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Duality and polyhedrality of cones for Mori dream spaces

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Abstract
Our goal is twofold. On one hand, we show that the cones of divisors ample in codimension k on a Mori dream space are rational polyhedral. On the other hand, we study the duality between such cones and the cones of k-moving curves by means of the Mori chamber decomposition of the former. We give a new proof of the weak duality property (already proved by Payne and Choi) and we exhibit an interesting family of examples for which strong duality holds.

Keywords Mori dream space · Mori chamber decomposition · Stable base locus decomposition · Weak duality · Strong duality

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1 Introduction

Let X be a normal $\mathbb{Q}$-factorial variety of dimension n and with irregularity zero, defined over an algebraically closed field of characteristic zero.

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For any integer $0 \leq k \leq n-1$, we consider the cone D_k of numerical classes of divisors on X whose stable base locus has codimension larger than k and we set $\mathcal{D}_k$ to be the closure of D_k in the Néron–Severi space $\mathrm{N}^1(X)_{\mathbb{R}}$. Following [36], we call $\mathcal{D}_k$ the cone of divisors that are *ample in codimension* k. Clearly $\mathcal{D}_0$ is the pseudo-effective cone $\overline{\mathrm{Eff}(X)}$, $\mathcal{D}_1$ is the cone of movable divisors, and $\mathcal{D}_{n-1}$ is the nef cone by Kleiman's theorem. When X is a Mori dream space, the cones $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_{n-1}$ are rational polyhedral, as shown in [25], moreover $\mathcal{D}_0$ and $\mathcal{D}_1$ admit a finite *Mori chamber decomposition* of which $\mathcal{D}_{n-1}$ is a chamber. In this article we prove that each cone $\mathcal{D}_k$ inherits the chamber decomposition and in particular it is rational polyhedral (and D_k is closed).

We have the obvious filtration of cones

$$\mathrm{Nef}(X) = \mathcal{D}_{n-1} \subseteq \cdots \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_0 = \overline{\mathrm{Eff}(X)};$$

on the other hand we shall consider an analogous filtration of cones of pseudo-effective 1-cycles:

$$\overline{\mathrm{NE}(X)} = \mathcal{C}_{n-1} \supseteq \cdots \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_0,$$

where $\mathcal{C}_k$ is the closure of the cone in $\mathrm{N}_1(X)_{\mathbb{R}}$ generated by classes of k-*moving* curves, i.e. classes of irreducible curves moving in a family that sweeps out an $(n-k)$-dimensional subvariety of X. Corresponding ends of the above filtrations are dual to each other with respect to the standard intersection pairing. Indeed the dual of the Mori cone $\mathcal{C}_{n-1}^{\vee}$ is the nef cone and the cone $\mathcal{C}_0$ is dual to the pseudo-effective cone $\overline{\mathrm{Eff}(X)}$, as shown in [5].

Since duality of the cones $\mathcal{D}_k$ and $\mathcal{C}_k$, for $1 \leq k \leq n-2$, is not to be expected in general, it is a natural question to ask under which conditions and for which integers $1 \leq k \leq n-2$ *strong duality* holds, that is $\mathcal{D}_k^{\vee} = \mathcal{C}_k$.

In this paper, we show that the answer is positive, for every k, for all blow-ups of $\mathbb{P}^n$ at s points in general position that are Mori dream spaces.

The answer to the duality question is not positive in general but a weaker version holds, as proved in [36] for toric varieties and in [14] for Mori dream spaces: for a Mori dream space X, $\mathcal{D}_k^{\vee}$ is generated by curves that move in families sweeping out the birational image, on a small modification of X, of $(n-k)$-dimensional subvarieties of X. In other terms, k-moving curves on X are not enough to reconstruct, by duality, the cone $\mathcal{D}_k$; instead one needs to consider k-moving curves on all small modifications of X, which are of finite number. We will call this property *weak duality*.

The first main theorem of this paper, Theorem 3.3, describes the combinatorial properties of the cones $\mathcal{D}_k$ for Mori dream spaces, providing a different perspective on the study of these cones and setting the bases for a new proof of the weak duality statement. As a crucial tool we use the stable base locus decomposition of the effective and movable cones, which is refined by the Mori chamber decomposition, and we prove that it is compatible with the filtration of cones $\mathcal{D}_k$.

Theorem 1.1 *If X is a Mori dream space, then the cones $\mathcal{D}_k$ are unions of Mori chambers, hence rational polyhedral, and the cones D_k are closed, i.e. $D_k = \mathcal{D}_k$, for every $0 \leq k \leq n-1$.*

The fact that the cones $\mathcal{D}_k$ are rational polyhedral in the case of toric varieties was proved by Payne in [36] where, furthermore, a constructive description was presented in terms of the boundary divisors.

Choi asked if the $\mathcal{D}_k$'s are rational polyhedral also for Mori dream spaces, see [14, Question 4.4] and [15, p. 1177]. The first statement of Theorem 1.1 answers affirmatively the question for divisors on Mori dream spaces. Moreover if we restrict to the toric case, it gives another proof via a new approach to the rational polyhedrality of the cones $\mathcal{D}_k$.

Our second main result, proved in Theorems 5.5 and 5.8, provides a first family of examples for which strong duality holds.

Theorem 1.2 *The strong duality holds for all Mori dream blow-ups of projective spaces at points in general position.*

We remark that the geometry of these spaces and their chamber decompositions are completely encoded by *Weyl cycles*. We will investigate more deeply the interplay between the birational geometry and the action of the Weyl group in forthcoming work, continuing the study started in [6, 7].

Moreover, in future work we will seek criteria for strong duality to hold for general Mori dream spaces.

Finally, we point out that ampleness in codimension k relates to different notions of cohomological ampleness for divisors. These notions were studied by Andreotti–Grauert and Sommese (naive ampleness), by Demailly–Peternell–Schneider (uniform ampleness) and by Totaro (T-ampleness), see [37] and references therein. In [37, Question 11.2], Totaro asked if, for Fano varieties in characteristic zero, the closure of the cone of cohomologically k-ample divisors is a union of rational polyhedral cones. In [9] the authors proved that this property holds for simplicial toric varieties.

The article is organised as follows. In Sect. 2 we recall some preliminaries on cones of curves and divisors and on chamber decompositions. Section 3 is dedicated to our first main result, Theorem 1.1, that establishes the polyhedrality of the cones $\mathcal{D}_k$ for all Mori dream spaces. Section 4 contains our proof of the weak duality property for $\mathcal{D}_k$ and the cones of curves $\mathcal{C}_k$ for Mori dream spaces, by means of the Mori chamber decomposition of the effective cone. We also include the discussion of a toric example for which strong duality does not hold. In Sect. 5 we prove our second main result, Theorem 1.2, which concerns a family of examples of Mori dream spaces satisfying strong duality.

2 Preliminaries

Throughout the paper, we assume that X is a normal $\mathbb{Q}$-factorial variety of dimension n, defined over an algebraically closed field of characteristic zero.

In this section we recall some preliminaries. As a general reference see [29].

2.1 Rational polyhedral cones

Let $V \subset \mathbb{R}^n$ be a convex cone. A subcone $W \subset V$ is *extremal* if $u+v \in W$ implies $u, v \in W$. A one-dimensional subcone is called a *ray*. A cone is *polyhedral* if it is the intersection of a finite number of half-spaces through the origin. By the Minkowski–Weyl theorem, a cone is polyhedral if and only if it is finite, i.e. it is the convex conical hull of a finite number of vectors. In particular a polyhedral cone is closed. A convex cone is *rational polyhedral* if it is generated by a finite set of integer vectors.

2.2 Curve classes and divisor classes

Let $\mathrm{Pic}(X)$ be the Picard group of X and let $\mathrm{A}_k(X)$ be the group of k-cycles on X, for $1 \leq k \leq n-1$. We denote with $\mathrm{N}^1(X)_{\mathbb{R}} = (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$ the Néron–Severi space of X and with $\mathrm{N}_1(X)_{\mathbb{R}} = (\mathrm{A}_1(X)/\equiv) \otimes \mathbb{R}$ its dual space with respect to the standard intersection pairing. The cones of divisors $\mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}(X)} \subset \mathrm{N}^1(X)_{\mathbb{R}}$ are, respectively, the nef and the pseudo-effective cones. The former is dual to the Mori cone $\overline{\mathrm{NE}(X)} \subset \mathrm{N}_1(X)_{\mathbb{R}}$, which is the closure of the cone of effective 1-cycles; the latter is dual to the closure of the cone of moving curves, that are numerical curve classes such that the irreducible representatives of the class pass through a general point of X, see [5, Theorem 0.2] or [30, 11.4.C].

2.3 Stable base locus decomposition

Given an effective divisor D on X, we denote by $\mathrm{Bs}(D)$ the base locus of the linear system $|D|$. We recall, from [29, Definition 2.1.20], that the stable base locus of D is the Zariski closed set

$$\mathbb{B}(D) = \bigcap_{m>0} \mathrm{Bs}(mD).$$

Under the assumption $q(X) = h^1(X, \mathcal{O}_X) = 0$, linear and numerical equivalence of $\mathbb{Q}$-divisors coincide. Thus the stable base locus is well defined for numerical equivalence classes of $\mathbb{Q}$-divisors on X, [23].

The *Stable base locus decomposition* is a wall-and-chamber decomposition of the pseudo-effective cone $\overline{\mathrm{Eff}(X)}$ of X, where the stable base locus is constant in the interior of the chambers, see e.g. [26, Sect 4.1.3] and [27] for more details.

We will call *stable base locus subvariety of* X any irreducible and reduced subvariety $Y \subset X$ that whenever it is a component of the base locus of an effective divisor on X, it is a component of its stable base locus.

A divisor D in X is called *movable* if its stable base locus has codimension at least two in X. The movable cone of X is the convex cone $\mathrm{Mov}(X)$ generated by classes of movable divisors. We have the following obvious inclusions of cones: $\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}(X)} \subseteq \overline{\mathrm{Eff}(X)}$.

2.4 Mori dream spaces and Mori chamber decomposition

We recall the definition of *Mori dream space* from [25, Definition 1.10]. Mori dream spaces include several classes of interesting varieties such as toric varieties, $\mathbb{Q}$-factorial spherical varieties [8], Fano varieties and varieties of Fano type [3] among others.

Definition 2.1 A normal projective $\mathbb{Q}$-factorial variety X is called a Mori dream space if the following conditions hold:

(1) $\mathrm{Pic}(X)$ is finitely generated, that is $h^1(X, \mathcal{O}_X) = 0$,
(2) $\mathrm{Nef}(X)$ is generated by the classes of finitely many semi-ample divisors,
(3) there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : S \dashrightarrow X_i$, such that each X_i satisfies (2), and

$$\mathrm{Mov}(X) = \bigcup f_i^*(\mathrm{Nef}(X_i)).$$

Recall that a small $\mathbb{Q}$-factorial modification (SQM) $f : X \dashrightarrow X'$ is a contracting birational map that is an isomorphism in codimension 1, with X' a normal projective and

$\mathbb{Q}$-factorial variety. If X is a Mori dream space, then every SQM of X factors as a finite sequence of flips and a regular contraction.

If X is a Mori dream space, then the effective cone $\mathrm{Eff}(X)$ is rational polyhedral, and in particular closed, see e.g. [35]. The section ring $R(X, D)$ of a divisor D on X is finitely generated and it induces a natural rational map

$$f_D : X \dashrightarrow X_D \subseteq \mathrm{Proj}(R(X, D)),$$

which is regular away from $\mathbb{B}(D)$. Two $\mathbb{Q}$-divisors D_1 and D_2 are *Mori equivalent* if f_{D_1} and f_{D_2} have the same Stein factorisation. Given an effective divisor D on X, the *Mori chamber* of D is the closure of the set of all divisors that are Mori equivalent to D. Mori chambers are convex cones and the set of such cones form a fan, the *Mori fan* of X. The maximal cones of this fan form a wall-and-chamber decomposition of the effective cone, called the *Mori chamber decomposition*. Such decomposition is a refinement of the stable base locus decomposition, see e.g. [25, 27, 35] for details.

2.5 Cones of k-moving curves

We recall the notion of k-moving curve, for $0 \leq k \leq n-1$, cf. [36].

Definition 2.2 An irreducible curve $C \subset X$ is *k-moving* if it belongs to an algebraic family of curves, whose irreducible elements cover a Zariski open subset of an effective cycle of dimension (at least) $n-k$.

We define C_k to be the cone generated by the classes of k-moving curves in $\mathrm{N}_1(X)_{\mathbb{R}}$ and we set $\mathcal{C}_k = \overline{C_k}$ to be the closure in $\mathrm{N}_1(X)_{\mathbb{R}}$.

Example 2.3 Let $p_1, \ldots, p_s \in \mathbb{P}^n$ be points in general position and let X^n_s be to the blow of $\mathbb{P}^n$ at these points. For every three distinct indices i, j, k, the strict transform on X^n_s of an irreducible conic through p_i, p_j, p_k sweeps out the strict transform of the linear span of the points and hence it is an $(n-2)$-moving curve. Similarly, the strict transform of an irreducible curve C of degree r through $r+1$ points is an $(n-r)$-moving curve since it sweeps out the strict transform of the r-dimensional linear span of the points.

It follows from the definitions that a k-moving curve class is also $(k+1)$-moving, for $0 \leq k \leq n-2$. We obtain the following filtration of cones of curves:

$$\overline{\mathrm{NE}(X)} = \mathcal{C}_{n-1} \hookleftarrow \mathcal{C}_{n-2} \hookleftarrow \cdots \hookleftarrow \mathcal{C}_1 \hookleftarrow \mathcal{C}_0. \tag{2.1}$$

Clearly $\mathcal{C}_{n-1} = \overline{\mathrm{NE}(X)}$ is the Mori cone of curves (dual to $\mathrm{Nef}(X)$). Furthermore, $\mathcal{C}_0$ is the closed cone of the moving curve classes (dual to $\overline{\mathrm{Eff}}(X)$).

2.6 Cones $\mathcal{D}_k$ of divisors that are ample in codimension k

There is a stratification of the pseudoeffective cone of divisors as follows.

Definition 2.4 Let $0 \leq k \leq n-1$. We define D_k to be the cone generated by the classes of effective divisors with no component of the stable base locus of dimension $\geq n-k$; we denote with $\mathcal{D}_k$ the closure of D_k in $\overline{\mathrm{Eff}}(X)$.

Using the same terminology as Payne [36], we call such divisors *ample in codimension k*, indeed $\mathcal{D}_k$ is also the closure of the cone of numerical classes of divisors D such that there is an open set U, with $\mathcal{O}(D)_{|U}$ ample, whose complement has codimension greater than k. We have the following obvious inclusions:

$$\mathrm{Nef}(X) = \mathcal{D}_{n-1} \hookrightarrow \mathcal{D}_{n-2} \hookrightarrow \cdots \hookrightarrow \mathcal{D}_1 \hookrightarrow \mathcal{D}_0 = \overline{\mathrm{Eff}(X)}. \tag{2.2}$$

Recall that if X is a Mori dream space, then $\mathrm{Eff}(X)$ is rational polyhedral, hence $D_0 = \mathcal{D}_0 = \mathrm{Eff}(X)$. Moreover, also $\mathcal{D}_{n-1} = \mathrm{Nef}(X)$ and $\mathcal{D}_1 = \mathrm{Mov}(X)$ are rational polyhedral. Choi posed the natural question as to whether all cones $\mathcal{D}_k$ are rational polyhedral for Mori dream spaces, see [14, Question 4.4] or [15, page 1177]. In Theorem 3.3 below (cf. Theorem 1.1), we answer this question affirmatively and we also prove that all the cones D_k are closed.

This is not true in general for non-Mori dream spaces, as the following example shows for $k = n - 1$.

Example 2.5 Let $X = \mathrm{Bl}_9(\mathbb{P}^2)$ be the blow up of the projective plane at nine general points, which is not a Mori dream space. Since the dimension of X is 2, we have $\mathcal{D}_1 = \mathrm{Nef}(X)$ and it is well-known that the nef cone is not finitely generated. Hence $\mathcal{D}_1$ is not rational polyhedral.

Now we show that $D_1 \neq \mathcal{D}_1$. Indeed the anticanonical curve F, which is the strict transform on X of a plane cubic through the nine points, is nef, that is $F \in \mathcal{D}_1$. In fact the Mori cone of X is generated by the (-1)-curves and by F, and we have $F^2 = 0$ and $F \cdot E = 1$ for every (-1)-curve E, see [34].

However $F \notin D_1$. In fact, since there is a unique plane cubic through nine points, the linear system $|F|$ has dimension 0. Even more: $|mF| = \{mF\}$, for every integer $m \geq 1$, and this goes back to the classical work of Castelnuovo [12]. This implies that F is the stable base locus of its own linear system $|F|$.

3 The geometry of the cones $\mathcal{D}_k$

In this section we show that the stable base locus decomposition of the pseudo-effective cone of a normal $\mathbb{Q}$-factorial projective variety X with irrationality zero is compatible with the filtration of cones (2.2), that is, every cone $\mathcal{D}_k$ decomposes as a union of stable base locus chambers. Moreover if X is a Mori dream space, since each stable base locus chamber is a union of Mori chambers, we obtain that $\mathcal{D}_k$ is a union of such chambers. Furthermore, it is possible to characterise the set of Mori chambers whose union is $\mathcal{D}_k$, for every $1 \leq k \leq n-1$.

Lemma 3.1 *Let X be a normal $\mathbb{Q}$-factorial projective variety and assume $h^1(X, \mathcal{O}_X) = 0$. Then, for every k, the cone D_k is a union of stable base locus chambers.*

Proof We shall prove that the stable base locus chamber decomposition of the pseudo-effective cone of X induces a chamber decomposition on each cone D_k, by showing that every stable base locus chamber is contained in the difference set $D_{k-1} \setminus D_k$, for some $1 \leq k \leq n - 1$.

In fact, let B be a stable base locus chamber, and let D be an effective divisor that lies in B. Let $n - k$ be the dimension of $\mathbb{B}(D)$ or, equivalently, the maximum dimension of the components of $\mathbb{B}(D)$. The inequality $\dim \mathbb{B}(D) \geq n - k$ implies that D does not lie in D_k, while since $\dim \mathbb{B}(D) \leq n - k$, D lies in D_{k-1}, that is, $D \in D_{k-1} \setminus D_k$. Since B is a stable base locus chamber, by definition the same holds for any other divisor in B. Thus, $B \subset D_{k-1} \setminus D_k$. □

In the next theorem, we propose a detailed description of the Mori chamber decomposition of the cones $\mathcal{D}_k$ for Mori dream spaces.

Notation 3.2 *We set $F_1 := \{Id : X \to X\}$. Let $2 \le r \le n-1$ and let F_r be the set of all small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ given by sequences of flips of subvarieties whose birational preimage on X has dimension $< r$.*

Theorem 3.3 *Let X be a Mori dream space and fix $1 \le k \le n-1$.*

(1) *The cone $\mathcal{D}_k$ is rational polyhedral and the following holds:*

$$\mathcal{D}_k = \bigcup_{f_i \in F_{n-k}} f_i^*(\mathrm{Nef}(X_i)).$$

(2) *The cone D_k is closed, that is: $D_k = \mathcal{D}_k$.*

Proof Recall that, for a Mori dream space, every closed stable base locus chamber is a finite union of Mori chambers, see Sect. 2.4. By Lemma 3.1, D_k is a union of stable base locus chambers and hence its closure $\mathcal{D}_k$ is a finite union of Mori chambers. We shall now characterise which are the Mori chambers appearing in the union as follows. Let $f_i : X \dashrightarrow X_i$ be a small $\mathbb{Q}$-factorial modification of X and let $f_i^*(\mathrm{Nef}(X_i))$ be the corresponding Mori chamber. Let $D \in f_i^*(\mathrm{Nef}(X_i))$ be any pull-back of an ample class on X_i: we have that D is movable on X. The stable base locus of D equals the indeterminacy locus of f_i (see e.g. [25, 35]). Hence, the latter is a union of subvarieties of X of dimension $< n-k$ if and only if D lies in D_k. This argument is independent of D but rather depends on its base locus, and it applies to any pull-back on X of an ample divisors on X_i. Therefore, the interior of the Mori chamber $f_i^*(\mathrm{Nef}(X_i))$ is contained in D_k. Finally, taking closures, we have that $f_i^*(\mathrm{Nef}(X_i)) \subseteq \mathcal{D}_k$ if any only if $f_i \in F_{n-k}$.

Since X is a Mori dream space, there are finitely many Mori chambers and each is a rational polyhedral cone. Therefore the above argument shows that for every k, the convex cone $\mathcal{D}_k$ is a union of finitely many rational polyhedral cones. Since $\mathcal{D}_k$ is a cone, it is a rational polyhedral cone. This proves (1).

We now prove (2), i.e. that $\mathcal{D}_k = D_k$. In order to do so, we show that every extremal ray of $\mathcal{D}_k$ lies in D_k. Let E be an effective divisor on X spanning an extremal ray. Then E spans an extremal ray of some Mori chamber $f_j^*(\mathrm{Nef}(X_j)) \subseteq \mathcal{D}_k$ and hence the pull-back of a semi-ample divisor E_j on X_j, where $f_j : X \dashrightarrow X_j$, with $f_j \in F_{n-k}$, by (1). Let $m \ge 1$ be an integer such that mE_j is basepoint free. The stable base locus of $mE = f_j^*(mE_j)$ is therefore the indeterminacy locus of f_j which is a union of subvarieties of X of dimension strictly less than $n-k$. Since $\mathbb{B}(E) = \mathbb{B}(mE)$, we conclude that E lies in D_k. □

Remark 3.4 For $k = 1$, Theorem 3.3 is just property (3) of the definition of Mori dream spaces. For $k = n-1$ it is always true by Kleiman's theorem.

For non-Mori dream spaces, none of the statements of Theorem 3.3 hold in general; in fact the properties may hold or fail as we vary k. The following is an instance of this.

Example 3.5 Let $X = X_9^5$ be the blow-up of $\mathbb{P}^5$ at nine points in general position. Recall that X is not a Mori dream space, in particular $\mathcal{D}_0 = \overline{\mathrm{Eff}(X)}$ is not finitely generated. On the other hand, the Mori cone $\mathcal{C}_4$ is finitely generated by lines, see e.g. [22, Theorem 3.2], hence its dual, $\mathrm{Nef}(X)$, is rational polyhedral. Moreover the following holds: $\mathcal{D}_4 = D_4 = \mathrm{Nef}(X)$. Indeed, the nef cone $\mathrm{Nef}(X)$ is generated by the following classes: H, $H - E_i$, $2H - \sum_{i \in I} E_i$, for every $3 \le |I| \le 9$ as one can easily check with the help of any computer software that can compute cone duals, such as *Magma* [4]. It is an easy exercise to show that each of them is basepoint free, hence it lies in D_4.

It would be interesting to determine which, if any, of the intermediate cones $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are finitely generated.

4 Weak duality

Duality of the cones $\mathcal{D}_k$ and $\mathcal{C}_k$ is not to be expected in general, but a weaker version for Mori dream spaces holds, that takes into account k-moving curves not just on X but also on its small modifications. We will refer to this as *weak duality*, as opposed to *strong duality* that holds whenever $\mathcal{C}_k = \mathcal{D}_k^\vee$, for every k.

It is easy to show that $\mathrm{Mov}(X) \subseteq \mathcal{C}_1^\vee$. In fact if D is in the dual of $\mathcal{C}_1$ then $D \cdot C \geq 0$ for every 1-moving curve C. If it is not in the dual, then there is a 1-moving curve C such that $D \cdot C < 0$, and so the divisor ($(n-1)$-cycle) spanned by the algebraic family of C is a fixed component of the base locus of D. If $n = 2$, then it holds that $\mathrm{Mov}(X) = \mathcal{C}_1^\vee$. In fact the first cone is the nef cone, the second cone is the Mori cone.

A similar inclusion of cones is valid in general.

Remark 4.1 If $C \in \mathcal{C}_k$, we know that C moves in a family that sweeps out an effective cycle V of dimension $n-k$. If $C \notin \mathcal{D}_k^\vee$, then there exists an effective divisor D such that $C \cdot D < 0$ and whose stable base locus does not have any component of dimension $\geq n-k$. But then V is in the stable base locus of D which is a contradiction. This proves that

$$\mathcal{C}_k \subseteq \mathcal{D}_k^\vee.$$

We adopt the definition of k-moving curves on small modifications proposed by Payne in [36].

Definition 4.2 We say that a curve class $c \in \mathrm{N}_1(X)_\mathbb{R}$ is *bir-k-moving* on X if there exists a small modification $f_i : X \dashrightarrow X_i$, such that there is a curve $C \subset X_i$, in the class corresponding to c, that moves in a family which sweeps out a cycle $f_i(V)$ birational to a cycle $V \subset X$ of codimension k.

We denote by $\mathcal{C}_k^{bir}$ the closure of the cone generated in $\mathrm{N}_1(X)_\mathbb{R}$ by all the bir-k-moving classes on X.

In [16], a version of the cone theorem for k-moving curves on small modifications of X is discussed, in the setting of the minimal model program.

Applying the same idea as in Remark 4.1, one can easily see that the inclusion $\mathcal{C}_k^{bir} \subseteq \mathcal{D}_k^\vee$ holds. Using the results of Sect. 3, we prove that the latter inequality holds with equality, hence recovering a result that was first proved by Payne for complete $\mathbb{Q}$-factorial toric varieties [36] and later generalised by Choi for log Fano varieties and for Mori dream spaces [14, Corollary 4.3], running the log minimal model program.

The main ingredient of our proof is the Mori chamber decomposition of the effective cone of X, and of $\mathcal{D}_k$, proved in Theorem 3.3.

Theorem 4.3 (Weak duality theorem for Mori dream spaces) *If X is a Mori dream space then weak duality holds, that is*

$$\mathcal{C}_k^{bir} = \mathcal{D}_k^\vee.$$

Proof We only have to prove that the following inclusion holds: $\mathcal{D}_k^\vee \subseteq \mathcal{C}_k^{bir}$. Every facet $\mathcal{F}$ of $\mathcal{D}_k$ corresponds, by duality, to an extremal ray of $\mathcal{D}_k^\vee$, spanned by a curve class $c = c(\mathcal{F}) \in \overline{\mathrm{NE}(X)}$. We shall prove that $c \in \mathcal{C}_k^{bir}$. We consider the following cases separately.

Assume first that $\mathcal{F}$ is a facet of $\mathcal{D}_0 = \overline{\mathrm{Eff}(X)}$. Then c is a moving curve class and the statement follows.

Now, let $\mathcal{F}$ be a facet of $\mathcal{D}_k$ but not a facet of $\mathcal{D}_0$. Then there is a Mori chamber inside $\mathcal{D}_k$ and adjacent to $\mathcal{F}$, that means that a facet of the chamber is contained or is equal to $\mathcal{F}$.

Let $f_-^*(\mathrm{Nef}(X_-)) \subset \mathcal{D}_k$ be one such chamber, for $f_- : X \dashrightarrow X_-$ a small modification of X. It follows that $c \in \mathrm{N}_1(X)_\mathbb{R} \cong \mathrm{N}_1(X_-)_\mathbb{R}$ generates an extremal ray of the Mori cone of X_-. Let $V_- \subset X_-$ be a subvariety of X_- of maximal dimension swept out by a family of irreducible curves of class c.

Now, let $f_+^*(\mathrm{Nef}(X_+)) * \mathrm{ex}(f_+)$ be a Mori chamber outside $\mathcal{D}_k$, see [25, Proposition 1.11], and adjacent to $\mathcal{F}$, that has a common facet with $f_-^*(\mathrm{Nef}(X_-))$ contained in $\mathcal{F}$. The contracting birational map $f_+ : X \dashrightarrow X_+$ can be decomposed as $f_+ = g \circ f_-$, where $g : X_- \dashrightarrow X_+$ corresponds to crossing the wall $\mathcal{F}$, and the indeterminacy locus of g is V_-. Take $D \in f_+^*(\mathrm{Nef}(X^+)) * \mathrm{ex}(f_+)$ a divisor on X whose ray lies in the relative interior of the Mori chamber. Since $D \notin \mathcal{D}_k$, then D must have a component of its stable base locus of codimension $\leq k$. But the stable base locus of D, as a divisor on X, is the indeterminacy locus of f_+, which is the union of the indeterminacy locus of f_- and $V := f_-^{-1}(V_-)$. Since by Theorem 3.3, the indeterminacy locus of f_- has codimension $> k$, then V must have codimension $p \leq k$. In particular, V is not in the indeterminacy locus of f_- and therefore V and V_- are birational. We conclude that $c \in \mathcal{C}_p^{bir} \subseteq \mathcal{C}_k^{bir}$. □

4.1 Losev–Manin's moduli 3-space

Strong duality is a rare property even for toric varieties, and it is not preserved under small modifications for dimension three or higher. An explicit example of a toric threefold for which strong duality does not hold was proposed by Payne in [36, Example 1].

In this section we shall discuss another interesting smooth threefold for which strong duality holds, i.e. such that $\mathcal{D}_1^\vee = \mathcal{C}_1$, but for which it fails, i.e. $\mathcal{D}_1^\vee \supsetneq \mathcal{C}_1$, as soon as we flop one or more curves. Let X be the 3-dimensional Losev–Manin moduli space introduced in [31] as a toric compactification of the moduli space of rational curves with six marked points. It can be interpreted as the blow-up of $\mathbb{P}^3$ at its four coordinate points and, subsequently, along the strict transforms of the six coordinate lines. We denote by H the class of the pull-back on X of a hyperplane of $\mathbb{P}^3$, by E_i, $i = 1, \dots, 4$, the strict transforms of the exceptional divisors of the points and by E_{ij}, $1 \leq i < j \leq 4$, the exceptional divisors of the lines, which together generate the Picard group of X. Similarly, we denote by $h = H^2$, $e_i = -E_i^2$, $e_{ij} = HE_{ij} = E_iE_{ij}$ the classes of a general line, a general line in E_i and a vertical fibre of E_{ij} respectively: they generate the Chow group of 1-cycles. The matrix of the intersection pairing $\mathrm{N}^1(X)_\mathbb{R} \times \mathrm{N}_1(X)_\mathbb{R} \to \mathbb{R}$, with respect to the above bases, is diagonal and defined by:

$$H \cdot h = 1, \;\; E_i \cdot e_i = E_{ij} \cdot e_{ij} = -1.$$

We denote by $E_{ijk} = H - E_i - E_j - E_k - E_{ij} - E_{ik} - E_{jk}$ the strict transform of a fixed hyperplane of $\mathbb{P}^3$ spanned by three coordinate points. The following divisors generate the extremal rays of the effective cone $\mathcal{D}_0$ of X:

$$E_i, E_{ij}, E_{ijk},$$

for all distinct i, j, k. Since the above classes correspond to the torus invariant prime divisors of X, they give rise to a set of generators of the Cox ring (see [2, Section 2.1.3]).

Now, using [2, Prop. 3.3.2.3], we can describe the movable cone of X via its defining inequalities. In other terms, one can show that the extremal rays of the dual cone $\mathcal{D}_1^\vee$ are generated by the following effective curve classes, up to permutations of indices:

- curves sweeping out E_i: $e_i - e_{ij}$, and $2e_i - e_{ij} - e_{ik} - e_{il}$,
- curves sweeping out E_{ij}: e_{ij} and $h - e_i - e_j + e_{ij}$,
- curves sweeping out E_{ijk}: $h - e_i - e_{jk}$ and $h - e_{ij} - e_{ik} - e_{jk}$.

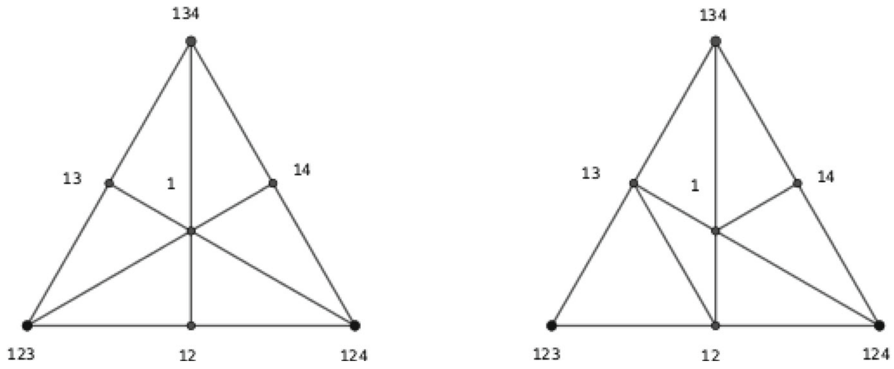

Fig. 1 A section of a portion of the fan Σ_X and of Σ_Y

This computation can be checked using any computer software that can compute cone duals, such as *Magma* [4]. While we are guaranteed that the effective cones $\mathcal{D}_0$ and the Mori cone $\mathcal{C}_0$ are dual to each other for all small $\mathbb{Q}$- factorial modifications of X, the same need not happen for the movable cone $\mathcal{D}_1$ and the cone of 1-moving curves $\mathcal{C}_1$; whether or not it does, it depends on the model. In fact we will now discuss a small modification of X for which $\mathcal{D}_1$ and $\mathcal{C}_1$ are not dual cones.

In order to do this, it is convenient to describe X, and its small modifications, using the language of toric varieties. For a general reference on toric varieties, see [17] or [24]. Since X is obtained from $\mathbb{P}^3$ by the iterated blow-up of torus invariant subvarieties, then X is toric. Let $N \cong \mathbb{Z}^3$ be a three-dimensional lattice; the rays of Σ_X are generated by the following primitive vectors:

$$\begin{aligned}
&v_1 = (1, 1, 1), v_2 = (0, 0, -1), v_3 = (0, -1, 0), v_4 = (-1, 0, 0),\\
&v_{12} = (1, 1, 0), v_{13} = (1, 0, 1), v_{14} = (0, 1, 1),\\
&v_{23} = (0, -1, -1), v_{24} = (-1, 0, -1), v_{34} = (-1, -1, 0),\\
&v_{123} = (1, 0, 0), v_{124} = (0, 1, 0), v_{134} = (0, 0, 1), v_{234} = (-1, -1, -1).
\end{aligned}$$

The ray generated by the vector v_I, which we will denote with ρ_I, corresponds to the boundary divisor E_I, for every index set $I \subset \{1, \dots, 4\}$ of cardinality $1 \leq |I| \leq 3$. The set of maximal cones, which are all simplicial, is the following:

$$\mathrm{Cone}(v_i, v_{ij}, v_{ijk}), \forall\{i\} \subset \{i, j\} \subset \{i, j, k\} \subset \{1, \dots, 4\}.$$

The fan of X is dual to a 3-dimensional permutohedron, which is a polytope with 8 hexagonal faces (corresponding to the boundary divisors E_{ijk} and E_i) and six quadrilateral faces (corresponding to the E_{ij}'s). Figure 1 displays a section of a portion of the fan of X lying in the plane of $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^3$ of equation $x + y + z = a$, with $a > 0$. Consider the cone $\mathrm{Cone}(v_1, v_{123})$, which is a common two dimensional face of $\mathrm{Cone}(v_1, v_{12}, v_{123})$ and $\mathrm{Cone}(v_1, v_{13}, v_{123})$. The corresponding invariant subvariety is a floppable curve whose numerical class is $e_1 - e_{12} - e_{13}$ and which corresponds to a fixed line inside E_1. Let $X \dashrightarrow Y$ be the flop of such curve. The fan Σ_Y differs from Σ_X only within the quadrilateral cone generated by $v_1, v_{12}, v_{13}, v_{123}$ and, in particular, the image $-e_1 + e_{12} + e_{13}$ corresponds to the cone $\mathrm{Cone}(v_{12}, v_{13}) \in \Sigma_Y$.

We argue that on Y strong duality does not hold. Consider the class $\gamma = e_1 - e_{14}$ which is 1-moving on X. The intersections $\gamma \cdot E_1 = -1$ and $\gamma \cdot E_{123} = 1$ imply that every representative of γ is contained in E_1 and it intersects the hyperplane E_{123}, where by abuse of notation we use the same symbols for cycles on X and their birational images on Y. Since the two divisors are disjoint in Y, no irreducible curve with class γ exists on Y. In particular $\gamma \notin \mathcal{C}_1(Y)$; thus strong duality does not hold for Y, that is $\mathcal{C}_1(Y) \subsetneq \mathcal{D}_1^\vee$.

5 Strong duality for blow ups of $\mathbb{P}^n$

In this section we consider $X = X_s^n$, i.e. the blow-up of $\mathbb{P}^n$ at s points in general position, for the pairs (n, s) for which X is a Mori dream space. These varieties are classified in [13, 32, 33] and they fit in the following list:

- X_s^2 for $s \leq 8$;
- X_s^3 for $s \leq 7$;
- X_s^4 for $s \leq 8$;
- X_s^n for $n \geq 5$ and $s \leq n + 3$.

The main result of this section is the proof of Theorem 1.2. More precisely, we show that the cone of k-moving curves is dual to the cone of divisors that are ample in codimension k, for every Mori dream X_s^n. In the surface case $n = 2$ there is nothing to prove. The case X_s^n with $s \leq n + 3$ and $n \geq 3$, will be treated in Sect. 5.1, while X_7^3 and X_8^4 will be considered in Sect. 5.2.

5.1 Blow up of $\mathbb{P}^n$ in $s \leq n + 3$ general points

Let $X = X_s^n$ be the blow up of $\mathbb{P}^n$ at $s \leq n + 3$ general points and assume that $n \geq 3$. We recall some notation from [6]. For any subset $I \subset \{1, \ldots, s\}$ we denote by L_I the strict transform of the linear span of the points indexed by I. For $s = n + 3$, we denote by C the strict transform of the unique rational normal curve of degree n passing through all points and by σ_t the strict transform of its t-secant variety, where we set $\sigma_1 := C$. Let $J(L_I, \sigma_t)$ denote the strict transform of the join of the linear span of the points indexed by I and the t-secant variety of the rational normal curve. We make the following identifications: $J(L_I, \sigma_t) = \emptyset$ if $t = |I| = 0$, $J(L_I, \sigma_t) = E_i$ if $t = 0$ and $I = \{i\}$, $J(L_I, \sigma_t) = L_I$ if $t = 0$ and $|I| > 1$, and $J(L_I, \sigma_t) = \sigma_t$ if $|I| = 0$. It is easy to see that the join $J(L_I, \sigma_t)$ has dimension $2t + |I| - 1$, unless $J(L_I, \sigma_t) = E_i$.

For $s \leq n + 3$, the inequalities cutting out the effective cone of X_s^n were obtained in [6] and a formula for the dimension of all linear systems of divisors on X_s^n was proved in [28]. Moreover, by[18, Proposition 4.2] and [6, Lemma 4.1], if H denotes the general hyperplane class in X_s^n and $E_1, \ldots, E_{n+3}$ denote the exceptional divisors, then for an effective divisor

$$D = dH - \sum_{i=1}^{n+3} m_i E_i, \tag{5.1}$$

the following integer

$$\kappa_{J(L_I, \sigma_t)} = -(|I| + (n + 1)t - 1)d + \sum_{i \in I} (t + 1)m_i + \sum_{i \notin I} t m_i, \tag{5.2}$$

when positive, is the exact multiplicity of containment of the join $J(L_I, \sigma_t)$ in the base locus of $|D|$. Notice that since equation (5.2) is linear in d and in the m_i's, then if the join $J(L_I, \sigma_t)$ is in the base locus of a linear system, it is a component of its stable base locus.

Formulas (5.2) induce the Mori chamber decomposition of the effective cone of X^n_s, already studied by Mukai. This turns out to coincide with its stable base locus decomposition. This is the content of the following result that already appeared in [10, Theorem 2.60], but that we include here for the sake of completeness.

Theorem 5.1 *For $X = X^n_{n+3}$, the Mori chamber decomposition and the stable base locus decomposition of the effective cone coincide and they are induced by the following hyperplane arrangement*

$$\{\kappa_{J(L_I,\sigma_t)} = 0 : 0 \le t \le \frac{n}{2}, 0 \le |I| \le n-2t\}, \tag{5.3}$$

where $\kappa_{J(L_I,\sigma_t)}$ is defined in (5.2).

Proof Notice that the hyperplane arrangement (5.3) is in correspondence with the set of subvarieties

$$\{J(L_I,\sigma_t) : 0 \le t \le \frac{n}{2}, 0 \le |I| \le n-2t\}. \tag{5.4}$$

More precisely, (5.3) induces a chamber decomposition of the effective cone of divisors of X, such that the divisors in the interior of any given chamber contain the same joins in the stable base locus. Such decomposition is refined by the stable base locus decomposition of the effective cone, because a priori effective divisors might have stable base locus that is not of the form $J(L_I,\sigma_t)$. This turns out not to be the case for X^n_{n+3}.

As explained in [1, Section 4.3], it follows from the work of Mukai ([33]) that the Mori chamber decomposition of the effective cone of X is given by (5.3). Since the Mori chamber decomposition refines the stable base locus decomposition, which is a further refinement of (5.3), this concludes the proof. □

Recall that for a Mori dream space we have $\mathcal{D}_k = D_k$, by Theorem 3.3. Using Theorem 5.1, we can describe the cones D_k of X^n_s $s \le n+3$, with their inequalities.

Proposition 5.2 *For any integer k with $1 \le k \le n-1$. The cone $\mathcal{D}_k$ of divisors on X^n_{n+3} that are ample in codimension k is cut out by the following inequalities:*

(I) $0 \le d$ *and* $m_i \le d$, *for every* $i = 1,\dots,n+3$*;*
(II) $-m_i + \sum_{j=1}^{n+3} m_j \le nd$, *for every* $i = 1,\dots,n+3$*;*
(III$_k$) $\kappa_{J(L_I,\sigma_t)} \le 0$, *for every* $0 \le t \le n/2$ *and* $|I| = n-2t-k+1$*;*
(IV) $0 \le m_i$, *for every* $i = 1,\dots,n+3$.

Proof For $k=1$ the statement is proved in [6, Theorem 5.3].

We proceed by induction on k. We want to prove that $\mathcal{D}_k$ is defined by the inequalities (I), (II), (III$_k$) and (IV), for $k \ge 2$. We know that $\mathcal{D}_k \subseteq \mathcal{D}_{k-1}$. Hence, by the induction hypothesis, we have that $\mathcal{D}_k$ is contained in the cone cut out by the inequalities (I), (II), (III$_{k-1}$) and (IV). Notice that the only inequalities to be added to the list are the ones excluding divisors with stable base locus of dimension exactly $n-k$. By Theorem 5.1, these are given by $\kappa_{J(L_I,\sigma_t)} \le 0$ for all joins $J(L_I,\sigma_t)$ of dimension $n-k$. Recalling that the dimension of any such join is $|I|+2t-1$, we get that these inequalities are precisely the ones of (III$_k$). We conclude by noticing that, by definition of $\kappa_{J(L_I,\sigma_t)}$ and the effectivity conditions (I) and (II), the inequalities (III$_k$) imply the ones of (III$_{k-1}$), so they become redundant and can be discarded. □

Remark 5.3 The effective cone $\mathcal{D}_0$ of X^n_{n+3} is described in [6, Theorem 5.1]. The defining inequalities in this case are only (I), (II), (III$_{k=0}$).

Corollary 5.4 *For $X = X^n_s$ with $1 \leq s \leq n+2$, the cone $\mathcal{D}_k$ is defined by the inequalities from Proposition 5.2 as follows:*

(1) *if $s = n+2$, by the inequalities (I), (II), (III)$_k$ (with $t = 0$) and (IV);*
(2) *if $s \leq n+1$, then*

   (a) *if $k \geq n-s+1$, by the inequalities (I), (III)$_k$ (with $t = 0$) and (IV),*
   (b) *if $k < n-s+1$, then we have $\mathcal{D}_k = \mathcal{D}_{k+1}$.*

Proof The statement follows from Proposition 5.2 by using the natural isomorphism between the Néron–Severi space of $X = X^n_s$ and the linear subspace of $\mathrm{N}^1(X^n_{n+3})$ defined by the equations $m_i = 0$, for $s+1 \leq i \leq n+3$. □

With a description of the cones $\mathcal{D}_k$ in X^n_s via their inequalities at hand, it is immediate to compute their dual cones with respect to the standard intersection pairing. Recall that $\mathrm{N}_1(X^n_s)$ is generated by the class of a general line in X^n_s, which we denote by h, along with the classes e_i, for every $i = 1, \ldots, n+3$, of a general line in the exceptional divisor E_i. Thus, the intersection product of a divisor $D \in \mathrm{N}^1(X^n_s)_{\mathbb{R}}$ of the form (5.1) and a curve of class

$$c = \delta h - \sum_{i=1}^{s} \mu_i e_i \in \mathrm{N}_1(X^n_s)_{\mathbb{R}} \tag{5.5}$$

is computed by

$$D \cdot c = d\delta - \sum m_i \mu_i. \tag{5.6}$$

In particular, to every hyperplane in $\mathrm{N}^1(X^n_s)_{\mathbb{R}}$, there corresponds a curve class in $\mathrm{N}_1(X^n_s)_{\mathbb{R}}$.

We will show that for X^n_s, with $s \leq n+3$, the dual of $\mathcal{D}_k$ is precisely $\mathcal{C}_k$ for every $0 \leq k \leq n-1$, that is: strong duality holds for every k. We need to introduce another ingredient before we can state and prove the result: the Weyl group action on $\mathrm{N}_1(X^n_s)_{\mathbb{R}}$, which is described in detail in [20, Proposition 2.5]. For an index set $\Gamma \subseteq \{1, \ldots, s\}$ of length $n+1$, the standard Cremona transformation $\mathrm{Cr}_\Gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ based at the points of $\mathbb{P}^n$ parametrized by Γ, lifts to an isomorphism of $\mathrm{N}_1(X^n_s)_{\mathbb{R}}$ sending the curve class $c \in \mathrm{N}_1(X^n_s)_{\mathbb{R}}$ of the form (5.5) to the following curve class:

$$\mathrm{Cr}_\Gamma(c) = (\delta - (n-1)a_\Gamma)h - \sum_{j \in \Gamma} (\mu_j - a_\Gamma)e_j - \sum_{i \notin \Gamma} \mu_i e_i, \tag{5.7}$$

where we set

$$a_\Gamma := \sum_{j \in \Gamma} \mu_j - \delta.$$

A composition of Cremona transformations is called a Weyl transformation and we say that two curve classes c and c' are in the same Weyl orbit if there is a Weyl transformation ϕ such that $\phi(c) = c'$.

We are now in position to prove the following theorem.

Theorem 5.5 (Strong duality theorem for X^n_{n+3}) *Let $1 \leq s \leq n+3$ and $X = X^n_s$, then*

$$\mathcal{C}_k = \mathcal{D}_k^\vee.$$

Proof Using the intersection pairing (5.6), for every k, the inequalities of Proposition 5.2 correspond to the following effective curve classes which generate the dual cone $\mathcal{D}_k^\vee$:

(I) h and $h - e_i$, for every $i = 1, \ldots, n+3$;

(II) $nh - \sum_{j=1}^{n+3} e_j + e_i$, for every $i = 1, \dots, n+3$;
(III_k) $(|I| + (n+1)t - 1)h - \sum_{i \in I}(t+1)e_i - \sum_{i \notin I} te_i$, for every $0 \leq t \leq n/2$ and $|I| = n - 2t - k + 1$;
(IV) e_i, for every $i = 1, \dots, n+3$.

Since the inclusion $\mathcal{C}_k \subseteq \mathcal{D}_k^\vee$ is always satisfied, in order to prove the strong duality statement, we only need to show that each one of the above curve classes belongs to $\mathcal{C}_k$.

From now on we will assume that $s = n+3$. For smaller s the proof is similar and uses Corollary 5.4, so we leave the details to the reader.

First of all notice that the curves classes (I) and (II) are dual to codimension one faces of the effective cone, by Remark 5.3. Hence, by duality, they are in $\mathcal{C}_0$ which is contained in $\mathcal{C}_k$ for every k.

The curves in the classes (IV) sweep out the exceptional divisors, so they belong to $\mathcal{C}_1$, which is contained in $\mathcal{C}_k$ for every $1 \leq k \leq n-1$.

It remains to deal with the curve classes (III_k). Given a pair (t, I) satisfying the bounds in (III_k), the curve class

$$c_{I,t} = (|I| + (n+1)t - 1)h - \sum_{i \in I}(t+1)e_i - \sum_{i \notin I} te_i \tag{5.8}$$

is dual to the wall $\kappa_{J(L_I, \sigma_t)} = 0$. This wall separates the divisors which do not contain the join $J(L_I, \sigma_t)$ in their stable base locus from those that do. By Theorem 4.3, we have that there is a family of curves of class $c_{I,t}$ that sweep out a subvariety of some small modification of X that is birational to $J(L_I, \sigma_t) \subset X$. We claim that such family lives on X and it sweeps out $J(L_I, \sigma_t)$. We now prove the claim. Assume that $t = 0$. In this case the class of $c_{I,0}$ reads as follows:

$$(n-k)h - \sum_{|I|=n-k+1} e_i. \tag{5.9}$$

This curve class has irreducible representatives which sweep out the linear cycle L_I, see Example 2.3.

Now, for any $t > 0$ we show that the curve class (5.8) is in the Weyl orbit of a curve class of type (5.9). Assume the statement is true for any pair $(t-1, I')$ satisfying the conditions in (III_k), and let $c_{I,t}$ be a curve class for a pair (t, I) as in (5.8). After relabelling the indexes if necessary, we may assume that $I = \{1, \dots, r_t\}$ where $r_t = n - 2t - k + 1$. Now take $\Gamma = \{1, \dots, r_t\} \cup \{r_t + 3, \dots, n+3\}$, we have that

$$\mathrm{Cr}_\Gamma(c_{I,t}) = (r_t + 2 + (n+1)(t-1) - 1)h - \sum_{i=1}^{r_t+2} te_i - \sum_{i=r_t+3}^{n+3} (t-1)e_i,$$

which is a curve class as in (III_k) for the pair $(t-1, I' = I \cup \{r_t+1, r_t+2\})$. By the induction hypothesis, there is a Weyl transformation ϕ' taking $\mathrm{Cr}_\Gamma(c_{I,t})$ to a curve with class as in (5.9). Thus, taking $\phi = \phi' \circ \mathrm{Cr}_\Gamma$, we see that $c_{I,t}$ is in the Weyl orbit of some curve class with irreducible representatives sweeping out a linear cycle of dimension equal to the dimension of $J(L_I, \sigma_t)$. This proves that $c_{I,t}$ has irreducible representatives on X, which are the images of the irreducible representatives of (5.9), and they sweep out $J(L_I, \sigma_t)$ on X. Hence, c belongs to $\mathcal{C}_k$. □

Remark 5.6 We recall that an (i)-curve class on a smooth n-dimensional variety X is defined as a smooth irreducible rational curve with normal bundle isomorphic to $\oplus_{i=1}^{n-1}\mathcal{O}(i)$, for $i \in \{-1, 0, 1\}$, see [21] and [20, Definition 1.1]. The extremal rays of $\mathcal{C}_0$ are examples of

(1)-curves and (0)-curves. The extremal rays of the Mori cone $\mathcal{C}_{n-1}$ of type $h - e_i - e_j$ and (for $s = n + 3$) the rational normal curve class C are all the (-1)-curves of X^n_s.

5.2 Blow up of $\mathbb{P}^4$ in 8 points and of $\mathbb{P}^3$ in 7 points

We now prove strong duality for the two remaining cases: X^3_7 and X^4_8.

We first recall the definition of Weyl cycles in X^n_s introduced in [7]. A *Weyl divisor* is an effective divisor $D \in \mathrm{Pic}(X^n_s)$ which belongs to the Weyl orbit of an exceptional divisor E_i. A *Weyl cycle* of codimension r is an irreducible component of the intersection of pairwise orthogonal Weyl divisors, where orthogonality is taken with respect to the Dolgachev–Mukai pairing.

Theorem 5.7 *For $X = X^3_7$ or $X = X^4_8$ the Mori chamber decomposition and the stable base locus decomposition coincide. In particular they are induced by the Weyl cycles of X.*

Proof The proof is similar to that of Theorem 5.1 and it uses two ingredients: on the one hand the classification of Weyl cycles and their corresponding base locus lemmata done in [7]; on the other hand the description of the Mori chamber decomposition. For the threefold X^3_7 the latter is easy to obtain and left to the reader (there are only floppable curves and contracting divisors). On the other hand, obtaining the Mori chamber decomposition for the fourfold X^4_8 is not trivial: it was done by Mukai [33] and by Casagrande, Codogni and Fanelli [11] by means of the Gale duality between X^4_8 and the del Pezzo surface $S := X^2_8$, a projective plane blown up in eight general points.

In [7] (see also [19] for an alternative approach) the Weyl cycles on X^3_7 and X^4_8 are classified and explicitly described. For X^3_7 these are (-1)-curves and Weyl divisors, (see [7, Section 4]), for X^4_8 we further have five types of Weyl surfaces (see [7, Section 5] for details): they are the strict transforms of the following surfaces of $\mathbb{P}^4$:

- $S^1_{i,j,k}$ a plane through three points,
- $S^3_{i,\widehat{j}} = J(C_{\widehat{j}}, p_i)$, a pointed cone over the rational normal curve passing through seven points $p_1, \dots, \widehat{p_j}, \dots, p_8$,
- $S^6_{i,j,k}$ a sextic surface with five triple points and simple at three points,
- $S^{10}_{i,j}$ a degree-10 surface with two sextuple points and five triple points,
- S^{15}_i a degree-15 surface with seven sextuple points and a triple point.

Associating to each of the above Weyl cycles a hyperplane in the Néron–Severi space, we define a hyperplane arrangement which induces a chamber decomposition (that we may call *Weyl chamber decomposition*) of the effective cone of divisors of X, such that the divisors in any chamber contain the same Weyl cycles in their stable base locus.

More precisely: the hyperplane associated to a Weil curve C is defined by the equation $D \cdot C = 0$; the hyperplane associated to a Weyl divisor E is defined by $\langle D, E \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the Dolgachev–Mukai pairing. The hyperplanes associated to the five surfaces have equations $D \cdot \mu = 0$, where μ is one of the following curve classes:

- $\mu^1_{i,j,k} = 2h - e_i - e_j - e_k$,
- $\mu^3_{i,\widehat{j}} = 5h - \sum_{k=1}^8 e_k - e_i + e_j$,
- $\mu^6_{i,j,k} = 8h - 2\sum_{l=1}^8 e_l + e_i + e_j + e_k$,
- $\mu^{10}_{i,j} = 11h - 2\sum_{k=1}^8 e_k - e_i - e_j$,
- $\mu^{15}_i = 14h - 3\sum_{j=1}^8 e_j + e_i$.

Now, the Weyl chamber decomposition is refined by the stable base locus decomposition, since effective divisors may have stable base locus which is not of Weyl type. Moreover, recall that the stable base locus decomposition is, in turn, refined by the Mori chamber decomposition, described in [11, Theorem 5.13]. Therefore, since the Weyl and Mori decompositions agree, we conclude. □

Theorem 5.8 *Let* $X = X_7^3$ *or* $X = X_8^4$*, then*

$$\mathcal{C}_k = \mathcal{D}_k^\vee.$$

Proof Following the proof of Theorem 5.5, one can prove the duality statement for $\mathcal{C}_1$ and $\mathcal{C}_2$. Since for the movable cone $\mathcal{D}_1$ the statement is easy, we just give a hint of the proof for the case of $\mathcal{D}_2(X_8^4)$.

Using the notation of the proof of Theorem 5.7, one first observes that $\mu^1_{1,2,3}$ belongs to $\mathcal{C}_2(X_8^4)$ (see also Example 2.3) and then verifies that the remaining four curve types $\mu^1_{1,2,3}$, $\mu^3_{1,\widehat{8}}$, $\mu^6_{1,2,3}$, $\mu^{10}_{1,2}$ and μ^{15}_1 may be transformed to $\mu^1_{i,j,k}$ via the Weyl action, for some indices i, j, k. The details are left to the reader. □

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