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ON THE SOLVABILITY OF PARAMETER-DEPENDENT ELLIPTIC FUNCTIONAL BVPS ON ANNULAR-LIKE DOMAINS

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ABSTRACT. We investigate the existence of nontrivial solutions of parameter-dependent elliptic equations with deviated argument in annular-like domains in \mathbb{R}^n , with $n \geq 2$, subject to functional boundary conditions. In particular we consider a boundary value problem that may be used to model heat-flow problems. We obtain an existence result by means of topological methods; in particular, we make use of a recent variant in affine cones of the celebrated Birkhoff–Kellogg theorem. Using an ODE argument, we illustrate in an example the applicability of our theoretical result.

1. Introduction. Our purpose is to study functional boundary value problems (BVPs) associated with elliptic equations in suitable “annular-like” domains; in particular we consider the parametric, functional BVP

$$\begin{cases} Lu = \lambda f(x, u, u_\sigma), & \text{in } \Omega, \\ u(x) = \psi(x), & \text{in } \overline{\Omega}_1, \\ u(x) = \lambda \zeta(x)B[u], & \text{on } \Gamma_2, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ with Ω_1, Ω_2 bounded domains with sufficiently regular boundary, $\overline{\Omega}_1 \subset \Omega_2$, $\partial\Omega_1 = \Gamma_1$, $\partial\Omega_2 = \Gamma_2$ and Γ_1, Γ_2 suitable manifolds. Moreover λ is a real parameter, L is a strongly uniformly elliptic operator, ψ and ζ are continuous, and B is a suitable functional (see Section 2 for details). Concerning the right-hand side appearing in the differential equation

$$Lu = \lambda f(x, u, u_\sigma), \quad x \in \Omega, \quad (2)$$

by f we mean a real-valued continuous function defined on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$, where u_σ incorporates a deviated argument that can take into account the “global” behaviour of u , including the “hole” Ω_1 ; in particular, this setting covers the interesting case of spatial delays.

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A motivation for studying this kind of BVPs is that they may occur in physical applications. We illustrate this fact with the annulus $\Omega = \{x \in \mathbb{R}^2 : 1 < \|x\|_2 < 2\}$, where $\|\cdot\|_2$ is the Euclidean norm and consider the BVP

$$\begin{cases} -\Delta u = \lambda f(x, u, u_\sigma), & \text{in } \Omega, \\ u(x) = \psi(x), & \text{if } \|x\|_2 \leq 1, \\ u(x) = \lambda u(\eta), & \text{if } \|x\|_2 = 2, \end{cases} \quad (3)$$

where $\eta \in \Omega$ is a given point in the interior of the annulus; this is illustrated in Figure 1.

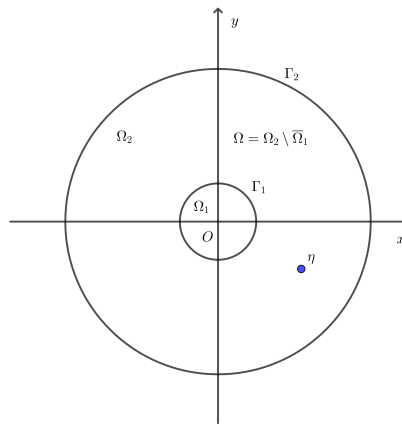


FIGURE 1.

The BVP (3) can be used as a model for the steady-states of the temperature of a heated annular plate. In this model, the heat source can depend also on the “global” behaviour of the system, namely the heat source in a point x of the annulus can also depend on the temperature measured at a point $\sigma(x)$ of the whole domain $\overline{\Omega}_2$ (and in particular it can take into account the values of the datum ψ). Moreover, a controller located in the exterior border of the annulus adds or removes heat in manner proportional to the temperature registered by a sensor located at the point η of the annulus. For one-dimensional similar heat-flow problems see for example [35] and references therein, for deviated arguments see [6, 9, 14], for the elliptic case see [3, 18].

Elliptic equations with non-local terms represent a widely investigated subject, also in view of applications. For example, we mention equations of Kirchhoff type, see for example the review by Ma [24] as well as the recent paper [19] and references therein. Concerning PDEs with deviated (or more general *functional*) arguments, we refer for example to [11, 19, 30, 31, 32, 33]. On the other hand, BVPs associated with ODEs with deviated arguments have been studied by many authors with different techniques: we refer for example to [6] and references therein.

Parametric BVPs similar to (1) have also been studied in association to elliptic functional differential equations subject to functional BCs (see for example [3, 19]). Note that in a local framework – that corresponds to the choice $\sigma(x) = x$ in (2) – and in the case of the Laplacian under Dirichlet boundary conditions, one gets the

classical Gelfand-type problem: see the Introduction of [4] for a recent review on this topic. In the context of PDEs, there is interest to the problem of parameter-dependent BCs, see for example a “nonlinear spectral problem” considered in Section 4.7 of the classical book by Pao [26].

We wish to point out that that the functional boundary operator B in (1) is fairly general and can be used to deal with nonlinear and nonlocal BCs; the latter topic is quite fashionable both in the context of ODEs and of PDEs, for example see the reviews [5, 12, 23, 25, 34, 37] and the papers [15, 16, 20, 21, 27, 28, 36].

Regarding the nature of the domain, that is the annular-like region, for the elliptic context we refer the reader to the classical work of Amann and López-Gómez [2] and the more recent paper of Kowalczyk and co-authors [22].

Our main tool is Theorem 2.2, which is a version in affine cones of the celebrated Birkhoff–Kellogg theorem. The proof of Theorem 2.2 is obtained by topological methods in the recent paper [7]. Applications of Theorem 2.2 in the context of ODEs have been considered in [8, 9, 10]. We remark that the setting of affine cones seems to be helpful when dealing with differential equations with delay effects; as far as we are aware, this is the first time that this approach is utilized in the context of PDEs. Here we obtain the existence of nontrivial solutions (u, λ) of the BVP (1) such that u belongs to a suitable translate of a cone of nonnegative functions.

We close the paper with an example, set in \mathbb{R}^2 , where the nonlinearity f that occurs is not radially symmetric; nevertheless we exploit the symmetry of the domain and use an ODE technique in order to construct suitable lower bounds that are a key ingredient to apply our theoretical result.

2. Some known facts and setting of the problem. Let $\Omega \subset \mathbb{R}^n, (n \geq 2)$ be an annular-like domain, that is, $\Omega = \Omega_2 \setminus \overline{\Omega}_1$, where Ω_1, Ω_2 are bounded domains with smooth boundary such that $\overline{\Omega}_1$ is strictly contained into Ω_2 . Denote by $\Gamma_1 = \partial\Omega_1$ and by $\Gamma_2 = \partial\Omega_2$; then, Γ_1 and Γ_2 are smooth compact manifolds without boundary (cf. [1, Sect. 4], see also [29, Chap. 2]). Given $\alpha \in (0, 1)$ we denote by $C^\alpha(\overline{\Omega})$ the space of all α -Hölder continuous functions $g : \overline{\Omega} \rightarrow \mathbb{R}$ and, for every $k \in \mathbb{N}$, we denote by $C^{k+\alpha}(\overline{\Omega})$ the space of all functions $g \in C^k(\overline{\Omega})$ such that all the partial derivatives of g of order k are α -Hölder continuous in $\overline{\Omega}$ (for more details see [1, Examples 1.13 and 1.14]).

We study the parametric, functional BVP

$$\begin{cases} Lu = \lambda f(x, u, u_\sigma), & \text{in } \Omega, \\ u(x) = \psi(x), & \text{on } \overline{\Omega}_1, \\ u(x) = \lambda \zeta(x) B[u], & \text{on } \Gamma_2, \end{cases} \tag{4}$$

where L is a strongly uniformly elliptic operator, namely

$$Lu(x) = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i}(x) + a(x)u(x), \quad \text{for } x \in \Omega$$

with coefficients $a_{ij}, a_i, a \in C^\alpha(\overline{\Omega})$ for $i, j = 1, 2, \dots, m, a(x) \geq 0$ on $\overline{\Omega}, a_{ij}(x) = a_{ji}(x)$ on $\overline{\Omega}$ for $i, j = 1, 2, \dots, m$; moreover there exists $\mu_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \mu_0 \|\xi\|_n^2 \quad \text{for } x \in \Omega \text{ and } \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n,$$

where $\|\cdot\|_n$ is the Euclidean norm.

Concerning the other terms appearing in (4), $\lambda > 0$ is a real parameter, f is a real-valued continuous function defined on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}$, $\psi : \bar{\Omega}_1 \rightarrow \mathbb{R}$ is continuous, and $\zeta : \Gamma_2 \rightarrow \mathbb{R}_+$ is continuous. Moreover, B is a suitable functional defined on the space $C(\bar{\Omega}_2)$, endowed with the standard supremum norm $\|\cdot\|_\infty$.

Finally, given a continuous map $\sigma : \bar{\Omega} \rightarrow \bar{\Omega}_2$ and $u \in C(\bar{\Omega}_2)$, with the notation u_σ we mean the composition

$$u_\sigma : \bar{\Omega} \rightarrow \mathbb{R} \quad u_\sigma(x) = u(\sigma(x));$$

hence $u_\sigma \in C(\bar{\Omega})$ is a map which may take into account the global behaviour of u .

We apply the classical elliptic theory to the following auxiliary Dirichlet problems. In fact, it is known that, under the previous conditions, a strong maximum principle holds (see [1], Section 4 of Chapter 1; see also Chapter 2 of [29]). Furthermore, given $f \in C(\bar{\Omega})$, the homogeneous Dirichlet problem on Ω

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma_1 \cup \Gamma_2, \end{cases} \quad (5)$$

admits a unique classical solution $u_f \in C^{1,\alpha}(\bar{\Omega})$. Moreover, the BVP

$$\begin{cases} Lu(x) = 0, & x \in \Omega, \\ u(x) = \psi(x), & x \in \Gamma_1, \\ u(x) = 0, & x \in \Gamma_2, \end{cases} \quad (6)$$

admits a unique solution $\delta \in C^{1,\alpha}(\bar{\Omega})$, and the BVP

$$\begin{cases} Lu(x) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \Gamma_1, \\ u(x) = \zeta(x), & x \in \Gamma_2, \end{cases} \quad (7)$$

also admits a unique solution $\gamma \in C^{1,\alpha}(\bar{\Omega})$ such that $\gamma(x) \geq 0$ for every $x \in \Omega$.

The solution operator associated to the BVP (5), $\mathcal{G} : C(\bar{\Omega}) \rightarrow C^1(\bar{\Omega})$, defined as $\mathcal{G}(f) = u_f$ is linear and continuous. Moreover, we observe that we can uniquely extend \mathcal{G} to another operator, denoted again by the same name, $\mathcal{G} : C(\bar{\Omega}) \rightarrow C(\bar{\Omega}_2)$, by considering a trivial continuous extension of a function $v \in C(\bar{\Omega})$ such that $v|_{\Gamma_1} \equiv 0$, that is

$$\tilde{v}(x) = \begin{cases} \tilde{v}(x) = v(x), & x \in \bar{\Omega}, \\ \tilde{v}(x) = 0, & x \in \bar{\Omega}_1. \end{cases}$$

For example notice that $\tilde{\gamma} \in C(\bar{\Omega}_2)$, where γ is the unique solution of the BVP (7). The operator \mathcal{G} is continuous, linear and compact and leaves the cone of positive functions invariant (see, for example [1], Section 4 of Chapter 1).

Some further notation is in order. We define

$$\phi(x) := \begin{cases} \psi(x), & x \in \bar{\Omega}_1, \\ \delta(x), & x \in \bar{\Omega} = \bar{\Omega}_2 \setminus \Omega_1, \end{cases} \quad (8)$$

where δ is the unique solution of the BVP (6). Observe that, by construction, the function $\phi \in C(\bar{\Omega}_2)$. The function ϕ will play a key role in the sequel, since it will be the vertex of the affine cone that we use when applying Theorem 2.2.

We denote by \mathcal{F} the *superposition (Nemytskii)* operator associated with maps f and σ as above; that is,

$$\mathcal{F} : C(\bar{\Omega}_2) \rightarrow C(\bar{\Omega}), \quad \mathcal{F}(u)(x) := f(x, u(x), u(\sigma(x))), \quad x \in \bar{\Omega}.$$

Note, in particular, that the operator $\mathcal{G} \circ \mathcal{F}$ is well-defined.

The above discussion justifies the following definition of *solution* of the problem (4).

Definition 2.1. We say that a function $u \in C(\overline{\Omega}_2)$ is a (*weak*) *solution* of the problem (4) if

$$u = \phi + \lambda \left(\mathcal{G} \circ \mathcal{F}(u) + \tilde{\gamma} B[u] \right).$$

In other words, u is a solution of the problem (4) if and only if it is a fixed point of the operator $\phi + \lambda \mathcal{T}$, where

$$\mathcal{T} : C(\overline{\Omega}_2) \rightarrow C(\overline{\Omega}_2), \quad \mathcal{T}(u) := \mathcal{G} \circ \mathcal{F}(u) + \tilde{\gamma} B[u].$$

Let now $(X, \|\cdot\|)$ be a real Banach space. A *cone* K of X is a closed set with $K + K \subset K$, $\mu K \subset K$ for all $\mu \geq 0$ and $K \cap (-K) = \{0\}$. For $y \in X$, the *translate* of the cone K is defined as

$$K_y := y + K = \{y + x : x \in K\}.$$

Given a bounded and open (in the relative topology) subset \mathcal{O} of K_y , we denote by $\overline{\mathcal{O}}$ and $\partial \mathcal{O}$ the closure and the boundary of \mathcal{O} relative to K_y . Given an open bounded subset D of X we denote $D_{K_y} = D \cap K_y$, an open subset of K_y .

With these ingredients we can now state a Birkhoff–Kellogg type result in affine cones.

Theorem 2.2 ([7], Corollary 2.4). *Let $(X, \|\cdot\|)$ be a real Banach space, $K \subset X$ be a cone and $D \subset X$ be an open bounded set with $y \in D_{K_y}$ and $\overline{D}_{K_y} \neq K_y$. Assume that $\mathcal{F} : \overline{D}_{K_y} \rightarrow K$ is a compact map and that*

$$\inf_{x \in \partial D_{K_y}} \|\mathcal{F}(x)\| > 0.$$

Then there exist $x^ \in \partial D_{K_y}$ and $\lambda^* \in (0, +\infty)$ such that $x^* = y + \lambda^* \mathcal{F}(x^*)$.*

The proof of Theorem 2.2 is derived from the classical fixed point index theory for compact maps; this is done in the context of affine cones, see also [13]. We refer a reader interested in the fixed point index to the review of Amann [1] and to the book by Guo and Lakshmikantham [17].

By K_0 we denote the following cone of non-negative functions in $C(\overline{\Omega}_2)$:

$$K_0 = \{u \in C(\overline{\Omega}_2) : u(x) \geq 0 \text{ for every } x \in \overline{\Omega}_2 \text{ and } u(x) = 0 \text{ for every } x \in \overline{\Omega}_1\}.$$

Note that $K_0 \neq \{0\}$ since the map $\tilde{\gamma}$ introduced above is in K_0 .

Given $\omega \in C(\overline{\Omega}_2)$, let K_ω be the following translate of the cone K_0 :

$$K_\omega = \omega + K_0 = \{\omega + u : u \in K_0\}.$$

Definition 2.3. Given $\omega \in C(\overline{\Omega}_2)$ and $\rho > 0$, we define the following subsets of $C(\overline{\Omega}_2)$:

$$K_{0,\rho} := \{u \in K_0 : \|u\|_\infty < \rho\}, \quad K_{\omega,\rho} := \omega + K_{0,\rho}.$$

3. Main result. The following theorem provides an existence result for the problem (4): here we obtain a non-trivial solution within the cone K_ϕ , where ϕ is as in (8).

Theorem 3.1. *Let $\rho \in (0, +\infty)$ and assume the following conditions hold.*

(a) *There exists $\underline{\ell}_\rho \in C(\overline{\Omega})$ such that*

$$f(x, u, v) \geq \underline{\ell}_\rho(x) \geq 0, \text{ for every } (x, u, v) \in \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \text{ with } \max\{u, |v|\} \leq \rho + \|\phi\|_\infty.$$

- (b) $B : \overline{K}_{\phi,\rho} \rightarrow \mathbb{R}_+$ is continuous and bounded, in particular there is $\underline{b}_\rho \in [0, +\infty)$ such that

$$B[u] \geq \underline{b}_\rho, \text{ for every } u \in \partial K_{\phi,\rho}.$$

- (c) There exists $d_\rho \in (0, +\infty)$ such that

$$\sup_{x \in \overline{\Omega}} |\mathcal{G}(\underline{\ell}_\rho)(x) + \underline{b}_\rho \gamma(x)| \geq d_\rho. \quad (9)$$

Then there exist $\lambda_\rho \in (0, +\infty)$ and $u_\rho \in \partial K_{\phi,\rho}$ that satisfy the problem (4).

Proof. First notice that, due to the assumptions above, the operator \mathcal{T} maps $\overline{K}_{\phi,\rho}$ into K_0 and is compact. In fact, since \mathcal{F} is continuous and bounded and \mathcal{G} is linear and compact, it follows that the operator $\mathcal{G} \circ \mathcal{F}$ is compact as well; moreover, given $u \in \overline{K}_{\phi,\rho}$, we have $\mathcal{G} \circ \mathcal{F}(u) \in K_0$ by (a) and the maximum principle. On the other hand, assumption (b) implies that B is a continuous, finite rank operator with nonnegative values and, as we already stressed, $\tilde{\gamma} \in K_0$.

Take $u \in \partial K_{\phi,\rho}$. Then for every $x \in \overline{\Omega}$ we have

$$\mathcal{G} \circ \mathcal{F}(u)(x) + \tilde{\gamma}(x)B[u] \geq \mathcal{G}(\underline{\ell}_\rho)(x) + \underline{b}_\rho \gamma(x).$$

Thus by (c) we obtain

$$\|\mathcal{G} \circ \mathcal{F}(u) + \tilde{\gamma}B[u]\|_\infty \geq \sup_{x \in \overline{\Omega}} |\mathcal{G} \circ \mathcal{F}(u)(x) + \tilde{\gamma}(x)B[u]| \geq \sup_{x \in \overline{\Omega}} |\mathcal{G}(\underline{\ell}_\rho)(x) + \underline{b}_\rho \gamma(x)| \geq d_\rho.$$

In particular, note that the RHS of the latter inequality does not depend on the particular u chosen. Therefore we have

$$\inf_{u \in \partial K_{\phi,\rho}} \|\mathcal{G} \circ \mathcal{F}(u) + \tilde{\gamma}B[u]\|_\infty \geq d_\rho > 0,$$

and the result follows by Theorem 2.2. \square

We illustrate now in an example the applicability of Theorem 3.1.

Example 3.2. Let us consider the following BVP in $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u(x_1, x_2) = \lambda(1 + x_1^2) e^{-u(x_1, x_2) - u(\frac{x_1}{2}, \frac{x_2}{2})}, & (x_1, x_2) \in \Omega, \\ u(x_1, x_2) = x_1^2 + x_2^2, & (x_1, x_2) \in \overline{\Omega}_1, \\ u(x_1, x_2) = \lambda \int_{\Omega_2} u^2 dx, & (x_1, x_2) \in \Gamma_2, \end{cases} \quad (10)$$

where

$$\Omega = \left\{ (x_1, x_2) : 1 \leq \sqrt{x_1^2 + x_2^2} \leq e \right\}.$$

Note that in this case the function that occurs in the PDE is not radially symmetric. Nevertheless, in our case, it is convenient to take as a lower bound the radially symmetric function

$$\underline{\ell}_\rho(x) = e^{-2(\rho+1)},$$

and solve the ‘‘torsion problem’’

$$\begin{cases} -\Delta u(x_1, x_2) = 1, & (x_1, x_2) \in \Omega, \\ u(x_1, x_2) = 0, & (x_1, x_2) \in \Gamma_1 \cup \Gamma_2. \end{cases} \quad (11)$$

The solution of the BVP (11) is obtained by transforming the BVP (11) into a corresponding ODE problem. Indeed, let $u(x_1, x_2) := \varphi(r)$ be a radial solution

of (11), with $r := \sqrt{x_1^2 + x_2^2}$. Then, φ solves

$$\begin{cases} -\varphi''(r) - \frac{1}{r}\varphi'(r) = 1, & r \in (1, e), \\ \varphi(1) = 0, \varphi(e) = 0, \end{cases}$$

which yields

$$u_0(x_1, x_2) = \frac{1}{8}[(e^2 - 1) \ln(x_1^2 + x_2^2) + 2(1 - x_1^2 - x_2^2)].$$

The same ODE argument can be used to solve the BVP

$$\begin{cases} -\Delta u(x_1, x_2) = 0, & (x_1, x_2) \in \Omega, \\ u(x_1, x_2) = 0, & (x_1, x_2) \in \Gamma_1, \\ u(x_1, x_2) = 1, & (x_1, x_2) \in \Gamma_2, \end{cases}$$

which yields

$$\gamma(x_1, x_2) = \frac{\ln(x_1^2 + x_2^2)}{2},$$

and the BVP

$$\begin{cases} -\Delta u(x_1, x_2) = 0, & (x_1, x_2) \in \Omega, \\ u(x_1, x_2) = 1, & (x_1, x_2) \in \Gamma_1, \\ u(x_1, x_2) = 0, & (x_1, x_2) \in \Gamma_2, \end{cases}$$

which gives

$$\delta(x_1, x_2) = 1 - \frac{\ln(x_1^2 + x_2^2)}{2},$$

so that

$$\phi(x_1, x_2) = \begin{cases} x_1^2 + x_2^2, & (x_1, x_2) \in \bar{\Omega}_1, \\ 1 - \frac{\ln(x_1^2 + x_2^2)}{2}, & (x_1, x_2) \in \bar{\Omega}_2 \setminus \bar{\Omega}_1. \end{cases}$$

A direct computation yields

$$\sup_{(x_1, x_2) \in \Omega} |u_0(x_1, x_2)| = \frac{1}{8} \left[(e^2 - 1) \ln\left(\frac{e^2 - 1}{2}\right) + 3 - e^2 \right]. \tag{12}$$

From (12) and the choice of $b_\rho = 0$ we obtain

$$\begin{aligned} \sup_{(x_1, x_2) \in \Omega} |\mathcal{G}(\ell_\rho)(x_1, x_2) + b_\rho \gamma(x_1, x_2)| &\geq e^{-2(\rho+1)} \cdot \sup_{(x_1, x_2) \in \Omega} |u_0(x_1, x_2)| = \\ &= e^{-2(\rho+1)} \frac{1}{8} \left[(e^2 - 1) \ln\left(\frac{e^2 - 1}{2}\right) + 3 - e^2 \right] =: d_\rho > 0, \end{aligned}$$

which implies that (9) is satisfied for every $\rho \in (0, +\infty)$.

Thus we can apply Theorem 3.1, obtaining uncountably many pairs of solutions and parameters (u_ρ, λ_ρ) for the BVP (10).

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