $L^r(\mathbb{R}^N)$ spaces, we deduce that, for all $r \in [2,\infty)$,

$$\zeta_n \to \zeta := (V_1 + C_0\eta) w(\cdot,0) + C_\eta C_0 w^{q-1}(\cdot,0) \quad \text{in } L^r(\mathbb{R}^N),$$

and that $|\zeta_n|_\infty \le C$ for all $n \in \mathbb{N}$. Let $z_n \in H^s(\mathbb{R}^N)$ be the unique solution to

$$(-\Delta+m^2)^s z_n = \zeta_n \quad \text{in } \mathbb{R}^N. \tag{5.32}$$

Thus, $z_n = \mathcal{G}_{2s,m} * \zeta_n$, where $\mathcal{G}_{2s,m}(x) := (2\pi)^{-\frac{N}{2}} \mathcal{F}^{-1}((|k|^2+m^2)^{-s})(x)$ is the Bessel kernel with parameter m and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The scaling property of the Fourier transform implies that $\mathcal{G}_{2s,m}(x) = m^{N-2s}\mathcal{G}_{2s,1}(mx)$. From [13, Chapter 2], we derive that

$$\mathcal{G}_{2s,m}(x) = \frac{1}{2^{\frac{N+2s-2}{2}} \pi^{\frac{N}{2}} \Gamma(s)} m^{\frac{N-2s}{2}} K_{\frac{N-2s}{2}}(m|x|) |x|^{\frac{2s-N}{2}},$$

and that the following properties hold:
$(\mathcal{G}1)$ $\mathcal{G}_{2s,m}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$,
$(\mathcal{G}2)$ $\mathcal{G}_{2s,m}(x) \le C(\chi_{B_2}(x)|x|^{2s-N} + \chi_{B_2^c}(x) e^{-c|x|})$ for all $x \in \mathbb{R}^N$, for some $C, c > 0$,
$(\mathcal{G}3)$ $\mathcal{G}_{2s,m} \in L^r(\mathbb{R}^N)$ for all $r \in [1, \frac{N}{N-2s})$.
Next we claim that

$$z_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}. \tag{5.33}$$

Take $\delta \in (0, \frac{1}{2})$. Then we write

$$\begin{aligned} z_n(x) = (\mathcal{G}_{2s,m} * \zeta_n)(x) &= \int_{B^c_{\frac{1}{\delta}}(x)} \mathcal{G}_{2s,m}(x-\bar{x})\zeta_n(\bar{x})\, d\bar{x} + \int_{B_{\frac{1}{\delta}}(x)} \mathcal{G}_{2s,m}(x-\bar{x})\zeta_n(\bar{x})\, d\bar{x} \\ &=: S_{n,\delta}(x) + T_{n,\delta}(x). \end{aligned} \tag{5.34}$$

In view of $(\mathcal{G}1)$ and $(\mathcal{G}2)$, we can see that

$$0 \le S_{n,\delta}(x) \le C|\zeta_n|_\infty \int_{B^c_{\frac{1}{\delta}}(x)} e^{-c|x-\bar{x}|}\, d\bar{x} \le C_1 \int_{\frac{1}{\delta}}^{\infty} e^{-cr} r^{N-1} dr =: C_1 A(\delta) \to 0 \quad \text{as } \delta \to 0. \tag{5.35}$$

On the other hand, it holds

$$0 \leq T_{n,\delta}(x) = \int_{B_{\frac{1}{\delta}}(x)} \mathcal{G}_{2s,m}(x-\bar{x})(\zeta_n(\bar{x}) - \zeta(\bar{x}))\, d\bar{x} + \int_{B_{\frac{1}{\delta}}(x)} \mathcal{G}_{2s,m}(x-\bar{x})\zeta(\bar{x})\, d\bar{x}.$$

Select $q_0 \in (1, \min\{\frac{N}{N-2s}, 2\})$ so that $q_0' > 2$, where $q_0' := \frac{q_0}{q_0-1}$. On account of $(\mathcal{G}3)$ and Hölder's inequality, we get

$$T_{n,\delta}(x) \leq |\mathcal{G}_{2s,m}|_{q_0} |\zeta_n - \zeta|_{q_0'} + |\mathcal{G}_{2s,m}|_{q_0} |\zeta|_{L^{q_0'}(B_{\frac{1}{\delta}}(x))}.$$

Exploiting $|\zeta_n - \zeta|_{q_0'} \to 0$ as $n \to \infty$ and $|\zeta|_{L^{q_0'}(B_{\frac{1}{\delta}}(x))} \to 0$ as $|x| \to \infty$, we can find $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$T_{n,\delta}(x) \leq C_2 \delta \quad \text{for all } |x| \geq R \text{ and } n \geq n_0. \tag{5.36}$$

Combining (5.35) and (5.36), we arrive at

$$z_n(x) = (\mathcal{G}_{2s,m} * \zeta_n)(x) \leq C_3(A(\delta) + \delta) \quad \text{for all } |x| \geq R \text{ and } n \geq n_0. \tag{5.37}$$

On the other hand, fixed $n \in \{1, \dots, n_0 - 1\}$, there exists $R_n > 0$ such that $|\zeta_n|_{L^{q_0'}(B_{\frac{1}{\delta}}(x))} < \delta$ for all $|x| \geq R_n$. Using this fact, (5.35) and Hölder's inequality, for all $|x| \geq R_n$ we have

$$\begin{aligned} z_n(x) = S_{n,\delta}(x) + T_{n,\delta}(x) &\leq C_1 A(\delta) + \int_{B_{\frac{1}{\delta}}(x)} \mathcal{G}_{2s,m}(x-\bar{x})\zeta_n(\bar{x})\, d\bar{x} \\ &\leq C_1 A(\delta) + |\mathcal{G}_{2s,m}|_{q_0} |\zeta_n|_{L^{q_0'}(B_{\frac{1}{\delta}}(x))} \\ &\leq C'(A(\delta) + \delta). \end{aligned} \tag{5.38}$$

Put $\bar{R} := \max\{R, R_1, \dots, R_{n_0-1}\}$. Then (5.37) and (5.38) lead to

$$0 \leq z_n(x) \leq C''(A(\delta) + \delta) \quad \text{for all } |x| \geq \bar{R} \text{ and } n \in \mathbb{N}. \tag{5.39}$$

Passing to the limit as $\delta \to 0$ in (5.39), we deduce that (5.33) is true.

From (5.31), (5.32), and the comparison principle for $(-\Delta + m^2)^s$ (see [9, Theorem 4.1]), it follows that $0 \leq w_n(\cdot, 0) \leq z_n$ in $\mathbb{R}^N$, and thanks to (5.33) we obtain that (5.13) holds. Finally, we prove (5.14). According to Lemma 5.1, there exist $r, \beta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{B_r} w_n^2(x, 0)\, dx \geq \beta \quad \text{for all } n \geq n_0.$$

This fact combined with (5.12) implies

$$0 < \beta \leq \int_{B_r} w_n^2(x, 0)\, dx \leq |B_r| |w_n(\cdot, 0)|_\infty^2 \quad \text{for all } n \geq n_0,$$

which shows that (5.14) is valid. The proof of the lemma is now complete. □

Finally, we provide the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 5.2, we can find $(y_n) \subset \mathbb{R}^N$ and $w \in X^s(\mathbb{R}^{N+1}_+) \setminus \{0\}$ such that, up to a subsequence, $w_n(x, y) := v_n(x + y_n, y) \to w$ in $X^s(\mathbb{R}^{N+1}_+)$ and $\varepsilon_n y_n \to x_0$ for some $x_0 \in M$. Hence, there is $r > 0$ such that for some subsequence, still denoted by itself, we get

$$B_r(\varepsilon_n y_n) \subset \Lambda \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$B_{\frac{r}{\varepsilon_n}}(y_n) \subset \Lambda_{\varepsilon_n} \quad \text{for all } n \in \mathbb{N},$$

or equivalently

$$\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(y_n) \quad \text{for all } n \in \mathbb{N}.$$

With the help of (5.13), we know that for some $R > 0$ it holds

$$w_n(x, 0) < a \quad \text{for all } |x| \geq R \text{ and } n \in \mathbb{N},$$

from which

$$v_n(x,0) = w_n(x-y_n,0) < a \quad \text{for all } x \in B^c_R(y_n) \text{ and } n \in \mathbb{N}.$$

On the other hand, there exists $n_0 \in \mathbb{N}$ such that

$$\Lambda^c_{\varepsilon_n} \subset B^c_{\frac{r}{\varepsilon_n}}(y_n) \subset B^c_R(y_n) \quad \text{for all } n \geq n_0.$$

Therefore,

$$v_n(x,0) < a \quad \text{for all } x \in \Lambda^c_{\varepsilon_n} \text{ and } n \geq n_0.$$

This means that there is $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0)$, (2.4) possesses a solution v_ε. On account of the weak Harnack inequality [10, Proposition 4.2.11], we have that $v_\varepsilon(\cdot,0) > 0$ in $\mathbb{R}^N$. Finally, we investigate the behavior of the maximum points of solutions to problem (1.1). Let $\varepsilon_n \to 0$ and consider a sequence of mountain pass solutions $(v_n) \subset X_{\varepsilon_n}$ to (3.2). Using Lemma 5.2, there exist $(y_n) \subset \mathbb{R}^N$ and $w \in X^s(\mathbb{R}^{N+1}_+) \setminus \{0\}$ such that, up to a subsequence, $w_n(x,y) := v_n(x+y_n,y) \to w$ in $X^s(\mathbb{R}^{N+1}_+)$ and $\varepsilon_n y_n \to x_0$ for some $x_0 \in M$. Let now q_n be a global maximum point of $w_n(\cdot,0)$. From (5.13) and (5.14), we derive that (q_n) is bounded in $\mathbb{R}^N$, that is, for some $\tilde{R} > 0$, it holds $|q_n| \leq \tilde{R}$ for all $n \in \mathbb{N}$. Because $x_n := q_n + y_n$ is a global maximum point of $v_n(\cdot,0)$ and $\varepsilon_n x_n \to x_0 \in M$, it follows from the continuity of V that

$$\lim_{n\to\infty} V(\varepsilon_n x_n) = V(x_0) = -V_0.$$

Next we establish a decay estimate for $v_n(\cdot,0)$. Pick $\delta \in (0, m^{2s} - V_1)$. Exploiting (5.13), (f_1), the definition of g and (5.16), we can find $R_0 > 0$ such that

$$K_n(x) g_n(x, w_n(x,0)) \leq \delta w_n(x,0) \quad \text{for all } |x| > R_0 \text{ and } n \in \mathbb{N}. \tag{5.40}$$

Reasoning as in [9, Theorem 1.1], there are a positive continuous function $\bar{w} \in H^s(\mathbb{R}^N)$ and a constant $R_1 > 0$ such that

$$(-\Delta + m^2)^s \bar{w} - (V_1 + \delta)\bar{w} = 0 \quad \text{in } \bar{B}^c_{R_1}, \tag{5.41}$$

and

$$0 < \bar{w}(x) \leq k_1 e^{-k_2|x|} \quad \text{for all } x \in \mathbb{R}^N, \tag{5.42}$$

for some $k_1, k_2 > 0$. Set $R_2 := \max\{R_0, R_1\}$. From (V_1) and (5.40), we deduce that

$$(-\Delta + m^2)^s w_n(\cdot,0) - (V_1 + \delta) w_n(\cdot,0) \leq 0 \quad \text{in } \bar{B}^c_{R_2}. \tag{5.43}$$

Let $\varrho := \min_{\bar{B}_{R_2}} \bar{w} > 0$, $\lambda := \sup_{n\in\mathbb{N}} |w_n(\cdot,0)|_\infty < \infty$ and $z_n := (\lambda+1)\bar{w} - \varrho w_n(\cdot,0)$. Clearly, $z_n \geq 0$ in $\bar{B}_{R_2}$, and using (5.41) and (5.43) we see that z_n satisfies

$$(-\Delta + m^2)^s z_n - (V_1 + \delta) z_n \geq 0 \quad \text{in } \bar{B}^c_{R_2}.$$

Applying [9, Theorem 4.1] with $\Omega = \bar{B}^c_{R_2}$ (note that $V_1 + \delta < m^{2s}$), we obtain that $z_n \geq 0$ in $\mathbb{R}^N$. This fact combined with (5.42) leads to

$$0 \leq w_n(x,0) \leq k_3\, e^{-k_2|x|} \quad \text{for all } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N},$$

where $k_3 := (\lambda+1)\varrho^{-1} k_1 > 0$. Since $v_n(x,0) = w_n(x-y_n,0)$, we infer that, for some $C_1, C_2 > 0$,

$$v_n(x,0) = w_n(x-y_n,0) \leq C_1 e^{-C_2|x-y_n|} \quad \text{for all } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}.$$

The proof of Theorem 1.1 is now complete. □

Remark 5.1. *If we assume that* (f_1)*-*(f_4) *hold and that* $V \in C(\mathbb{R}^N)$ *satisfies the following conditions:*
(V_1') *there exists* $V_0 \in (0, m^{2s})$ *such that* $-V_0 := \inf_{x\in\mathbb{R}^N} V(x)$,
(V_2') *there exists a bounded open set* $\Lambda \subset \mathbb{R}^N$ *such that*

$$-V_0 < \min_{x\in\partial\Lambda} V(x) \quad \textit{and} \quad 0 \in M := \{x \in \Lambda : V(x) = -V_0\},$$

then we can follow the approach in [6] (see also [9]), with the appropriate modifications, to demonstrate that for every $\delta > 0$ *satisfying*

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_\delta)$, (1.1) *has at least* $cat_{M_\delta}(M)$ *positive solutions. Moreover, if* u_ε *denotes one of these solutions and* x_ε *is a global maximum point of* u_ε*, then we have*

$$\lim_{\varepsilon \to 0} V(\varepsilon\, x_\varepsilon) = -V_0.$$

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