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Ground state solutions for a (p, q)-Choquard equation with a general nonlinearity

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Abstract

In this paper, we study the existence of ground state solutions for the following (p, q)-Choquard equation:

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

where $2 \leq p < q < N$, Δ_s is the s-Laplacian operator, with $s \in \{p, q\}$, I_α is the Riesz potential of order $\alpha \in ((N-2q)^+, N)$, $F \in C^1(\mathbb{R}, \mathbb{R})$ is a general nonlinearity of Berestycki-Lions type and $F' = f$. Furthermore, we analyze the regularity, symmetry and decay properties of these solutions. In particular, we extend the results in [33] to the (p, q)-Laplacian setting.

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Keywords: (p, q)-Laplacian operator; Choquard equation; Pohozaev identity; Variational methods; Regularity; Qualitative properties of solutions

1. Introduction

In this paper, we focus on the following (p, q)-Choquard equation:

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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where $2 \leq p < q < N$, $\Delta_s u := \mathrm{div}(|\nabla u|^{s-2}\nabla u)$ is the s-Laplacian operator, with $s \in \{p, q\}$, and I_α denotes the Riesz potential of order $\alpha \in ((N-2q)^+, N)$ defined by

$$I_\alpha(x) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha}\frac{1}{|x|^{N-\alpha}} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

We assume that $f \in C(\mathbb{R}, \mathbb{R})$ fulfills the following conditions:

(f_1)

$$\limsup_{|t|\to 0} \frac{|tf(t)|}{|t|^{\frac{N+\alpha}{N}\frac{p}{2}}} < \infty \quad \text{and} \quad \limsup_{|t|\to \infty} \frac{|tf(t)|}{|t|^{\frac{N+\alpha}{N-q}\frac{q}{2}}} < \infty,$$

(f_2) the antiderivative $F(t) := \int_0^t f(\tau)\,d\tau$ of f satisfies

$$\lim_{t\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha}{N}\frac{p}{2}}} = 0 \quad \text{and} \quad \lim_{t\to \infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-q}\frac{q}{2}}} = 0,$$

(f_3) there exists $t_0 \in \mathbb{R} \setminus \{0\}$ such that $F(t_0) \neq 0$.

The interest in (1.1) is due to the fact that two phenomena occur in it: the nonhomogeneity of the (p, q)-Laplacian operator $\Delta_p + \Delta_q$ and the nonlocal character of the nonlinear term. In what follows, we recall the main motivations and some fundamental works regarding (p, q)-Laplacian problems and Choquard equations, respectively.

The study of (p, q)-Laplacian problems stems from the following general reaction-diffusion system

$$u_t = \mathrm{div}(\mathcal{D}(u)\nabla u) + f(x, u) \quad x \in \mathbb{R}^N, t > 0,$$

where $\mathcal{D}(u) := |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system finds applications in biophysics, plasma physics and chemical reaction design; in these contexts u stands for a concentration, the term $\mathrm{div}(\mathcal{D}(u)\nabla u)$ corresponds to the diffusion with a diffusion coefficient $\mathcal{D}(u)$, and $f(x, u)$ represents the reaction term which relates to source and loss processes (see [16]). For results concerning (p, q)–Laplacian problems in both bounded domains and in the whole of $\mathbb{R}^N$, we refer to [7,10,19–21,24,36,38] as well as the references therein. In particular, Pomponio and Watanabe [38] proved the existence of a positive ground state solution to

$$-\Delta_p u - \Delta_q u = g(u) \quad \text{in } \mathbb{R}^N,$$

where g is a Berestycki-Lions type nonlinearity [12]; see also [7,9] for related results.

We emphasize that the integral functional associated with the (p, q)-Laplacian operator is connected to the well-known double-phase functional

$$\mathfrak{F}_{p,q}(u; \Omega) := \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)\,dx,$$

where $\Omega \subset \mathbb{R}^N$ is an open set and $0 \le a(\cdot) \in L^\infty(\Omega)$, which was first considered by Zhikov [47,48] within the context of homogenizing highly anisotropic materials and in order to provide new examples of the Lavrentiev phenomenon. In elasticity theory, the weight function $a(\cdot)$ emerges as the modulating coefficient and it dictates the geometry of the composite made of two different materials with distinct power hardening exponents p and q. For what concerns the regularity theory, $\mathfrak{F}_{p,q}$ belongs to the category of nonuniformly elliptic functionals with nonstandard growth conditions of (p,q)-type, according to Marcellini's terminology; see the recent surveys in [29,30] for more details.

When $p = q = 2$, $N = 3$, $\alpha = 2$ and $F(t) = t^2$, equation (1.1) reduces to the so-called Choquard-Pekar equation

$$-\Delta u + u = \left(\frac{1}{4\pi|x|} * u^2\right) u \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

This equation was introduced in 1954 by Pekar [37] to describe the quantum mechanics of a polaron at rest. In 1976, Choquard employed equation (1.2) to investigate stationary states of the one-component plasma approximation within Hartree-Fock theory [25]. Later, Penrose proposed (1.2) as a model of self-gravitating matter and is known in that context as the Schrödinger-Newton equation [31]. The early existence and symmetry results can be addressed to the fundamental papers by Lieb [25] and Lions [27]. Ma and Zhao [28] demonstrated that all positive solutions of (1.2) must be radially symmetric and monotone decreasing about some fixed point. Moroz and Van Schaftingen [33] established the existence and qualitative properties of positive ground state solutions to

$$-\Delta u + u = (I_\alpha * F(u)) f(u) \quad \text{in } \mathbb{R}^N,$$

under almost necessary conditions on the nonlinearity F in the spirit of Berestycki and Lions [12]. Further interesting results for nonlinear Choquard type equations can be found in [34]. When $p = q \neq 2$, equation (1.1) falls in the realm of the following class of quasilinear Choquard equations involving the p-Laplacian operator:

$$-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = \frac{1}{\varepsilon^{N-\mu}} \left(\frac{1}{|x|^\mu} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\varepsilon > 0$, $\mu \in (0, p)$ and V is a positive continuous potential. For $\varepsilon > 0$ small enough, the existence and multiplicity of positive solutions to (1.3) has been object of study by Alves and Yang [2–4], whereas a regularity result and a Pohozaev type identity for (1.3) with $\varepsilon = V(\cdot) = 1$ have been recently obtained by the first author in [6].

Concerning nonlinear Choquard equations driven by the (p,q)-Laplacian operator, Zhang et al. [46] used Nehari manifold method and Ljusternik-Schnirelmann category theory to investigate the multiplicity and concentration phenomena of positive solutions for the equation

$$-\varepsilon^p \Delta_p u - \varepsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \frac{1}{\varepsilon^{N-\mu}} \left(\frac{1}{|x|^\mu} * G(u)\right) g(u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter, $\mu \in (0, p)$, g is a C^1 subcritical nonlinearity satisfying an Ambrosetti-Rabinowitz type condition [5] and $V \in C(\mathbb{R}^N, \mathbb{R})$ fulfills the following condition introduced by Rabinowitz [39]:

$$0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x) < \infty.$$

Xie et al. [45] applied the mountain pass theorem [5] to show the existence of nontrivial solutions to (1.4) with $\varepsilon = 1$, g is a subcritical continuous nonlinearity satisfying an Ambrosetti-Rabinowitz type condition, and $V \in C(\mathbb{R}^N, \mathbb{R})$ verifies the following Bartsch-Wang condition [11]: there exist constants $a, d_0 > 0$ such that $V(x) \geq a$ and

$$\lim_{|y|\to\infty} \text{meas}\left\{x \in \mathbb{R}^N : |x - y| \leq d_0,\ V(x) \leq M\right\} = 0 \quad \text{for all } M > 0.$$

We also mention [8] where the author explored the multiplicity and concentration of positive solutions for a Choquard equation involving the fractional (p, q)-Laplacian operator.

Motivated by the above papers, in this work we focus on the existence of ground state solutions to (1.1) with general nonlinearities, and we examine the qualitative properties of these solutions. In particular, we extend the results in [33] to the (p, q)-Laplacian setting. Next we list our main theorems. The first one concerns the regularity of weak solutions to (1.1) and a Pohozaev type identity. Here we say that $u \in \mathcal{W}_{p,q}$ is a weak solution to (1.1) if for every $\varphi \in \mathcal{W}_{p,q}$ we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \int_{\mathbb{R}^N} |u|^{p-2}u\,\varphi\,dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\nabla\varphi\,dx + \int_{\mathbb{R}^N} |u|^{q-2}u\,\varphi\,dx \\ &\quad= \int_{\mathbb{R}^N} (I_\alpha * F(u))\,f(u)\,\varphi\,dx,\end{aligned} \tag{1.5}$$

where $\mathcal{W}_{p,q}$ is defined in Section 2.

Theorem 1.1. *Assume that f fulfills (f_1)–(f_2). If $u \in \mathcal{W}_{p,q}$ is a weak solution to (1.1), then $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\lambda}_{loc}(\mathbb{R}^N)$ for some $\lambda \in (0, 1)$. Moreover, u satisfies the following Pohozaev identity:*

$$\begin{aligned}&\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{N}{p}\int_{\mathbb{R}^N} |u|^p\,dx + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q\,dx \\ &\quad= \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\,F(u)\,dx.\end{aligned} \tag{1.6}$$

The proof of Theorem 1.1 is divided into two main steps. We first obtain the L^∞-regularity of solutions to (1.1) by combining a suitable Moser iteration [35] and a De Giorgi type argument [17]. We stress that the restriction $\alpha \in ((N - 2q)^+, N)$ is only used to implement these techniques; see Proposition 3.1. In view of Young's inequality for convolutions, we see that the term $I_\alpha * F(u)$ is bounded, and so the $C^{1,\lambda}_{loc}$-regularity is a direct consequence of [21, Theorem 1].

We emphasize that the presence of the nonlocal term $I_\alpha * F(u)$ prevents a direct application of the regularity results in [21] for local (p,q)-equations. We recall that when $p = q = 2$ in (1.1), the authors in [33] proved a nonlocal version of the Brezis-Kato regularity result [14, Theorem 2.3] to deduce that every weak solution $u \in H^1(\mathbb{R}^N)$ is in $L^r(\mathbb{R}^N)$ for all $r \in [2, \frac{N}{\alpha}\frac{2N}{N-2})$, and then they used the Calderon-Zygmund theory to infer that $u \in W^{2,s}_{loc}(\mathbb{R}^N)$ for all $s \in [1,\infty)$; see [33, Theorem 2]. Due to the nonhomogeneity of the (p,q)-Laplacian operator, the previous arguments do not work in our context and thus a different approach is performed in the present paper. In light of the above $C^{1,\lambda}_{loc}$-regularity result, we then show that every weak solution to (1.1) satisfies the Pohozaev identity (1.6) by exploiting a variational identity for locally Lipschitz continuous solutions of a general class of quasilinear equations found in [18]. Observe that (1.6) can be seen as a generalization of [33, formula (1.5)] to the (p,q)-Laplacian case.

Our second result guarantees the existence of a ground state solution to (1.1), that is, a function $u \in \mathcal{W}_{p,q}$ such that $\mathcal{I}(u) = m$ and $\mathcal{I}'(u) = 0$, where the ground state energy level m is given by

$$m := \inf\left\{\mathcal{I}(v) \,:\, v \in \mathcal{W}_{p,q} \setminus \{0\},\, \mathcal{I}'(v) = 0\right\}, \tag{1.7}$$

and $\mathcal{I} : \mathcal{W}_{p,q} \to \mathbb{R}$ is the energy functional associated with (1.1) defined as

$$\mathcal{I}(u) := \frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p\right) dx + \frac{1}{q}\int_{\mathbb{R}^N} \left(|\nabla u|^q + |u|^q\right) dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\, F(u)\, dx.$$

Theorem 1.2. *Assume that f satisfies (f_1)–(f_3). Then (1.1) has a ground state solution.*

We sketch below the main lines of the proof of Theorem 1.2 which is inspired by [33, Theorem 1]. Note that the assumptions on f, Sobolev embeddings for $\mathcal{W}_{p,q}$ and the Hardy-Littlewood-Sobolev inequality [26, Theorem 4.3] show that $\mathcal{I}$ has a mountain pass geometry [5], and so we can define the mountain pass level

$$b := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t)), \tag{1.8}$$

where

$$\Gamma := \left\{\gamma \in C([0,1], \mathcal{W}_{p,q}) \,:\, \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0\right\}.$$

Utilizing the auxiliary functional $\mathcal{I}(u(\cdot/e^\theta)) : \mathbb{R} \times \mathcal{W}_{p,q} \to \mathbb{R}$ introduced in [22], we produce a Pohozaev-Palais-Smale sequence at level b, that is, a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{p,q}$ such that, as $n \to \infty$,

$$\begin{aligned} &\mathcal{I}(u_n) \to b, \\ &\mathcal{I}'(u_n) \to 0 \text{ in } \mathcal{W}^*_{p,q}, \\ &\mathcal{P}(u_n) \to 0, \end{aligned} \tag{1.9}$$

where $\mathcal{W}^*_{p,q}$ denotes the dual space of $\mathcal{W}_{p,q}$ and the Pohozaev functional $\mathcal{P} : \mathcal{W}_{p,q} \to \mathbb{R}$ is given by

$$\mathcal{P}(u) := \frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \frac{N}{p}\int_{\mathbb{R}^N} |u|^p\,dx + \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx + \frac{N}{q}\int_{\mathbb{R}^N} |u|^q\,dx$$
$$- \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))\,F(u)\,dx.$$

In particular, we derive from (1.9) that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$; see Lemma 4.1. Then we develop a concentration-compactness type argument to prove that $(u_n)_{n\in\mathbb{N}}$ converges, up to translations and extraction of a subsequence, strongly in $\mathcal{W}_{p,q}$ to a weak solution u to (1.1); see Proposition 4.2. Due to the presence of the (p,q)-Laplacian operator, differently from [33], we need to establish an almost everywhere convergence of the gradient of the Pohozaev-Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$; see Lemma 4.2. This result will be also useful to apply the Brezis-Lieb Lemma [15, Theorem 1] and deduce the compactness, modulo translations, of the set of ground state solutions to (1.1); see Proposition 4.4. In order to show that u is a ground state solution to (1.1), we take advantage of Theorem 1.1, which ensures that every weak solution to (1.1) satisfies the Pohozaev identity (1.6), to construct an optimal path in the spirit of [23, Lemma 2.1]; see Proposition 4.3. Consequently, we can conclude that the mountain pass level b coincides with the ground state energy level m. We stress that Theorem 1.2 improves [46, Lemma 3.1] because we consider more general nonlinearities. For instance, we cover the case $f(t) = |t|^{r-2}t$ with $r \in (\frac{N+\alpha}{N}\frac{p}{2}, \frac{N+\alpha}{N-q}\frac{q}{2})$, which has not been investigated in the literature.

Thirdly, by using a polarization argument, we obtain the following version of [33, Theorem 4] concerning the sign and symmetry of ground state solutions to (1.1).

Theorem 1.3. *If f satisfies (f_1)–(f_2) and, in addition, f is odd and has constant sign on $(0,\infty)$, then every ground state solution to (1.1) has constant sign and is radially symmetric with respect to some point in $\mathbb{R}^N$.*

Finally, if the condition near zero in (f_1) is replaced by

$$\limsup_{|t|\to 0} \frac{|tf(t)|}{|t|^p} < \infty,$$

then we are able to demonstrate the exponential decay of positive solutions to (1.1).

Theorem 1.4. *If f satisfies (f_1')–(f_2), where*
(f_1')

$$\limsup_{|t|\to 0} \frac{|tf(t)|}{|t|^p} < \infty \quad \textit{and} \quad \limsup_{|t|\to \infty} \frac{|tf(t)|}{|t|^{\frac{N+\alpha}{N-q}\frac{q}{2}}} < \infty,$$

and u is a positive weak solution to (1.1), then there exist $C_1, C_2 > 0$ such that

$$u(x) \le C_1 e^{-C_2|x|} \quad \textit{for all } x \in \mathbb{R}^N.$$

Roughly speaking, the idea of the proof of Theorem 1.4 relies on the facts that $(I_\alpha * F(u))(x) \to 0$ as $|x| \to \infty$ (by Young's inequality for convolutions) and $u(x) \to 0$ as $|x| \to \infty$

(because u is Lipschitz continuous and belongs to $L^p(\mathbb{R}^N)$), and utilizing the new assumption near the origin we find that $(I_\alpha * F(u))f(u)(x) = o(u^{p-1}(x) + u^{q-1}(x))$ as $|x| \to \infty$. Then we can use a comparison argument to achieve the desired exponential estimate.

The paper is organized as follows. In Section 2 we give some preliminary results. In Section 3 we study the regularity of weak solutions to (1.1) and we show that they satisfy a Pohozaev type identity. In Section 4 we provide the proof of Theorem 1.2. Section 5 is devoted to the symmetry properties of ground state solutions of (1.1). In Section 6 we prove the exponential decay of solutions to (1.1).

2. Preliminaries

Let $A \subset \mathbb{R}^N$ be a measurable set and $A^c := \mathbb{R}^N \setminus A$ its complement. For $t \in [1, \infty]$, we denote by $\|u\|_{L^t(A)}$ the $L^t(A)$-norm of $u : \mathbb{R}^N \to \mathbb{R}$. With $B_r(x_0)$ we indicate the ball in $\mathbb{R}^N$ centered at $x_0 \in \mathbb{R}^N$ with radius $r > 0$. When $x_0 = 0$, we simply write B_r instead of $B_r(0)$.

Let $t \in (1, \infty)$. The Sobolev space $W^{1,t}(\mathbb{R}^N)$ is the set of all functions $u \in L^t(\mathbb{R}^N)$ whose distributional first-order derivatives belong to $L^t(\mathbb{R}^N)$. The space $W^{1,t}(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_{W^{1,t}(\mathbb{R}^N)} := \left(\int\limits_{\mathbb{R}^N} \left(|\nabla u|^t + |u|^t\right) dx \right)^{\frac{1}{t}}.$$

It is well-known that $W^{1,t}(\mathbb{R}^N)$ is a reflexive Banach space [1, Theorems 3.3 and 3.6], and that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,t}(\mathbb{R}^N)$ [1, Corollary 3.23]. We also recall the classical Sobolev embeddings.

Theorem 2.1. *[1, Theorems 4.12, 4.31 and 6.3] Let $t \in (1, N)$. Then there exists a sharp constant $K_* = K_*(N, t) > 0$ such that*

$$K_* \|u\|^t_{L^{t^*}(\mathbb{R}^N)} \leq \|\nabla u\|^t_{L^t(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{1,t}(\mathbb{R}^N),$$

where $\mathcal{D}^{1,t}(\mathbb{R}^N)$ is the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_{L^t(\mathbb{R}^N)}$. Moreover, $W^{1,t}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for all $r \in [t, t^*]$, and compactly embedded in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, t^*)$.

We have the following useful inequality.

Lemma 2.1. *Let $t \in (1, N)$ and $s \in [t, t^*)$. Then there exists $C > 0$ such that*

$$\|u\|^s_{L^s(\mathbb{R}^N)} \leq C \left(\sup_{x_0 \in \mathbb{R}^N} \|u\|^s_{L^s(B_1(x_0))} \right)^{1-\frac{t}{s}} \|u\|^t_{W^{1,t}(\mathbb{R}^N)} \quad \text{for all } u \in W^{1,t}(\mathbb{R}^N).$$

Proof. Fix $u \in W^{1,t}(\mathbb{R}^N)$. If $s = t$ then the proof is obvious. Let $s \in (t, t^*)$ and take

$$r := \frac{N(s-t)}{t} < s.$$

Set $\beta_1 := \frac{r}{s-t}$ and $\beta_2 := \frac{t^*}{t}$, and note that

$$s = \frac{r}{\beta_1} + \frac{t^*}{\beta_2}.$$

Using the Hölder inequality and Theorem 2.1, we have

$$\begin{aligned}
\|u\|^s_{L^s(B_1(x_0))} &\le \left(\int_{B_1(x_0)} |u|^r\, dx\right)^{\frac{1}{\beta_1}} \left(\int_{B_1(x_0)} |u|^{t^*}\, dx\right)^{\frac{1}{\beta_2}} \\
&\le C_1 \left[\|u\|^r_{L^s(B_1(x_0))} |B_1(x_0)|^{\frac{s-r}{s}}\right]^{\frac{s-t}{r}} \|u\|^t_{W^{1,t}(B_1(x_0))} \\
&= C_2 \left(\|u\|^s_{L^s(B_1(x_0))}\right)^{1-\frac{t}{s}} \|u\|^t_{W^{1,t}(B_1(x_0))} \\
&\le C_2 \left(\sup_{x_0\in\mathbb{R}^N} \|u\|^s_{L^s(B_1(x_0))}\right)^{1-\frac{t}{s}} \|u\|^t_{W^{1,t}(B_1(x_0))}.
\end{aligned}$$

Covering $\mathbb{R}^N$ by balls with radius 1 in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce the conclusion. □

Let $1 < p < q < \infty$. In order to examine (1.1), we consider the space

$$\mathcal{W}_{p,q} := W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_{\mathcal{W}_{p,q}} := \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,q}(\mathbb{R}^N)}.$$

It is easy to verify that $\mathcal{W}_{p,q}$ is a reflexive Banach space and $C^\infty_c(\mathbb{R}^N)$ is dense in $\mathcal{W}_{p,q}$. When $q < N$, we derive from Theorem 2.1 that $\mathcal{W}_{p,q}$ is continuously embedded in $L^r(\mathbb{R}^N)$ for all $r \in [p, q^*]$, and compactly embedded in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$. We will denote by $\mathcal{W}^*_{p,q}$ the dual of $\mathcal{W}_{p,q}$ and by $\|\cdot\|_*$ the dual norm on $\mathcal{W}^*_{p,q}$.

Finally, we recall the Hardy-Littlewood-Sobolev inequality which will be frequently used throughout the paper.

Theorem 2.2. *[26, Theorem 4.3] Let $r, s \in (1,\infty)$ and $\mu \in (0, N)$ with $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{s} = 2$. Let $f \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \mu, r) > 0$, independent of f and h, such that*

$$\left|\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\mu}\, dxdy\right| \le C(N,\mu,r)\|f\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}.$$

Remark 2.1. As a consequence of Theorem 2.2, if $\alpha \in (0, N)$ and $s \in (1, \frac{N}{\alpha})$, then the map $f \in L^s(\mathbb{R}^N) \mapsto I_\alpha * f \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ is a bounded linear operator.

3. Proof of Theorem 1.1

In this section we analyze the regularity of solutions to (1.1) and we demonstrate the Pohozaev identity (1.6).

3.1. Regularity of solutions

We begin by studying the regularity of solutions to (1.1).

Proposition 3.1. *Assume that f satisfies (f_1)–(f_2). If $u \in \mathcal{W}_{p,q}$ is a weak solution to (1.1), then $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\lambda}_{loc}(\mathbb{R}^N)$ for some $\lambda \in (0, 1)$.*

Proof. We first prove that

$$u \in L^\nu(\mathbb{R}^N) \quad \text{for all } \nu \in [p, \infty). \tag{3.1}$$

Set $z := |u|$ and $z_\varepsilon := \sqrt{u^2 + \varepsilon^2} - \varepsilon$. Note that $z_\varepsilon \to z$ in $\mathcal{W}_{p,q}$ as $\varepsilon \to 0$. We claim that

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z \nabla \phi \, dx + \int\limits_{\mathbb{R}^N} z^{p-1}\phi \, dx + \int\limits_{\mathbb{R}^N} |\nabla z|^{q-2}\nabla z \nabla \phi \, dx + \int\limits_{\mathbb{R}^N} z^{q-1}\phi \, dx \\ &\leq \int\limits_{\mathbb{R}^N} |I_\alpha * F(u)||f(u)|\phi \, dx \end{aligned} \tag{3.2}$$

for all $\phi \in \mathcal{W}_{p,q}$ such that $\phi \geq 0$. To confirm this, fix $\phi \in C^\infty_c(\mathbb{R}^N)$ such that $\phi \geq 0$. Observing that for all $s \in \{p, q\}$ it holds

$$\begin{aligned} \int\limits_{\mathbb{R}^N} |\nabla z|^{s-2}\nabla z_\varepsilon \nabla \phi \, dx &= \int\limits_{\mathbb{R}^N} |\nabla u|^{s-2}\nabla u \frac{u}{\sqrt{u^2+\varepsilon^2}} \nabla \phi \, dx \\ &= \int\limits_{\mathbb{R}^N} |\nabla u|^{s-2}\nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi \right) dx - \int\limits_{\mathbb{R}^N} |\nabla u|^s \frac{\varepsilon^2}{(u^2+\varepsilon^2)^{3/2}}\phi \, dx \\ &\leq \int\limits_{\mathbb{R}^N} |\nabla u|^{s-2}\nabla u \nabla \left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi \right) dx, \end{aligned}$$

we see that

$$\int\limits_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z_\varepsilon \nabla \phi \, dx + \int\limits_{\mathbb{R}^N} |\nabla z|^{q-2}\nabla z_\varepsilon \nabla \phi \, dx$$

$$\le \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \nabla\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx$$

$$= -\int_{\mathbb{R}^N} |u|^{p-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx - \int_{\mathbb{R}^N} |u|^{q-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx$$

$$+ \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx$$

$$\le -\int_{\mathbb{R}^N} |u|^{p-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx - \int_{\mathbb{R}^N} |u|^{q-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx$$

$$+ \int_{\mathbb{R}^N} |(I_\alpha * F(u))||f(u)|\phi\, dx,$$

that is,

$$\int_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z_\varepsilon \nabla\phi\, dx + \int_{\mathbb{R}^N} |\nabla z|^{q-2}\nabla z_\varepsilon \nabla\phi\, dx$$

$$\le -\int_{\mathbb{R}^N} |u|^{p-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx - \int_{\mathbb{R}^N} |u|^{q-2}u\left(\frac{u}{\sqrt{u^2+\varepsilon^2}}\phi\right) dx$$

$$+ \int_{\mathbb{R}^N} |(I_\alpha * F(u))||f(u)|\phi\, dx.$$

Letting $\varepsilon \to 0$ in the above relation, it follows from the dominated convergence theorem that (3.2) holds for every $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi \ge 0$. By density, (3.2) is valid for all $\phi \in \mathcal{W}_{p,q}$ such that $\phi \ge 0$.

Now we proceed as in [6, Theorem 1.2]. Let $M > 1$ and $z_M := \min\{z, M\}$. Taking $\phi = z_M^{kq+1}$, with $k \ge 0$, in (3.2), we find

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z \nabla(z_M^{kq+1})\, dx + \int_{\mathbb{R}^N} z^{p-1} z_M^{kq+1}\, dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla z|^{q-2}\nabla z \nabla(z_M^{kq+1})\, dx + \int_{\mathbb{R}^N} z^{q-1} z_M^{kq+1}\, dx \\
&\quad \le \int_{\mathbb{R}^N} |(I_\alpha * F(u))||f(u)| z_M^{kq+1}\, dx.
\end{aligned} \tag{3.3}$$

Note that

$$\int_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z \nabla(z_M^{kq+1})\,dx = (kq+1)\int_{\mathbb{R}^N} |\nabla z_M|^p z_M^{kq}\,dx \geq 0, \tag{3.4}$$

and that the Sobolev inequality in Theorem 2.1 gives

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla z|^{q-2}\nabla z \nabla(z_M^{kq+1})\,dx &= \frac{kq+1}{(k+1)^q}\int_{\mathbb{R}^N} |\nabla(z_M^{k+1})|^q\,dx \\ &\geq S_* \frac{kq+1}{(k+1)^q}\left[\int_{\mathbb{R}^N} z_M^{(k+1)q^*}\,dx\right]^{\frac{q}{q^*}},\end{aligned} \tag{3.5}$$

where $S_* = S_*(N,q) > 0$ is the best Sobolev constant of the embedding $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L^{q^*}(\mathbb{R}^N)$. On the other hand, because of $0 \leq z_M \leq z$, we have

$$\int_{\mathbb{R}^N} z^{p-1} z_M^{kq+1}\,dx + \int_{\mathbb{R}^N} z^{q-1} z_M^{kq+1}\,dx \geq \int_{\{|u|<1\}} z_M^{p+kq}\,dx. \tag{3.6}$$

Now, using the continuity of f and (f_1), we know that

$$|F(t)| \leq C(|t|^{\frac{N+\alpha}{N}\frac{p}{2}} + |t|^{\frac{N+\alpha}{N-q}\frac{q}{2}}) \quad \text{for all } t \in \mathbb{R}. \tag{3.7}$$

Combining Theorem 2.2 and (3.7) yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} |(I_\alpha * F(u))||f(u)|z_M^{kq+1}\,dx\right| \\ &\leq C' \|F(u)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|f(u)z_M^{kq+1}\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &\leq C'' \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \|u\|^{q^*}_{L^{q^*}(\mathbb{R}^N)}\right)^{\frac{N+\alpha}{2N}} \|f(u)z_M^{kq+1}\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\ &=: \tilde{C}\,\|f(u)z_M^{kq+1}\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}.\end{aligned} \tag{3.8}$$

Fix $\varepsilon > 0$. From (f_2) and L'Hôpital's rule, it follows that

$$\lim_{t\to 0} \frac{f(t)}{|t|^{\frac{N+\alpha}{N}\frac{p}{2}-1}} = 0,$$

and so there exists $\delta_0 > 0$ such that

$$|f(t)| \leq \varepsilon\,|t|^{\frac{N+\alpha}{N}\frac{p}{2}-1} \quad \text{for all } 0 < |t| < \delta_0. \tag{3.9}$$

Let $0 < \delta < \min\{\delta_0, 1\}$. Thanks to (f_1), there are $K_0 > 0$ and $C_1 > 0$ such that

$$|f(t)| \leq C_1 |t|^{\frac{N+\alpha}{N-q}\frac{q}{2}-1} \quad \text{for all } |t| > K_0. \tag{3.10}$$

Let $K > \max\{K_0, 1\}$. Fix $\mu > K$. Since f is continuous in $\mathbb{R}$, we can find $C_{\delta,\mu} > 0$ such that

$$|f(t)| \leq C_{\delta,\mu} |t|^{\frac{N+\alpha}{N-q}\frac{q}{2}-1} \quad \text{for all } \delta \leq |t| \leq \mu. \tag{3.11}$$

Then, in view of $0 \leq z_M \leq |u|$, (3.9), (3.10), (3.11), and the inequality

$$(t_1 + t_2 + t_3)^a \leq t_1^a + t_2^a + t_3^a \quad \text{for all } t_1, t_2, t_3 \geq 0 \text{ and } a \in (0,1),$$

we obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^N} |f(u) z_M^{kq+1}|^{\frac{2N}{N+\alpha}}\, dx\right]^{\frac{N+\alpha}{2N}} \\
&= \left[\int_{\{|u|<\delta\}} |f(u) z_M^{kq+1}|^{\frac{2N}{N+\alpha}}\, dx + \int_{\{\delta<|u|<\mu\}} |f(u) z_M^{kq+1}|^{\frac{2N}{N+\alpha}}\, dx + \int_{\{|u|>\mu\}} |f(u) z_M^{kq+1}|^{\frac{2N}{N+\alpha}}\, dx\right]^{\frac{N+\alpha}{2N}} \\
&\leq \Bigg[\varepsilon^{\frac{2N}{N+\alpha}} \int_{\{|u|<\delta\}} ||u|^{\frac{N+\alpha}{N}\frac{p}{2}+kq}|^{\frac{2N}{N+\alpha}}\, dx + C_{\delta,\mu}^{\frac{2N}{N+\alpha}} \int_{\{\delta<|u|<\mu\}} ||u|^{\frac{N+\alpha}{N-q}\frac{q}{2}+kq}|^{\frac{2N}{N+\alpha}}\, dx \\
&\quad + C_1^{\frac{2N}{N+\alpha}} \int_{\{|u|>\mu\}} ||u|^{\frac{N+\alpha}{N-q}\frac{q}{2}+kq}|^{\frac{2N}{N+\alpha}}\, dx\Bigg]^{\frac{N+\alpha}{2N}} \\
&\leq \varepsilon \left(\int_{\{|u|<\delta\}} |u|^{(\frac{N+\alpha}{N}\frac{p}{2}+kq)\frac{2N}{N+\alpha}}\, dx\right)^{\frac{N+\alpha}{2N}} + C_{\delta,\mu} \left(\int_{\{\delta<|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}}\, dx\right)^{\frac{N+\alpha}{2N}} \\
&\quad + C_1 \left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}}\, dx\right)^{\frac{N+\alpha}{2N}} \\
&\leq \varepsilon \left(\int_{\{|u|<\delta\}} |u|^p |u|^{\frac{2N}{N+\alpha}kq}\, dx\right)^{\frac{N+\alpha}{2N}} + C_{\delta,\mu} \left(\int_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}}\, dx\right)^{\frac{N+\alpha}{2N}} \\
&\quad + C_1 \left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}}\, dx\right)^{\frac{N+\alpha}{2N}}.
\end{aligned}$$

Note that if $|u| < \delta < 1$ then $|u|^{\frac{2N}{N+\alpha}kq} \leq |u|^{kq}$ (since $\frac{2N}{N+\alpha} > 1$). Hence, exploiting the inequality $t^a \leq t + 1$ for all $t \geq 0$ and $a \in (0, 1)$, we get

$$\varepsilon \left(\int\limits_{\{|u|<\delta\}} |u|^p |u|^{\frac{2N}{N+\alpha}kq} \, dx \right)^{\frac{N+\alpha}{2N}} \leq \varepsilon \left(\int\limits_{\{|u|<\delta\}} |u|^{p+kq} \, dx \right)^{\frac{N+\alpha}{2N}} \leq \varepsilon \int\limits_{\{|u|<\delta\}} |u|^{p+kq} \, dx + \varepsilon \, .$$

Consequently,

$$\begin{aligned} &\left[\int\limits_{\mathbb{R}^N} |f(u) z_M^{kq+1}|^{\frac{2N}{N+\alpha}} \, dx \right]^{\frac{N+\alpha}{2N}} \\ &\leq \varepsilon \int\limits_{\{|u|<\delta\}} |u|^{p+kq} \, dx + \varepsilon + C_{\delta,\mu} \left(\int\limits_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \\ &\quad + C_1 \left(\int\limits_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} . \end{aligned} \tag{3.12}$$

Let us observe that $z_M = |u|$ on $\{|u| < \delta\}$ (because $0 < \delta < 1 < M$), and thus

$$\int\limits_{\{|u|<\delta\}} |u|^{p+kq} \, dx = \int\limits_{\{|u|<\delta\}} z_M^{p+kq} \, dx.$$

Combining (3.3), (3.4), (3.5), (3.6), (3.8), and (3.12), we find

$$\begin{aligned} &S_* \frac{kq+1}{(k+1)^q} \left[\int\limits_{\mathbb{R}^N} z_M^{(k+1)q^*} \, dx \right]^{\frac{q}{q^*}} + (1 - \tilde{C}\,\varepsilon) \int\limits_{\{|u|<\delta\}} z_M^{p+kq} \, dx \\ &\leq \tilde{C}\,\varepsilon + \tilde{C} C_{\delta,\mu} \left(\int\limits_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} + \tilde{C} C_1 \left(\int\limits_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} . \end{aligned}$$

Letting first $\varepsilon \to 0$ and then $M \to \infty$, we obtain

$$
\begin{aligned}
&S_* \frac{kq+1}{(k+1)^q} \left[\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right]^{\frac{q}{q^*}} \\
&\leq \tilde{C} C_{\delta,\mu} \left(\int_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} + \tilde{C} C_1 \left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} .
\end{aligned}
\tag{3.13}
$$

Now, we note that

$$
\begin{aligned}
\left(\int_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} &= \left(\int_{\{|u|<\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}-q+(k+1)q)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \\
&\leq \mu^{\frac{N+\alpha}{N-q}\frac{q}{2}-q} \left(\int_{\{|u|<\mu\}} |u|^{(k+1)\frac{2Nq}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \\
&\leq \mu^{\frac{N+\alpha}{N-q}\frac{q}{2}-q} \left(\int_{\mathbb{R}^N} |u|^{(k+1)\frac{2Nq}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} .
\end{aligned}
\tag{3.14}
$$

On the other hand, exploiting Hölder's inequality, we see that

$$
\begin{aligned}
\left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}+kq)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} &= \left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}-q+(k+1)q)\frac{2N}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \\
&= \left(\int_{\{|u|>\mu\}} |u|^{(\frac{N+\alpha}{N-q}\frac{q}{2}-q)\frac{2N}{N+\alpha}} |u|^{(k+1)\frac{2Nq}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \\
&\leq \left(\int_{\{|u|>\mu\}} |u|^{q^*} \, dx \right)^{\frac{\alpha+2q-N}{2N}} \left(\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right)^{\frac{q}{q^*}} \\
&=: D(\mu) \left(\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right)^{\frac{q}{q^*}} .
\end{aligned}
\tag{3.15}
$$

Thanks to (3.13), (3.14), and (3.15), we arrive at

$$S_* \frac{kq+1}{(k+1)^q} \left[\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right]^{\frac{q}{q^*}}$$
$$\leq \tilde{C} C_{\delta,\mu} \, \mu^{\frac{N+\alpha}{N-q}\frac{q}{2}-q} \left(\int_{\mathbb{R}^N} |u|^{(k+1)\frac{2Nq}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} + \tilde{C} C_1 D(\mu) \left(\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right)^{\frac{q}{q^*}}.$$

Since $\alpha + 2q - N > 0$ implies that $D(\mu) \to 0$ as $\mu \to \infty$, we can select $\mu > K$ sufficiently large such that

$$0 \leq D(\mu) < \vartheta \frac{S_*}{\tilde{C} C_1} \frac{kq+1}{(k+1)^q} \quad \text{ with } \vartheta \in (0,1).$$

Therefore,

$$(1-\vartheta) S_* \frac{kq+1}{(k+1)^q} \left[\int_{\mathbb{R}^N} |u|^{(k+1)q^*} \, dx \right]^{\frac{q}{q^*}} \leq \tilde{C} C_{\delta,\mu} \, \mu^{\frac{N+\alpha}{N-q}\frac{q}{2}-q} \left(\int_{\mathbb{R}^N} |u|^{(k+1)\frac{2Nq}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}},$$

and thus

$$\|u\|_{L^{(k+1)q^*}(\mathbb{R}^N)} \leq C_*^{\frac{1}{k+1}} \left[\frac{k+1}{(kq+1)^{\frac{1}{q}}} \right]^{\frac{1}{k+1}} \|u\|_{L^{(k+1)\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)}, \tag{3.16}$$

for some constant $C_* > 0$ depending on k. Put $\tau := \frac{2Nq}{N+\alpha}$. Note that $q^* > \tau$ due to $\alpha > N - 2q$. Now we use a bootstrap argument. Because $u \in L^{q^*}(\mathbb{R}^N)$, we can apply (3.16) with $k+1 = \frac{q^*}{\tau}$ to deduce that $u \in L^{(k+1)q^*}(\mathbb{R}^N) = L^{\frac{(q^*)^2}{\tau}}(\mathbb{R}^N)$. Again employing (3.16), after m iterations, we have that $u \in L^{q^*(\frac{q^*}{\tau})^m}(\mathbb{R}^N)$ and so $u \in L^\nu(\mathbb{R}^N)$ for all $\nu \in [q^*, \infty)$. In light of $u \in L^s(\mathbb{R}^N)$ for all $s \in [p, q^*]$, we can conclude that (3.1) is valid.

Next we show that

$$u \in L^\infty(\mathbb{R}^N). \tag{3.17}$$

Exploiting (3.1) and (3.7), it follows from Young's inequality for convolutions that

$$\mathcal{K} := I_\alpha * F(u) \in C_0(\mathbb{R}^N) := \{ v \in C(\mathbb{R}^N) : |v(x)| \to 0 \text{ as } |x| \to \infty \}.$$

Then u solves

$$-\Delta_p u - \Delta_q u = \psi(x,u) \quad \text{ in } \mathbb{R}^N, \tag{3.18}$$

where

$$\psi(x,u) := -|u|^{p-2}u - |u|^{q-2}u + \mathcal{K}(x)f(u)$$

satisfies

$$|\psi(x,t)| \le C(1+|t|^{r-1}) \quad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.19}$$

for some $r \in (q, q^*)$ and $C > 0$. Note that (3.19) is a consequence of (f_1), $p < q$, $\mathcal{K} \in L^\infty(\mathbb{R}^N)$, and $\alpha \in ((N-2q)^+, N)$. Next we develop a De Giorgi type iteration [17] to prove that (3.17) is true. Assume that $u^+ \neq 0$. Let

$$\rho \ge \max\{1, \|u\|^{-1}_{L^r(\mathbb{R}^N)}\},$$

and define

$$v := (\rho\|u\|_{L^r(\mathbb{R}^N)})^{-1}u.$$

Set $w_k := (v - 1 + 2^{-k})^+$ for $k \in \mathbb{N}$ and $w_0 := v^+$. Evidently, $w_k \in \mathcal{W}_{p,q}$ and $0 \le w_{k+1} \le w_k$ a.e. in $\mathbb{R}^N$. Let $U_k := \|w_k\|^r_{L^r(\mathbb{R}^N)}$. Put

$$\Omega_k := \{x \in \mathbb{R}^N : w_k(x) > 0\}.$$

Observe that $\Omega_{k+1} \subset \{w_k > 2^{-(k+1)}\}$, $v(x) < 2^{k+1}w_k(x)$ for $x \in \Omega_{k+1}$ and $|\Omega_{k+1}| \le 2^{(k+1)r}U_k$. Testing (3.18) with w_{k+1}, and using $\rho\|u\|_{L^r(\mathbb{R}^N)} \ge 1$ and (3.19), we can see that

$$\begin{aligned}
\|\nabla w_{k+1}\|^q_{L^q(\mathbb{R}^N)} &\le (\rho\|u\|_{L^r(\mathbb{R}^N)})^{p-q}\|\nabla w_{k+1}\|^p_{L^p(\mathbb{R}^N)} + \|\nabla w_{k+1}\|^q_{L^q(\mathbb{R}^N)} \\
&= (\rho\|u\|_{L^r(\mathbb{R}^N)})^{1-q}\left[\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla w_{k+1}\,dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\nabla w_{k+1}\,dx\right] \\
&= (\rho\|u\|_{L^r(\mathbb{R}^N)})^{1-q}\left[\int_{\mathbb{R}^N} \psi(x,u)w_{k+1}\,dx\right] \\
&\le C(\rho\|u\|_{L^r(\mathbb{R}^N)})^{1-q}\left[\int_{\Omega_{k+1}\cap\{|u|\le 1\}} w_{k+1}\,dx + \int_{\Omega_{k+1}\cap\{|u|>1\}} |u|^{r-1}w_k\,dx\right] \\
&\le C(\rho\|u\|_{L^r(\mathbb{R}^N)})^{1-q}\left[2^{-(k+1)}|\Omega_{k+1}| + (\rho\|u\|_{L^r(\mathbb{R}^N)})^{r-1}2^{(k+1)(r-1)}U_k\right] \\
&\le 2C2^{(k+1)(r-1)}(\rho\|u\|_{L^r(\mathbb{R}^N)})^{r-q}U_k,
\end{aligned} \tag{3.20}$$

where we have used

$$w_{k+1}(x) \le 2^{-(k+1)} \quad \text{for } x \in \Omega_{k+1} \cap \{|u| \le 1\},$$

and

$$0 \le u(x) < \rho \|u\|_{L^r(\mathbb{R}^N)} 2^{k+1} w_k(x) \quad \text{for } x \in \Omega_{k+1}.$$

By virtue of (3.20) and utilizing the Hölder and Sobolev inequalities, we arrive at

$$U_{k+1} \le \|w_{k+1}\|^r_{L^{q^*}(\mathbb{R}^N)} |\Omega_{k+1}|^{1-\frac{r}{q^*}} \le \bar{C}^k (\rho \|u\|_{L^r(\mathbb{R}^N)})^{\frac{r^2}{q}-r} U_k^{1+\frac{r}{N}}, \tag{3.21}$$

for some $\bar{C} > 1$ independent of k. Let $\omega := \bar{C}^{-\frac{N}{r}}$. Clearly, $0 < \omega < 1$. Select

$$\rho := \max \left\{ 1, \|u\|^{-1}_{L^r(\mathbb{R}^N)}, \left(\|u\|^{\frac{r^2}{q}-r}_{L^r(\mathbb{R}^N)} \omega^{-1} \right)^{\frac{1}{\beta}} \right\},$$

where

$$\beta := \frac{r^2}{N} + r - \frac{r^2}{q} > 0.$$

Arguing by induction, we will prove that

$$U_k \le \frac{\omega^k}{\rho^r} \quad \text{for all } k \in \mathbb{N} \cup \{0\}. \tag{3.22}$$

If $k = 0$, then $U_0 = \|v^+\|^r_{L^r(\mathbb{R}^N)} \le \|v\|^r_{L^r(\mathbb{R}^N)} = \rho^{-r}$. Next we assume that (3.22) is true for some $k \ge 0$ and we show that it also holds for $k + 1$. Using (3.21), the induction hypothesis, $(\bar{C}\omega^{\frac{r}{N}})^k = 1$, the definition of β, and that $\omega \ge \|u\|^{\frac{r^2}{q}-r}_{L^r(\mathbb{R}^N)} \rho^{-\beta}$, we find

$$\begin{aligned} U_{k+1} &\le \bar{C}^k (\rho \|u\|_{L^r(\mathbb{R}^N)})^{\frac{r^2}{q}-r} U_k^{1+\frac{r}{N}} \\ &\le \bar{C}^k (\rho \|u\|_{L^r(\mathbb{R}^N)})^{\frac{r^2}{q}-r} \left(\frac{\omega^k}{\rho^r} \right)^{1+\frac{r}{N}} \\ &= \frac{\omega^k}{\rho^r} \|u\|^{\frac{r^2}{q}-r}_{L^r(\mathbb{R}^N)} \rho^{-\beta} \\ &\le \frac{\omega^{k+1}}{\rho^r}. \end{aligned}$$

This completes the proof of (3.22). In particular, we deduce that $U_k \to 0$ as $k \to \infty$. Because $w_k \to (v-1)^+$ a.e. in $\mathbb{R}^N$ as $k \to \infty$ and $w_k \le v \in L^r(\mathbb{R}^N)$, it follows from the dominated convergence theorem that $\|(v-1)^+\|_{L^r(\mathbb{R}^N)} = 0$. Therefore, $v \le 1$ a.e. in $\mathbb{R}^N$, namely, $\|u\|_{L^\infty(\mathbb{R}^N)} \le \rho \|u\|_{L^r(\mathbb{R}^N)}$. Hence, (3.17) is valid. Thus, $u \in \mathcal{W}_{p,q} \cap L^\infty(\mathbb{R}^N)$ is a weak solution to (3.18) with $\psi \in L^\infty(\mathbb{R}^N)$, and we can invoke [21, Theorem 1] to infer that $u \in C^{1,\lambda}_{loc}(\mathbb{R}^N)$ for some $\lambda \in (0, 1)$. □

3.2. A Pohozaev type identity

Now we show that every weak solution to (1.1) fulfills the Pohozaev identity (1.6). Firstly, we recall the following useful result.

Lemma 3.1. *[18, Lemma 1] Let $\Omega \subset \mathbb{R}^N$ be an open set, $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a function of class C^1 and $g \in L^{\infty}_{loc}(\Omega)$. Assume also that $\xi \mapsto \mathcal{L}(x, s, \xi)$ is strictly convex for each $(x, s) \in \Omega \times \mathbb{R}$. Let $u : \Omega \times \mathbb{R}$ be a locally Lipschitz solution of*

$$-\operatorname{div}\{\nabla_\xi \mathcal{L}(x, u, \nabla u)\} + D_s \mathcal{L}(x, u, \nabla u) = g \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$\sum_{i,j=1}^{N} \int_{\Omega} D_i h_j D_{\xi_i} \mathcal{L}(x, u, \nabla u) D_j u \, dx - \int_{\Omega} [(\operatorname{div} h)\mathcal{L}(x, u, \nabla u) + h \nabla_x \mathcal{L}(x, u, \nabla u)] \, dx = \int_{\Omega} (h \nabla u) g \, dx \tag{3.23}$$

for every $h \in C^1_c(\Omega; \mathbb{R}^N)$.

Proposition 3.2. *Assume that f satisfies (f_1)–(f_2). If $u \in \mathcal{W}_{p,q}$ is a weak solution to (1.1), then u obeys (1.6).*

Proof. Let us choose $\mathcal{L}(x, s, \xi) = \frac{1}{p}|\xi|^p + \frac{1}{q}|\xi|^q$ and $g = -|u|^{p-2}u - |u|^{q-2}u + (I_\alpha * F(u)) f(u)$ in Lemma 3.1. Take $\varphi \in C^1_c(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ in $\mathbb{R}^N$, $\varphi(x) = 1$ for $|x| \le 1$ and $\varphi(x) = 0$ for $|x| \ge 2$. Set

$$h(x) := \varphi\left(\frac{x}{k}\right) x \in C^1(\mathbb{R}^N; \mathbb{R}^N).$$

Note that if $h_j(x) := \varphi\left(\frac{x}{k}\right) x_j$, for $j = 1, \dots, N$, then

$$D_i h_j(x) = D_i \varphi\left(\frac{x}{k}\right) \frac{x_j}{k} + \varphi\left(\frac{x}{k}\right) \delta_{ij}, \qquad \text{for all } x \in \mathbb{R}^N,\ j = 1, \dots, N,$$
$$\operatorname{div} h(x) = \nabla \varphi\left(\frac{x}{k}\right) \frac{x}{k} + N \varphi\left(\frac{x}{k}\right) \qquad \text{for all } x \in \mathbb{R}^N,$$

where δ_{ij} denotes the Kronecker delta symbol. Moreover,

$$\left| D_i \varphi\left(\frac{x}{k}\right) \frac{x_j}{k} \right| \le C \qquad \text{for all } x \in \mathbb{R}^N,\ i, j = 1, \dots, N. \tag{3.24}$$

In light of (3.23), we see that

$$\begin{aligned}&\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} D_i\varphi\left(\frac{x}{k}\right)\frac{x_j}{k}D_{\xi_i}\mathcal{L}(x,u,\nabla u)D_j u\,dx+\int_{\mathbb{R}^N}\varphi\left(\frac{x}{k}\right)D_{\xi}\mathcal{L}(x,u,\nabla u)\nabla u\,dx\\&\quad-\int_{\Omega}\left[\nabla\varphi\left(\frac{x}{k}\right)\frac{x}{k}\mathcal{L}(x,u,\nabla u)+N\varphi\left(\frac{x}{k}\right)\mathcal{L}(x,u,\nabla u)\right]dx\\&\quad=\int_{\mathbb{R}^N}\left(\varphi\left(\frac{x}{k}\right)x\nabla u\right)g\,dx.\end{aligned}\tag{3.25}$$

From (3.24), $\varphi\left(\frac{x}{k}\right)\to 1$ and $\nabla\varphi(\frac{x}{k})\frac{x}{k}\to 0$ as $k\to\infty$, $(|\nabla u|^p+|\nabla u|^q)\in L^1(\mathbb{R}^N)$, and applying the dominated convergence theorem, we obtain that

$$\begin{aligned}&\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} D_i\varphi\left(\frac{x}{k}\right)\frac{x_j}{k}D_{\xi_i}\mathcal{L}(x,u,\nabla u)D_j u\,dx+\int_{\mathbb{R}^N}\varphi\left(\frac{x}{k}\right)D_{\xi}\mathcal{L}(x,u,\nabla u)\nabla u\,dx\\&\quad-\int_{\Omega}\left[\nabla\varphi\left(\frac{x}{k}\right)\frac{x}{k}\mathcal{L}(x,u,\nabla u)+N\varphi\left(\frac{x}{k}\right)\mathcal{L}(x,u,\nabla u)\right]dx\\&\quad\to\left(1-\frac{N}{p}\right)\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\left(1-\frac{N}{q}\right)\int_{\mathbb{R}^N}|\nabla u|^q\,dx,\end{aligned}\tag{3.26}$$

as $k\to\infty$. On the other hand, an integration by parts and the dominated convergence theorem show that, for all $s\in\{p,q\}$,

$$\begin{aligned}&-\int_{\mathbb{R}^N}\left(\varphi\left(\frac{x}{k}\right)x\nabla u\right)|u|^{s-2}u\,dx\\&=-\int_{\mathbb{R}^N}\varphi\left(\frac{x}{k}\right)x\nabla\left(\frac{|u|^s}{s}\right)dx\\&=\int_{\mathbb{R}^N}\left[N\varphi\left(\frac{x}{k}\right)+\nabla\varphi\left(\frac{x}{k}\right)\frac{x}{k}\right]\frac{|u|^s}{s}\,dx\to\frac{N}{s}\int_{\mathbb{R}^N}|u|^s\,dx,\end{aligned}\tag{3.27}$$

as $k\to\infty$. Finally, an integration by parts yields

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\varphi\left(\frac{x}{k}\right)x\nabla u\right)(I_\alpha*F(u))f(u)\,dx\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(F\circ u)(y)I_\alpha(x-y)\varphi\left(\frac{x}{k}\right)x\nabla(F\circ u)(x)\,dxdy\\&=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}I_\alpha(x-y)\left((F\circ u)(y)\varphi\left(\frac{x}{k}\right)x\nabla(F\circ u)(x)+(F\circ u)(x)\varphi\left(\frac{y}{k}\right)y\nabla(F\circ u)(y)\right)dxdy\end{aligned}$$

$$= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u(y))I_\alpha(x-y)\left[N\varphi\left(\frac{x}{k}\right)+x\nabla\varphi\left(\frac{x}{k}\right)\right]F(u(x))\,dxdy$$

$$+\frac{N-\alpha}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u(y))I_\alpha(x-y)\frac{(x-y)(x\varphi\left(\frac{x}{k}\right)-y\varphi\left(\frac{y}{k}\right))}{|x-y|^2}F(u(x))\,dxdy,$$

and invoking the dominated convergence theorem we deduce that

$$\int_{\mathbb{R}^N}\left(\varphi\left(\frac{x}{k}\right)x\nabla u\right)(I_\alpha * F(u))f(u)\,dx \to -\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)\,dx, \tag{3.28}$$

as $k\to\infty$. Combining (3.25), (3.26), (3.27) and (3.28), we arrive at (1.6). This concludes the proof of Proposition 3.2. □

Proof of Theorem 1.1. This is a direct consequence of Propositions 3.1 and 3.2. □

4. Proof of Theorem 1.2

In this section we focus on the existence of a ground state solution to (1.1). We start by proving the existence of a Pohozaev-Palais-Smale sequence at the level b defined in (1.8).

Proposition 4.1. *Assume that f satisfies (f_1) and (f_3). Then there exists a sequence $(u_n)_{n\in\mathbb{N}}\subset \mathcal{W}_{p,q}$ satisfying (1.9).*

Proof. First we show that $\Gamma\neq\emptyset$. In view of the definition of Γ, it is sufficient to construct $u\in\mathcal{W}_{p,q}$ such that $\mathcal{I}(u)<0$. Let $\mathcal{F}:\mathcal{W}_{p,q}\to\mathbb{R}$ be defined as

$$\mathcal{F}(u) := \int_{\mathbb{R}^N}(I_\alpha * F(u))\,F(u)\,dx.$$

From (f_1), we know that $\mathcal{F}$ is continuous in $L^p(\mathbb{R}^N)\cap L^{q^*}(\mathbb{R}^N)$. Put

$$w(x) := t_0\,\chi_{B_1}(x),$$

where t_0 is given by (f_3). Note that

$$\mathcal{F}(w) = F(t_0)^2\int_{B_1}\int_{B_1} I_\alpha(x-y)\,dxdy > 0. \tag{4.1}$$

Exploiting the fact that $\mathcal{W}_{p,q}$ is dense in $L^p(\mathbb{R}^N)\cap L^{q^*}(\mathbb{R}^N)$ and (4.1), we can find $v\in\mathcal{W}_{p,q}$ such that $\mathcal{F}(v)>0$. Now, for all $\tau>0$ and $x\in\mathbb{R}^N$, we set

$$v_\tau(x) := v\left(\frac{x}{\tau}\right).$$

Since $N-q<N-p<N<N+\alpha$, we have

$$\mathcal{I}(v_\tau)=\frac{\tau^{N-p}}{p}\|\nabla v\|^p_{L^p(\mathbb{R}^N)}+\frac{\tau^N}{p}\|v\|^p_{L^p(\mathbb{R}^N)}$$
$$+\frac{\tau^{N-q}}{q}\|\nabla v\|^q_{L^q(\mathbb{R}^N)}+\frac{\tau^N}{q}\|v\|^q_{L^q(\mathbb{R}^N)}-\frac{\tau^{N+\alpha}}{2}\mathcal{F}(v)\to-\infty \quad \text{as } \tau\to\infty.$$

Then, taking τ large enough, we arrive at $\mathcal{I}(v_\tau)<0$. Hence, $\Gamma\neq\emptyset$. In particular we deduce that $b<\infty$.

Next we prove that $b>0$. Using Theorem 2.2, (3.7), the inequality

$$(a_1+a_2)^r\leq 2^{r-1}(a_1^r+a_2^r) \quad \text{for all } a_1,a_2\geq 0, r\geq 1, \tag{4.2}$$

and Theorem 2.1, we see that, for all $u\in\mathcal{W}_{p,q}$,

$$\begin{aligned}\mathcal{F}(u)&\leq C_0\|F(u)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}\\&\leq C_1\left(\|u\|^p_{L^p(\mathbb{R}^N)}+\|u\|^{q^*}_{L^{q^*}(\mathbb{R}^N)}\right)^{1+\frac{\alpha}{N}}\\&\leq C_2\left[\|u\|^{p(1+\frac{\alpha}{N})}_{L^p(\mathbb{R}^N)}+\|u\|^{q^*(1+\frac{\alpha}{N})}_{L^{q^*}(\mathbb{R}^N)}\right]\\&\leq C_3\left[\|u\|^{p(1+\frac{\alpha}{N})}_{W^{1,p}(\mathbb{R}^N)}+\|u\|^{q(\frac{N+\alpha}{N-q})}_{W^{1,q}(\mathbb{R}^N)}\right].\end{aligned}$$

Consequently, for all $u\in\mathcal{W}_{p,q}$,

$$\begin{aligned}\mathcal{I}(u)&=\frac{1}{p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)}+\frac{1}{q}\|u\|^q_{W^{1,q}(\mathbb{R}^N)}-\frac{1}{2}\mathcal{F}(u)\\&\geq\frac{1}{p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)}+\frac{1}{q}\|u\|^q_{W^{1,q}(\mathbb{R}^N)}-\frac{C_3}{2}\|u\|^{p(1+\frac{\alpha}{N})}_{W^{1,p}(\mathbb{R}^N)}-\frac{C_3}{2}\|u\|^{q(\frac{N+\alpha}{N-q})}_{W^{1,q}(\mathbb{R}^N)}.\end{aligned}$$

Pick $\rho>0$ such that

$$0<\rho<\min\left\{1,\frac{1}{(pC_3)^{\frac{N}{\alpha p}}},\frac{1}{(qC_3)^{\frac{N-q}{q(\alpha+q)}}}\right\}.$$

Let $u\in\mathcal{W}_{p,q}$ be such that $\|u\|_{\mathcal{W}_{p,q}}\leq\rho$. Then we obtain

$$\mathcal{I}(u)\geq\frac{1}{2p}\|u\|^p_{W^{1,p}(\mathbb{R}^N)}+\frac{1}{2q}\|u\|^q_{W^{1,q}(\mathbb{R}^N)}.$$

Let us observe that $1<p<q$ and $\|u\|_{W^{1,p}(\mathbb{R}^N)}<1$ imply that $\|u\|^p_{W^{1,p}(\mathbb{R}^N)}\geq\|u\|^q_{W^{1,p}(\mathbb{R}^N)}$. This fact together with (4.2) yields

$$
\begin{aligned}
\mathcal{I}(u) &\geq \frac{1}{2q}\left(\|u\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u\|^q_{W^{1,q}(\mathbb{R}^N)}\right) \\
&\geq \frac{1}{2q}\left(\|u\|^q_{W^{1,p}(\mathbb{R}^N)} + \|u\|^q_{W^{1,q}(\mathbb{R}^N)}\right) \\
&\geq C_4\left(\|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,q}(\mathbb{R}^N)}\right)^q \\
&= C_4\|u\|^q_{\mathcal{W}_{p,q}}.
\end{aligned}
$$

Note that, if $\gamma \in \Gamma$, then

$$
\|\gamma(1)\|_{\mathcal{W}_{p,q}} > \rho.
$$

Indeed, if by contradiction $\|\gamma(1)\|_{\mathcal{W}_{p,q}} \leq \rho$, then

$$
\mathcal{I}(\gamma(1)) \geq C_4\|\gamma(1)\|^q_{\mathcal{W}_{p,q}} \geq 0,
$$

which is in contrast with $\gamma \in \Gamma$. Hence,

$$
\|\gamma(0)\|_{\mathcal{W}_{p,q}} = 0 < \rho < \|\gamma(1)\|_{\mathcal{W}_{p,q}},
$$

and applying the intermediate value theorem we can find $\bar{\tau} \in (0,1)$ such that $\|\gamma(\bar{\tau})\|_{\mathcal{W}_{p,q}} = \rho$. Thus,

$$
C_4\rho^q \leq \mathcal{I}(\gamma(\bar{\tau})) \leq \max_{t\in[0,1]} \mathcal{I}(\gamma(t)).
$$

Due to the arbitrariness of $\gamma \in \Gamma$, we can deduce that

$$
b \geq C_4\rho^q > 0.
$$

Now we prove that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{p,q}$ satisfying (1.9). As in [22], we consider the map $\Phi : \mathbb{R} \times \mathcal{W}_{p,q} \to \mathcal{W}_{p,q}$ given by

$$
\Phi(\theta, u)(x) := u(e^{-\theta}x) \quad \text{ for } (\theta, u) \in \mathbb{R} \times \mathcal{W}_{p,q}.
$$

Here $\mathbb{R} \times \mathcal{W}_{p,q}$ is equipped with the norm

$$
\|(\theta, u)\|_{\mathbb{R}\times\mathcal{W}_{p,q}} := |\theta| + \|u\|_{\mathcal{W}_{p,q}}.
$$

Set $\tilde{\mathcal{I}} := \mathcal{I} \circ \Phi$. For every $(\theta, u) \in \mathbb{R} \times \mathcal{W}_{p,q}$, we have

$$
\begin{aligned}
\tilde{\mathcal{I}}(\theta, u) = &\frac{e^{(N-p)\theta}}{p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{e^{N\theta}}{p}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&+ \frac{e^{(N-q)\theta}}{q}\|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \frac{e^{N\theta}}{q}\|u\|^q_{L^q(\mathbb{R}^N)}
\end{aligned}
$$

$$-\frac{e^{(N+\alpha)\theta}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))\,F(u)\,dx.$$

It is easy to check that $\tilde{\mathcal{I}} \in C^1(\mathbb{R}\times\mathcal{W}_{p,q},\mathbb{R})$. In view of the previous arguments, $\tilde{\mathcal{I}}$ has a mountain pass geometry and so we can define the mountain pass level of $\tilde{\mathcal{I}}$ by setting

$$\tilde{b} := \inf_{\tilde{\gamma}\in\tilde{\Gamma}}\max_{t\in[0,1]}\tilde{\mathcal{I}}(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma} := \left\{\tilde{\gamma}\in C([0,1];\mathbb{R}\times\mathcal{W}_{p,q}) : \tilde{\gamma}(0)=(0,0) \text{ and } \tilde{\mathcal{I}}(\tilde{\gamma}(1))<0\right\}.$$

One readily verifies that $b=\tilde{b}$. Invoking the general minimax principle [43, Theorem 2.8], we can produce a sequence $((\theta_n,v_n))_{n\in\mathbb{N}}\subset\mathbb{R}\times\mathcal{W}_{p,q}$ such that, as $n\to\infty$,

(i) $\tilde{\mathcal{I}}(\theta_n,v_n)\to b$,
(ii) $\tilde{\mathcal{I}}'(\theta_n,v_n)\to 0$ in $(\mathbb{R}\times\mathcal{W}_{p,q})^*$,
(iii) $\theta_n\to 0$.

In fact, thanks to (1.8), there exists $(\gamma_n)_{n\in\mathbb{N}}\subset\Gamma$ such that

$$\max_{t\in[0,1]}\mathcal{I}(\gamma_n(t))\le b+\frac{1}{n^2}.$$

Define $\tilde{\gamma}_n(t):=(0,\gamma_n(t))\in\tilde{\Gamma}$. Therefore,

$$\max_{t\in[0,1]}\tilde{\mathcal{I}}(\tilde{\gamma}_n(t))=\max_{t\in[0,1]}\mathcal{I}(\gamma_n(t))\le b+\frac{1}{n^2}.$$

From [43, Theorem 2.8], we can find $((\theta_n,v_n))_{n\in\mathbb{N}}\subset\mathbb{R}\times\mathcal{W}_{p,q}$ such that (i) and (ii) are fulfilled, and

$$\mathrm{dist}_{\mathbb{R}\times\mathcal{W}_{p,q}}((\theta_n,v_n),\{0\}\times\gamma_n([0,1]))\to 0,$$

which yields (iii). Here we have used the notation

$$\mathrm{dist}_{\mathbb{R}\times\mathcal{W}_{p,q}}((\theta,u),A):=\inf_{(t,v)\in A}\left(|\theta-t|+\|u-v\|_{\mathcal{W}_{p,q}}\right)\quad\text{for all } A\subset\mathbb{R}\times\mathcal{W}_{p,q}.$$

Now, for all $(h,w)\in\mathbb{R}\times\mathcal{W}_{p,q}$, it holds

$$\langle\tilde{\mathcal{I}}'(\theta_n,v_n),(h,w)\rangle=\langle\mathcal{I}'(\Phi(\theta_n,v_n)),\Phi(\theta_n,w)\rangle+\mathcal{P}(\Phi(\theta_n,v_n))h. \tag{4.3}$$

Put $u_n:=\Phi(\theta_n,v_n)$. Owing to (i), we obtain that $\mathcal{I}(u_n)\to b$. Taking $h=1$ and $w=0$ in (4.3), and utilizing (ii), we get

$$\mathcal{P}(u_n) \to 0.$$

Fix $\phi \in \mathcal{W}_{p,q}$. Applying (4.3) with $h = 0$ and $w(x) = \phi(e^{\theta_n} x)$, and using (ii) and (iii), we deduce that

$$\langle \mathcal{I}'(u_n), \phi \rangle = o_n(1) \|\phi(e^{\theta_n} \cdot)\|_{\mathcal{W}_{p,q}} = o_n(1) \|\phi\|_{\mathcal{W}_{p,q}},$$

and thus $\mathcal{I}'(u_n) \to 0$ in $\mathcal{W}^*_{p,q}$. Accordingly, the sequence $(u_n)_{n\in\mathbb{N}}$ has the required properties. □

Lemma 4.1. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{p,q}$ be a sequence such that, as $n \to \infty$,*

$$\begin{aligned} &(\mathcal{I}(u_n))_{n\in\mathbb{N}} \text{ is bounded}, \\ &\mathcal{I}'(u_n) \to 0 \text{ in } \mathcal{W}^*_{p,q}, \\ &\mathcal{P}(u_n) \to 0. \end{aligned} \tag{4.4}$$

Then, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$. Moreover, there exists $u \in \mathcal{W}_{p,q}$ such that, up to a subsequence, as $n \to \infty$,

$$\begin{aligned} &u_n \rightharpoonup u && \text{in } \mathcal{W}_{p,q}, \\ &u_n \to u && \text{in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, q^*), \\ &u_n \to u && \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.5}$$

Proof. Since, for all $n \in \mathbb{N}$, it holds

$$\begin{aligned} \mathcal{I}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n) = &\frac{\alpha+p}{p(N+\alpha)} \|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)} \|u_n\|^p_{L^p(\mathbb{R}^N)} \\ &+ \frac{\alpha+q}{q(N+\alpha)} \|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)} \|u_n\|^q_{L^q(\mathbb{R}^N)}, \end{aligned}$$

it follows from (4.4) that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$. Hence, employing the reflexivity of $\mathcal{W}_{p,q}$ and that $\mathcal{W}_{p,q}$ is compactly embedded in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$, we can infer that (4.5) is true. □

The next result guarantees the almost everywhere convergence of the gradients of Pohozaev-Palais-Smale sequences.

Lemma 4.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{W}_{p,q}$ such that $\mathcal{I}'(u_n) \to 0$ in $\mathcal{W}^*_{p,q}$ as $n \to \infty$. Up to a subsequence, we assume that (4.5) is valid for some $u \in \mathcal{W}_{p,q}$. Then, up to a subsequence, as $n \to \infty$,*

$$\begin{aligned} &\nabla u_n \to \nabla u && \text{a.e. in } \mathbb{R}^N, \\ &|\nabla u_n|^{s-2}\nabla u_n \rightharpoonup |\nabla u|^{s-2}\nabla u && \text{in } (L^{\frac{s}{s-1}}(\mathbb{R}^N))^N \text{ for all } s \in \{p, q\}. \end{aligned} \tag{4.6}$$

Proof. The proof is inspired by [7, Proposition 4.1] (see also [13, Theorem 2.1]). Fix $\varepsilon > 0$ and define the truncation function $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ at height ε by setting

$$T_\varepsilon(t) := \begin{cases} t & \text{if } |t| \le \varepsilon, \\ \varepsilon \frac{t}{|t|} & \text{if } |t| \ge \varepsilon. \end{cases}$$

Take $R > 0$ and let $\psi_R \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \le \psi_R \le 1$ in $\mathbb{R}^N$, $\psi_R = 1$ in B_R and $\psi_R = 0$ in B_{2R}^c. We can write

$$\begin{aligned}
&\int_{\mathbb{R}^N} \psi_R \Big[|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{p-2}\nabla u - |\nabla u|^{q-2}\nabla u\Big] \nabla T_\varepsilon(u_n - u)\,dx \\
&= -\int_{\mathbb{R}^N} T_\varepsilon(u_n - u)\Big[|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n\Big]\nabla \psi_R\,dx \\
&\quad - \int_{\mathbb{R}^N} \psi_R \Big[|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\Big]\nabla T_\varepsilon(u_n - u)\,dx + \langle \mathcal{I}'(u_n), \psi_R T_\varepsilon(u_n - u)\rangle \qquad (4.7) \\
&\quad - \int_{\mathbb{R}^N} (|u_n|^{p-2} + |u_n|^{q-2}) u_n \psi_R T_\varepsilon(u_n - u)\,dx \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n) \psi_R T_\varepsilon(u_n - u)\,dx.
\end{aligned}$$

Because of (4.5), we see that $T_\varepsilon(u_n - u) \rightharpoonup 0$ in $\mathcal{W}_{p,q}$ and $T_\varepsilon(u_n - u) \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$, as $n \to \infty$. These facts together with $\mathcal{I}'(u_n) \to 0$ in $\mathcal{W}^*_{p,q}$ imply that the first three terms of the right-hand side of (4.7) go to zero as $n \to \infty$. On the other hand, using $|T_\varepsilon(t)| \le \varepsilon$ for all $t \in \mathbb{R}$, $0 \le \psi_R \le 1$, the Hölder inequality, and the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, we get

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} |u_n|^{s-2} u_n \psi_R T_\varepsilon(u_n - u)\,dx \right| &\le \varepsilon \int_{\mathbb{R}^N} |u_n|^{s-1} |\psi_R|\,dx \\
&\le \varepsilon \left(\int_{\mathbb{R}^N} |u_n|^s\,dx \right)^{\frac{s-1}{s}} \left(\int_{\mathbb{R}^N} |\psi_R|^s\,dx \right)^{\frac{1}{s}} \\
&\le C'_R\, \varepsilon \qquad \text{for } s \in \{p, q\},
\end{aligned}$$

for some constant $C'_R > 0$. Finally, exploiting Theorem 2.2, (f_1), $|T_\varepsilon(t)| \le \varepsilon$ for all $t \in \mathbb{R}$, the Hölder inequality, and the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$, we can see that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} (I_\alpha * F(u_n))\, f(u_n)\psi_R T_\varepsilon(u_n-u)\,dx\right| \\
&\le C_1\left(\int_{\mathbb{R}^N} |F(u_n)|^{\frac{2N}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{2N}}\left(\int_{\mathbb{R}^N} |f(u_n)\psi_R T_\varepsilon(u_n-u)|^{\frac{2N}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{2N}} \\
&\le C_2\left(\|u_n\|^p_{L^p(\mathbb{R}^N)}+\|u_n\|^{q^*}_{L^{q^*}(\mathbb{R}^N)}\right)^{\frac{N+\alpha}{2N}}\left[\int_{\mathbb{R}^N} |u_n|^{\frac{(N+\alpha)p-2N}{N+\alpha}}|\psi_R|^{\frac{2N}{N+\alpha}}|T_\varepsilon(u_n-u)|^{\frac{2N}{N+\alpha}}\,dx\right. \\
&\qquad\left.+\int_{\mathbb{R}^N} |u_n|^{\frac{((N+\alpha)q-2(N-q))N}{(N-q)(N+\alpha)}}|\psi_R|^{\frac{2N}{N+\alpha}}|T_\varepsilon(u_n-u)|^{\frac{2N}{N+\alpha}}\,dx\right]^{\frac{N+\alpha}{2N}} \\
&\le C_3\left[\varepsilon^{\frac{2N}{N+\alpha}}\left(\int_{\mathbb{R}^N}|u_n|^p\,dx\right)^{\frac{(N+\alpha)p-2N}{(N+\alpha)p}}\left(\int_{\mathbb{R}^N}|\psi_R|^p\,dx\right)^{\frac{2N}{(N+\alpha)p}}\right. \\
&\qquad\left.+\varepsilon^{\frac{2N}{N+\alpha}}\left(\int_{\mathbb{R}^N}|u_n|^{q^*}\,dx\right)^{\frac{(N+\alpha)q-2(N-q)}{(N+\alpha)q}}\left(\int_{\mathbb{R}^N}|\psi_R|^{q^*}\,dx\right)^{\frac{2(N-q)}{(N+\alpha)q}}\right]^{\frac{N+\alpha}{2N}} \\
&\le C''_R\,\varepsilon,
\end{aligned}
$$

for some constant $C''_R > 0$. From the definition of T_ε, the fact that $\psi_R \ge 0$ in $\mathbb{R}^N$, and recalling that for all $r > 1$ it holds

$$
(|\xi_1|^{r-2}\xi_1 - |\xi_2|^{r-2}\xi_2)(\xi_1-\xi_2) > 0 \quad \text{for all } \xi_1,\xi_2 \in \mathbb{R}^N \text{ such that } \xi_1 \ne \xi_2, \tag{4.8}
$$

we derive that the integrand of the left-hand side in (4.7) is nonnegative. Then, utilizing that $\psi_R = 1$ in B_R, we discover that

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{B_R}\left[(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) + (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)\right]\nabla T_\varepsilon(u_n-u)\,dx \\
&\le C_R\,\varepsilon,
\end{aligned} \tag{4.9}
$$

for some constant $C_R > 0$. Define

$$
\begin{aligned}
e_n(x) := {}&(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x)) \\
&+ (|\nabla u_n(x)|^{q-2}\nabla u_n(x) - |\nabla u(x)|^{q-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x)).
\end{aligned}
$$

Evidently, $e_n \geq 0$ in $\mathbb{R}^N$ thanks to (4.8). Moreover, $(e_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^N)$ since $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $(L^s(\mathbb{R}^N))^N$ and $(|\nabla u_n|^{s-2}\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $(L^{\frac{s}{s-1}}(\mathbb{R}^N))^N$, for all $s \in \{p, q\}$. Take $\theta \in (0, 1)$ and consider the sets

$$\mathbf{A}_R^\varepsilon := \{x \in B_R : |u_n(x) - u(x)| \leq \varepsilon\} \text{ and } \mathbf{B}_R^\varepsilon := \{x \in B_R : |u_n(x) - u(x)| \geq \varepsilon\}.$$

Applying the Hölder inequality, we have

$$\begin{aligned}\int_{B_R} e_n^\theta \,dx &= \int_{\mathbf{A}_R^\eta} e_n^\theta \,dx + \int_{\mathbf{B}_R^\eta} e_n^\theta \,dx \\ &\leq \left(\int_{\mathbf{A}_R^\eta} e_n \,dx\right)^\theta |\mathbf{A}_R^\eta|^{1-\theta} + \left(\int_{\mathbf{B}_R^\eta} e_n \,dx\right)^\theta |\mathbf{B}_R^\eta|^{1-\theta}.\end{aligned} \tag{4.10}$$

Observing that, for $\varepsilon > 0$ fixed, $|\mathbf{B}_R^\varepsilon| \to 0$ as $n \to \infty$, and using the boundedness of $(e_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}^N)$, it follows from (4.9) and (4.10) that

$$\limsup_{n\to\infty} \int_{B_R} e_n^\theta \,dx \leq (C_R\, \varepsilon)^\theta |B_R|^{1-\theta}.$$

Letting $\varepsilon \to 0$, we obtain that $e_n^\theta \to 0$ in $L^1(B_R)$, and so, up to a subsequence, $e_n \to 0$ a.e. in B_R. Since $R > 0$ is arbitrary, up to a subsequence, $e_n \to 0$ a.e. in $\mathbb{R}^N$. This fact and (4.8) imply that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Because $(|\nabla u_n|^{s-2}\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $(L^{\frac{s}{s-1}}(\mathbb{R}^N))^N$, for all $s \in \{p, q\}$, we deduce that, up to a subsequence, $|\nabla u_n|^{s-2}\nabla u_n \rightharpoonup |\nabla u|^{s-2}\nabla u$ in $(L^{\frac{s}{s-1}}(\mathbb{R}^N))^N$, for all $s \in \{p, q\}$. The proof of Lemma 4.2 is now complete. □

Now we study the convergence of Pohozaev-Palais-Smale sequences.

Proposition 4.2. *Assume that f fulfills (f_1)–(f_2). Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{p,q}$ be a sequence satisfying* (4.4). *Then,*

(1) *either up to a subsequence, $u_n \to 0$ in $\mathcal{W}_{p,q}$ as $n \to \infty$,*
(2) *or there exist $u \in \mathcal{W}_{p,q} \setminus \{0\}$ such that $\mathcal{I}'(u) = 0$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot - x_n) \rightharpoonup u$ in $\mathcal{W}_{p,q}$ as $n \to \infty$.*

Proof. First we note that, by Lemma 4.1, $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$. Let us assume that (1) does not hold. Then, without loss of generality, we may suppose that

$$\liminf_{n\to\infty} \|u_n\|_{W^{1,q}(\mathbb{R}^N)} > 0. \tag{4.11}$$

Observe that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))\, F(u_n)\, dx$$
$$= \frac{2}{N+\alpha}\left[\frac{N-p}{p}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N-q}{q}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{N}{q}\|u_n\|^q_{L^q(\mathbb{R}^N)} - \mathcal{P}(u_n)\right],$$

and utilizing $\mathcal{P}(u_n) \to 0$ as $n \to \infty$ and (4.11), we can see that

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))\, F(u_n)\, dx \\
&= \liminf_{n\to\infty} \frac{2}{N+\alpha}\left\{\frac{N-p}{p}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N-q}{q}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{N}{q}\|u_n\|^q_{L^q(\mathbb{R}^N)} - \mathcal{P}(u_n)\right\} \\
&= \liminf_{n\to\infty} \frac{2}{N+\alpha}\left\{\frac{N-p}{p}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N}{p}\|u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{N-q}{q}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{N}{q}\|u_n\|^q_{L^q(\mathbb{R}^N)}\right\} \\
&\geq \liminf_{n\to\infty} \frac{2}{N+\alpha}\left[\frac{N-q}{q}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{N}{q}\|u_n\|^q_{L^q(\mathbb{R}^N)}\right] > 0.
\end{aligned} \tag{4.12}$$

Next we claim that, for every $r \in (q, q^*)$, we have

$$\liminf_{n\to\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} > 0. \tag{4.13}$$

Assume by contradiction that (4.13) is not true, that is, for some $r \in (q, q^*)$,

$$\liminf_{n\to\infty} \sup_{x_0\in\mathbb{R}^N} \|u_n\|_{L^r(B_1(x_0))} = 0. \tag{4.14}$$

Since F is continuous and satisfies (f_2), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(t)|^{\frac{2N}{N+\alpha}} \leq \varepsilon\left(|t|^p + |t|^{q^*}\right) + C_\varepsilon |t|^r \quad \text{for all } t \in \mathbb{R}. \tag{4.15}$$

Exploiting (4.15), the fact that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$, and applying Lemma 2.1, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \|F(u_n)\|^{\frac{2N}{N+\alpha}}_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} &\leq \liminf_{n\to\infty}\left[\varepsilon\left(\|u_n\|^p_{L^p(\mathbb{R}^N)} + \|u_n\|^{q^*}_{L^{q^*}(\mathbb{R}^N)}\right) + C_\varepsilon \|u_n\|^r_{L^r(\mathbb{R}^N)}\right] \\
&\leq C_1\,\varepsilon + C'_\varepsilon \liminf_{n\to\infty}\left(\sup_{x_0\in\mathbb{R}^N} \|u_n\|^r_{L^r(B_1(x_0))}\right)^{1-\frac{q}{r}}.
\end{aligned} \tag{4.16}$$

Then, using (4.14), (4.16), and since $\varepsilon > 0$ is arbitrary, we get

$$\liminf_{n\to\infty} \|F(u_n)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = 0. \tag{4.17}$$

Employing (4.12), Theorem 2.2 and (4.17), we arrive at

$$0 < \liminf_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))\, F(u_n)\, dx \leq C_2 \liminf_{n\to\infty} \|F(u_n)\|^2_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} = 0,$$

that is a contradiction. Hence, (4.13) is proved. Thus, up to a translation, we may assume that, for some $r \in (q, q^*)$,

$$\liminf_{n\to\infty} \|u_n\|_{L^r(B_1)} > 0.$$

From this and Lemma 4.1, up to a subsequence, we may suppose that (4.5) holds for some $u \in \mathcal{W}_{p,q} \setminus \{0\}$. Next we verify that u is a weak solution to (1.1). Utilizing the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{W}_{p,q}$ and $\mathcal{I}'(u_n) \to 0$ in $\mathcal{W}^*_{p,q}$, we can apply Lemma 4.2 to deduce that (4.6) is valid. Taking into account that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$, it follows from (f_1) that $(F(u_n))_{n\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Because F is continuous, $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^N$. Therefore, $F(u_n) \rightharpoonup F(u)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Recalling that the Riesz potential I_α is a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ (see Remark 2.1), we have that $I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. On the other hand, thanks to (f_1) and (4.5), we know that $f(u_n) \to f(u)$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, \frac{2Nq}{(N+\alpha)q-2(N-q)})$. Consequently,

$$(I_\alpha * F(u_n)) f(u_n) \rightharpoonup (I_\alpha * F(u)) f(u) \text{ in } L^r(\mathbb{R}^N) \quad \text{for all } r \in \left[1, \frac{Nq}{Nq - N + q}\right). \tag{4.18}$$

Combining (4.5), (4.6), and (4.18), we see that, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla\varphi + |u|^{p-2}u\,\varphi\right) dx + \int_{\mathbb{R}^N} \left(|\nabla u|^{q-2}\nabla u \nabla\varphi + |u|^{q-2}u\,\varphi\right) dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * F(u))\, f(u)\,\varphi\, dx \\
&= \lim_{n\to\infty} \Bigg[\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi + |u_n|^{p-2}u_n\,\varphi\right) dx \\
&\quad + \int_{\mathbb{R}^N} \left(|\nabla u_n|^{q-2}\nabla u_n \nabla\varphi + |u_n|^{q-2}u_n\,\varphi\right) dx
\end{aligned}$$

$$- \int_{\mathbb{R}^N} (I_\alpha * F(u_n))\, f(u_n)\, \varphi \, dx \Bigg] = 0.$$

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{W}_{p,q}$, we conclude that u is a weak solution to (1.1). □

In order to establish the existence of a ground state solution to (1.1), we exploit Proposition 3.2 to construct an optimal path in the spirit of [23, Lemma 2.1].

Proposition 4.3. *Assume that f satisfies (f_1)–(f_2), and let $u \in \mathcal{W}_{p,q} \setminus \{0\}$ be a weak solution to* (1.1)*. Then, there exists $L > 1$ (sufficiently large but fixed) such that the path defined by*

$$\gamma(t)(x) := \begin{cases} u\left(\dfrac{x}{Lt}\right) & \textit{if } t \in (0,1], \\ 0 & \textit{if } t = 0, \end{cases}$$

satisfies

$$\begin{aligned} &\gamma(0) = 0, \\ &\gamma(1/L) = u, \\ &\gamma \in C([0,1], \mathcal{W}_{p,q}), \\ &\mathcal{I}(\gamma(t)) < \mathcal{I}(u) \quad \textit{for all } t \in [0,1] \setminus \{1/L\}, \\ &\mathcal{I}(\gamma(1)) < 0, \end{aligned}$$

Proof. Let $\tilde{\gamma} : [0, \infty) \to \mathcal{W}_{p,q}$ be the path given by

$$\tilde{\gamma}(t)(x) := \begin{cases} u\left(\dfrac{x}{t}\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Clearly, $\tilde{\gamma} \in C([0,\infty); \mathcal{W}_{p,q})$. Note that, using (1.6), we have

$$\begin{aligned} \mathcal{I}(\tilde{\gamma}(t)) &= \frac{t^{N-p}}{p} \|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{t^N}{p} \|u\|^p_{L^p(\mathbb{R}^N)} + \frac{t^{N-q}}{q} \|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \frac{t^N}{q} \|u\|^q_{L^q(\mathbb{R}^N)} \\ &\quad - \frac{t^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))\, F(u)\, dx \\ &= \left(\frac{t^{N-p}}{p} - \frac{(N-p)}{(N+\alpha)} \frac{t^{N+\alpha}}{p} \right) \|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \left(\frac{t^N}{p} - \frac{N}{(N+\alpha)} \frac{t^{N+\alpha}}{p} \right) \|u\|^p_{L^p(\mathbb{R}^N)} \\ &\quad + \left(\frac{t^{N-q}}{q} - \frac{(N-q)}{(N+\alpha)} \frac{t^{N+\alpha}}{q} \right) \|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \left(\frac{t^N}{q} - \frac{N}{(N+\alpha)} \frac{t^{N+\alpha}}{q} \right) \|u\|^q_{L^q(\mathbb{R}^N)}. \end{aligned}$$

Therefore,

$$\frac{d}{dt}(\mathcal{I}(\tilde{\gamma}(t))) = t^{N+\alpha-1}\left\{\left[\left(\frac{1}{t^{\alpha+p}}-1\right)\frac{N-p}{p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \left(\frac{1}{t^{\alpha}}-1\right)\frac{N}{p}\|u\|^p_{L^p(\mathbb{R}^N)}\right]\right.$$
$$\left. + \left[\left(\frac{1}{t^{\alpha+q}}-1\right)\frac{N-q}{q}\|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \left(\frac{1}{t^{\alpha}}-1\right)\frac{N}{q}\|u\|^q_{L^q(\mathbb{R}^N)}\right]\right\},$$

which combined with $N > q > p$ yields

$$\frac{d}{dt}(\mathcal{I}(\tilde{\gamma}(t))) = 0 \text{ if } t = 1, \; \frac{d}{dt}(\mathcal{I}(\tilde{\gamma}(t))) > 0 \text{ if } t \in (0,1), \; \frac{d}{dt}(\mathcal{I}(\tilde{\gamma}(t))) < 0 \text{ if } t \in (1,\infty).$$

Hence, $\mathcal{I}(\tilde{\gamma}(t))$ achieves its strict global maximum at $t = 1$. Because

$$\lim_{t\to\infty}\mathcal{I}(\tilde{\gamma}(t)) = -\infty,$$

we can deduce that the path γ can be defined by a suitable change of variable. □

Now we are ready to provide the proof of the main result of this section.

Proof of Theorem 1.2. According to Propositions 4.1 and 4.2, we can find a Pohozaev-Palais-Smale sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{W}_{p,q}$ at the mountain pass level $b > 0$ such that (4.5) and (4.6) hold for some $u \in \mathcal{W}_{p,q} \setminus \{0\}$ which satisfies (1.1). By Proposition 3.2, we know that $\mathcal{P}(u) = 0$. In light of these facts, and exploiting $\mathcal{I}(u_n) \to b$ and $\mathcal{P}(u_n) \to 0$ as $n \to \infty$, we see that

$$\begin{aligned}
\mathcal{I}(u) &= \mathcal{I}(u) - \frac{1}{N+\alpha}\mathcal{P}(u) \\
&= \frac{\alpha+p}{p(N+\alpha)}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\le \liminf_{n\to\infty}\left[\frac{\alpha+p}{p(N+\alpha)}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u_n\|^p_{L^p(\mathbb{R}^N)}\right. \\
&\quad \left. + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u_n\|^q_{L^q(\mathbb{R}^N)}\right] \\
&= \liminf_{n\to\infty}\left(\mathcal{I}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n)\right) \\
&= \liminf_{n\to\infty}\mathcal{I}(u_n) = b.
\end{aligned} \tag{4.19}$$

Since u is a nontrivial solution to (1.1), it follows from (1.7) and (4.19) that

$$m \le \mathcal{I}(u) \le b. \tag{4.20}$$

Now, let $v \in \mathcal{W}_{p,q} \setminus \{0\}$ be a weak solution of (1.1) such that $\mathcal{I}(v) \le \mathcal{I}(u)$. Applying Proposition 4.3 to v, we can find $\gamma \in \Gamma$ such that $\max_{t\in[0,1]}\mathcal{I}(\gamma(t)) = \mathcal{I}(v)$. Thus, thanks to (1.8), $\mathcal{I}(v) \ge b$. This combined with (4.20) gives $\mathcal{I}(v) \ge b \ge \mathcal{I}(u)$. Therefore, $\mathcal{I}(v) = \mathcal{I}(u) = b = m$, and this concludes the proof of Theorem 1.2. □

Finally, we prove an interesting result that will be useful to obtain the compactness, modulo translations, of the set of ground state solutions.

Corollary 4.1. *Under the assumptions of Proposition 4.2, if we assume that*

$$\liminf_{n\to\infty} \|u_n\|_{\mathcal{W}_{p,q}} > 0,$$

and

$$\limsup_{n\to\infty} \mathcal{I}(u_n) \le m,$$

then there exist $u \in \mathcal{W}_{p,q} \setminus \{0\}$ such that $\mathcal{I}'(u) = 0$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that, up to a subsequence, $u_n(\cdot - x_n) \to u$ in $\mathcal{W}_{p,q}$ as $n \to \infty$.

Proof. By Proposition 4.2, up to a subsequence and translations, we may suppose that $u_n \rightharpoonup u$ in $\mathcal{W}_{p,q}$ for some $u \in \mathcal{W}_{p,q} \setminus \{0\}$ that solves (1.1). Then, arguing as in the proof of Theorem 1.2, and using $\limsup_{n\to\infty} \mathcal{I}(u_n) \le m$, we have

$$\begin{aligned}
m \le \mathcal{I}(u) &= \mathcal{I}(u) - \frac{1}{N+\alpha}\mathcal{P}(u) \\
&= \frac{\alpha+p}{p(N+\alpha)}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\le \liminf_{n\to\infty}\Bigg[\frac{\alpha+p}{p(N+\alpha)}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u_n\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u_n\|^q_{L^q(\mathbb{R}^N)}\Bigg] \\
&\le \limsup_{n\to\infty}\Bigg[\frac{\alpha+p}{p(N+\alpha)}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u_n\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u_n\|^q_{L^q(\mathbb{R}^N)}\Bigg] \\
&= \limsup_{n\to\infty}\left(\mathcal{I}(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n)\right) \\
&= \limsup_{n\to\infty}\mathcal{I}(u_n) \le m,
\end{aligned}$$

from which

$$\begin{aligned}
&\frac{\alpha+p}{p(N+\alpha)}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u\|^q_{L^q(\mathbb{R}^N)}
\end{aligned} \tag{4.21}$$

$$= \lim_{n\to\infty}\left[\frac{\alpha+p}{p(N+\alpha)}\|\nabla u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha}{p(N+\alpha)}\|u_n\|^p_{L^p(\mathbb{R}^N)} + \frac{\alpha+q}{q(N+\alpha)}\|\nabla u_n\|^q_{L^q(\mathbb{R}^N)} + \frac{\alpha}{q(N+\alpha)}\|u_n\|^q_{L^q(\mathbb{R}^N)}\right].$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{W}_{p,q}$, and utilizing the facts that $u_n \to u$ a.e. in $\mathbb{R}^N$ and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, it follows from the Brezis-Lieb Lemma [15, Theorem 1] and (4.21) that $u_n \to u$ in $\mathcal{W}_{p,q}$. □

As a byproduct of Corollary 4.1, we obtain the following result.

Proposition 4.4. *Up to translations in $\mathbb{R}^N$, the set*

$$\mathcal{S}_m := \left\{u \in \mathcal{W}_{p,q} : \mathcal{I}(u) = m,\ \mathcal{I}'(u) = 0\right\}$$

is compact in $\mathcal{W}_{p,q}$ endowed with the strong topology.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{S}_m$. By Theorem 1.1, we know that $\mathcal{P}(u_n) = 0$ for all $n \in \mathbb{N}$. Hence, $(u_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 4.2. We derive from $\mathcal{I}(u_n) = m > 0$ for all $n \in \mathbb{N}$ and the continuity of $\mathcal{I}$ that $u_n \nrightarrow 0$ in $\mathcal{W}_{p,q}$. Thus, by Corollary 4.1, up to subsequences and translations, $u_n \to u$ in $\mathcal{W}_{p,q}$ for some $u \in \mathcal{W}_{p,q} \setminus \{0\}$ such that $\mathcal{I}'(u) = 0$. Because $\mathcal{I}$ is continuous, we deduce that $\mathcal{I}(u_n) \to \mathcal{I}(u)$, and so $u \in \mathcal{S}_m$. □

5. Proof of Theorem 1.3

In this section, we investigate qualitative properties of ground state solutions to (1.1). We start by showing that optimal paths yield critical points.

Lemma 5.1. *Assume that f satisfies (f_1). Let $\gamma \in \Gamma$ and $t_* \in (0, 1)$ such that*

$$\mathcal{I}(\gamma(t)) < \mathcal{I}(\gamma(t_*)) = b \quad \text{for all } t \in [0, 1] \setminus \{t_*\}. \tag{5.1}$$

Then, $\mathcal{I}'(\gamma(t_*)) = 0$.

Proof. Suppose by contradiction that $\mathcal{I}'(\gamma(t_*)) \neq 0$. By continuity, we can find $\varepsilon, \delta > 0$ such that

$$\inf\left\{\|\mathcal{I}'(v)\|_* : \|v - \gamma(t_*)\|_{\mathcal{W}_{p,q}} \le \delta\right\} > \frac{8\,\varepsilon}{\delta}.$$

Applying [43, Lemma 2.3] with $X = \mathcal{W}_{p,q}$, $S = \{\gamma(t_*)\}$ and $c = b$, there exists $\eta \in C([0, 1] \times \mathcal{W}_{p,q}; \mathcal{W}_{p,q})$ such that

- (i) $\eta(t, u) = u$ if $t = 0$ or if $u \notin \mathcal{I}^{-1}([b - 2\,\varepsilon, b + 2\,\varepsilon]) \cap S_{2\delta}$,
- (ii) $\eta(1, \mathcal{I}^{b+\varepsilon} \cap S) \subset \mathcal{I}^{b-\varepsilon}$,
- (iii) $\eta(t, \cdot)$ is an homeomorphism of $\mathcal{W}_{p,q}$, for all $t \in [0, 1]$,
- (iv) $\|\eta(t, u) - u\|_{\mathcal{W}_{p,q}} \le \delta$ for all $u \in \mathcal{W}_{p,q}$ and $t \in [0, 1]$,

(v) $\mathcal{I}(\eta(\cdot, u))$ is non increasing for all $u \in \mathcal{W}_{p,q}$,
(vi) $\mathcal{I}(\eta(t, u)) < b$ for all $u \in \mathcal{I}^b \cap S_\delta$ and $t \in (0, 1]$.

Here we have used the notations $\mathcal{I}^d(u) := \{u \in \mathcal{W}_{p,q} : \mathcal{I}(u) \leq d\}$ and $S_\delta := \{u \in \mathcal{W}_{p,q} : \|u\|_{\mathcal{W}_{p,q}} = \delta\}$ for $d \in \mathbb{R}$ and $\delta > 0$. Let $\phi : [0, 1] \to \mathcal{W}_{p,q}$ be defined by

$$\phi(t) := \eta(1, \gamma(t)) \quad \text{for } t \in [0, 1].$$

Let us observe that:

- $\phi \in \Gamma$; indeed combining $\gamma \in \Gamma$ with (i), we see

  $$\phi(0) = \eta(1, \gamma(0)) = \eta(1, 0) = 0,$$

  whereas (i) and (v) imply that

  $$\mathcal{I}(\phi(1)) = \mathcal{I}(\eta(1, \gamma(1))) \leq \mathcal{I}(\eta(0, \gamma(1))) = \mathcal{I}(\gamma(1)) < 0.$$

- $\mathcal{I}(\phi(t)) < b$ for all $t \in [0, 1]$; in fact, using (v), (i), and (5.1), we obtain

  $$\mathcal{I}(\eta(1, \gamma(t))) \leq \mathcal{I}(\eta(0, \gamma(t))) = \mathcal{I}(\gamma(t)) < b \quad \text{for all } t \in [0, 1] \setminus \{t_*\},$$

  whereas (ii) yields

  $$\mathcal{I}(\eta(1, \gamma(t_*))) \leq b - \varepsilon < b.$$

Consequently,

$$\max_{t \in [0,1]} \mathcal{I}(\phi(t)) < b,$$

which contradicts the definition of b. Hence, $\mathcal{I}'(\gamma(t_*)) = 0$. □

Next we verify that ground state solutions to (1.1) do not change sign whenever f is odd and has constant sign.

Proposition 5.1. *Assume that f satisfies (f_1)–(f_2). If f is odd and does not change sign on $(0, \infty)$, then every ground state solution $u \in \mathcal{W}_{p,q}$ of (1.1) has constant sign.*

Proof. Without loss of generality, we may assume that $f \geq 0$ on $(0, \infty)$. Let $u \in \mathcal{W}_{p,q}$ be a ground state solution of (1.1). Note that, because f is odd, F is even. Thus,

$$\mathcal{I}(|v|) = \mathcal{I}(v) \quad \text{for all } v \in \mathcal{W}_{p,q}. \tag{5.2}$$

By Proposition 4.3, we know that there exist $L > 1$ and $\gamma \in \Gamma$ such that

$$\begin{aligned}&\gamma(1/L)=u,\\&\mathcal{I}(\gamma(t))<\mathcal{I}(u)\quad \text{for all } t\in[0,1]\setminus\{1/L\}.\end{aligned}\tag{5.3}$$

Exploiting (5.2) and (5.3), we have

$$\mathcal{I}(|\gamma(t)|)=\mathcal{I}(\gamma(t))<\mathcal{I}(u)=m=b\quad \text{for all } t\in[0,1]\setminus\{1/L\},$$

which combined with Lemma 5.1 ensures that $\mathcal{I}'(|u|)=0$. Therefore, $\mathcal{I}(|u|)=\mathcal{I}(u)=m$ and $\mathcal{I}'(|u|)=0$, that is, $|u|$ is a ground state solution to (1.1). As a result, $|u|$ weakly solves

$$-\Delta_p|u|-\Delta_q|u|+|u|^{p-1}+|u|^{q-1}=(I_\alpha * F(|u|))f(|u|)\quad \text{in } \mathbb{R}^N.$$

Since $|u|\in L^\infty(\mathbb{R}^N)\cap C^{1,\lambda}_{loc}(\mathbb{R}^N)$ for some $\lambda\in(0,1)$ (by Theorem 1.1), it follows from the Harnack inequality [41, Theorem 1.2] that $|u|>0$ in $\mathbb{R}^N$. In conclusion, u has constant sign, as desired. □

Finally, we study the symmetry of ground state solutions to (1.1). For this purpose, we recall some useful facts about the theory of polarization; see [44, Chapter 8] for more details.

Let $H\subset\mathbb{R}^N$ be a closed half-space and denote by σ_H the reflection with respect to the frontier of H. The polarization (with respect to H) of a function $u:\mathbb{R}^N\to\mathbb{R}$ is defined by

$$u^H(x):=\begin{cases}\max\{u(x),u(\sigma_H(x))\} & \text{if } x\in H,\\ \min\{u(x),u(\sigma_H(x))\} & \text{if } x\in H^c.\end{cases}$$

The following results will be crucial to accomplish our aim.

Proposition 5.2. *[44, Propositions 8.3.7 and 8.3.12] Let $t\in[1,\infty)$ and $u\in W^{1,t}(\mathbb{R}^N)$. Then $u^H\in W^{1,t}(\mathbb{R}^N)$ and it holds*

$$\|\nabla u^H\|_{L^t(\mathbb{R}^N)}=\|\nabla u\|_{L^t(\mathbb{R}^N)} \text{ and } \|u^H\|_{L^t(\mathbb{R}^N)}=\|u\|_{L^t(\mathbb{R}^N)}.$$

Lemma 5.2. *[32, Lemma 5.3] Let $\alpha\in(0,N)$, $u\in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ and $H\subset\mathbb{R}^N$ be a closed half-space. If $u\geq 0$ then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u(x)u(y)}{|x-y|^{N-\alpha}}\,dxdy\leq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^H(x)u^H(y)}{|x-y|^{N-\alpha}}\,dxdy,$$

with equality if and only if $u^H=u$ or $u^H=u\circ\sigma_H$.

Lemma 5.3. *[32, Lemma 5.4] Let $t\in[1,\infty)$ and $u\in L^t(\mathbb{R}^N)$ be such that $u\geq 0$. Then there exist $x_0\in\mathbb{R}^N$ and a nonincreasing function $v:(0,\infty)\to\mathbb{R}$ such that $u(x)=v(|x-x_0|)$ for a.e. $x\in\mathbb{R}^N$ if and only if $u^H=u$ or $u^H=u\circ\sigma_H$ for every closed half-space $H\subset\mathbb{R}^N$.*

The proposition below concerns the sign and symmetry of ground state solutions to (1.1).

Proposition 5.3. *Assume that f satisfies (f_1)–(f_2). If f is odd and does not change sign on $(0,\infty)$, then every ground state solution $u \in \mathcal{W}_{p,q}$ to (1.1) is radially symmetric about a point.*

Proof. Suppose that $f \geq 0$ on $(0,\infty)$. Let $u \in \mathcal{W}_{p,q}$ be a ground state solution to (1.1) and let $H \subset \mathbb{R}^N$ be a closed half-space. In view of Proposition 5.1, we may assume that $u > 0$. Because F is nondecreasing, it holds

$$F(u^H) = F(u)^H. \tag{5.4}$$

From Proposition 4.3, we can find $L > 1$ and $\gamma \in \Gamma$ such that

$$\begin{aligned} &\gamma(t) \geq 0 \quad \text{for all } t \in [0,1], \\ &\gamma(1/L) = u, \\ &\mathcal{I}(\gamma(t)) < \mathcal{I}(u) \quad \text{for all } t \in [0,1] \setminus \{1/L\}. \end{aligned} \tag{5.5}$$

Define $\gamma^H : [0,1] \to \mathcal{W}_{p,q}$ by setting

$$\gamma^H(t) := (\gamma(t))^H.$$

Thanks to [42, Corollary 2.40], $\gamma^H \in C([0,1]; \mathcal{W}_{p,q})$. By Lemma 5.2 and Proposition 5.2, we obtain that

$$\mathcal{I}(\gamma^H(t)) \leq \mathcal{I}(\gamma(t)) \quad \text{for all } t \in [0,1]. \tag{5.6}$$

Hence, $\gamma^H \in \Gamma$, and using (1.8), we see that

$$\max_{t\in[0,1]} \mathcal{I}(\gamma^H(t)) \geq b. \tag{5.7}$$

Taking (5.5) and (5.6) into account, we get

$$\mathcal{I}(\gamma^H(t)) \leq \mathcal{I}(\gamma(t)) < \mathcal{I}(u) = b \quad \text{for all } t \in [0,1] \setminus \{1/L\}. \tag{5.8}$$

Therefore, (5.5), (5.7), and (5.8) give

$$\mathcal{I}(\gamma^H(1/L)) = b = \mathcal{I}(\gamma(1/L)), \tag{5.9}$$

that is, $\mathcal{I}(u^H) = \mathcal{I}(u)$. This fact combined with Proposition 5.2, Lemma 5.2 and (5.4), implies that either $F(u)^H = F(u)$ or $F(u^H) = F(u \circ \sigma_H)$.

First we consider the case $F(u)^H = F(u)$. Since F is non decreasing, for every $x \in H$ we have

$$\int_{u(\sigma_H(x))}^{u(x)} f(\tau)\,d\tau = F(u(x)) - F(u(\sigma_H(x))) = F(u(x))^H - F(u(\sigma_H(x))) \geq 0.$$

Hence, either $u(\sigma_H(x)) \le u(x)$ or $f = 0$ in $[u(x), u(\sigma_H(x))]$. In particular,

$$f(u^H) = f(u) \quad \text{on } \mathbb{R}^N. \tag{5.10}$$

Note that (5.8) and (5.9) yield

$$\mathcal{I}(\gamma^H(t)) < \mathcal{I}(\gamma^H(1/L)) = b \quad \text{for all } t \in [0,1] \setminus \{1/L\},$$

and so, applying Lemma 5.1, we discover that $\mathcal{I}'(u^H) = 0$. This together with (5.4) and (5.10) ensures that u^H is a weak solution to

$$\begin{aligned} &-\Delta_p u^H - \Delta_q u^H + |u^H|^{p-2}u^H + |u^H|^{q-2}u^H \\ &\quad = (I_\alpha * F(u^H))f(u^H) = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

that is, for all $\varphi \in \mathcal{W}_{p,q}$,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u^H|^{p-2}\nabla u^H \nabla\varphi\, dx + \int_{\mathbb{R}^N} |u^H|^{p-2}u^H\, \varphi\, dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla u^H|^{q-2}\nabla u^H \nabla\varphi\, dx + \int_{\mathbb{R}^N} |u^H|^{q-2}u^H\, \varphi\, dx \\ &\quad = \int_{\mathbb{R}^N} (I_\alpha * F(u))\, f(u)\, \varphi\, dx. \end{aligned} \tag{5.11}$$

Taking $\varphi = u - u^H$ in (1.5) and (5.11), respectively, by subtracting we obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u - |\nabla u^H|^{p-2}\nabla u^H\right)\left(\nabla u - \nabla u^H\right) dx \\ &+ \int_{\mathbb{R}^N} \left(|u|^{p-2}u - |u^H|^{p-2}u^H\right)\left(u - u^H\right) dx \\ &+ \int_{\mathbb{R}^N} \left(|\nabla u|^{q-2}\nabla u - |\nabla u^H|^{q-2}\nabla u^H\right)\left(\nabla u - \nabla u^H\right) dx \\ &+ \int_{\mathbb{R}^N} \left(|u|^{q-2}u - |u^H|^{q-2}u^H\right)\left(u - u^H\right) dx. \end{aligned}$$

Recalling the well-known inequality (see [40, formula (2.2)])

$$(|x|^{t-2}x - |y|^{t-2}y)(x-y) \ge C_t|x-y|^t \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } t \in [2, \infty),$$

we arrive at

$$C_p\|u-u^H\|^p_{W^{1,p}(\mathbb{R}^N)}+C_q\|u-u^H\|^q_{W^{1,q}(\mathbb{R}^N)}\leq 0,$$

which implies $u=u^H$.

Secondly, if $F(u^H)=F(u\circ\sigma_H)$, then we can argue as before to infer that $u^H=u\circ\sigma_H$. Since H is arbitrary, we can invoke Lemma 5.3 to conclude that u is radially symmetric and radially decreasing. □

Proof of Theorem 1.3. The desired assertions follow directly from Propositions 5.1 and 5.3. □

6. Proof of Theorem 1.4

By Proposition 3.1, we know that $u\in L^\infty(\mathbb{R}^N)$ and $\mathcal{K}=I_\alpha * F(u)\in C_0(\mathbb{R}^N)$. Therefore, $u\in\mathcal{W}_{p,q}\cap L^\infty(\mathbb{R}^N)$ solves

$$-\Delta_p u-\Delta_q u=\tilde{\psi}\quad \text{in } \mathbb{R}^N,$$

with $\tilde{\psi}\in L^\infty(\mathbb{R}^N)$ given by $\tilde{\psi}(x):=-u^{p-1}(x)-u^{q-1}(x)+\mathcal{K}(x)f(u(x))$, and using [21, Theorem 1] we deduce that $\|\nabla u\|_{L^\infty(\mathbb{R}^N)}\leq C$ for some $C>0$. Hence, u is Lipschitz continuous in $\mathbb{R}^N$ and so uniformly continuous in $\mathbb{R}^N$. This fact and $u\in L^p(\mathbb{R}^N)$ shows that $u(x)\to 0$ as $|x|\to\infty$. From this, $\limsup_{|t|\to 0}\frac{|tf(t)|}{|t|^p}<\infty$, $p<q$ and $|\mathcal{K}(x)|\to 0$ as $|x|\to\infty$, we derive that

$$\mathcal{K}(x)f(u)=\mathcal{K}(x)\frac{f(u)}{u^{p-1}+u^{q-1}}(u^{p-1}+u^{q-1})\leq\frac{1}{2}(u^{p-1}+u^{q-1})\quad \text{in } B_R^c,$$

for some $R>0$. Thus u satisfies

$$-\Delta_p u-\Delta_q u+\frac{1}{2}(u^{p-1}+u^{q-1})\leq 0\quad \text{in } B_R^c. \tag{6.1}$$

Let us now define $\phi(x):=\Upsilon e^{\kappa R}e^{-\kappa|x|}$ where $\kappa,\Upsilon>0$ are such that

$$\kappa<\min\left\{\left(\frac{1}{2(p-1)}\right)^{\frac{1}{p}},\left(\frac{1}{2(q-1)}\right)^{\frac{1}{q}}\right\},$$

and $\|u\|_{L^\infty(\mathbb{R}^N)}\leq\Upsilon$. Clearly, $u(x)\leq\phi(x)$ for all $|x|\leq R$. It is readily seen that

$$\begin{aligned}
&-\Delta_p\phi-\Delta_q\phi+\frac{1}{2}(\phi^{p-1}+\phi^{q-1})\\
&=\phi^{p-1}\left(\frac{1}{2}-\kappa^p(p-1)+\frac{N-1}{|x|}\kappa^{p-1}\right)\\
&\quad+\phi^{q-1}\left(\frac{1}{2}-\kappa^q(q-1)+\frac{N-1}{|x|}\kappa^{q-1}\right)>0\quad \text{in } B_R^c.
\end{aligned} \tag{6.2}$$

Subtracting (6.2) from (6.1) and choosing $\zeta:=(u-\phi)^+\in W^{1,p}_0(B_R^c)\cap W^{1,q}_0(B_R^c)$ as test function, we obtain

$$0 \geq \int\limits_{\{|x|>R\,:\,u(x)>\phi(x)\}} \Bigg(\Big[(|\nabla u|^{p-2}\nabla u - |\nabla \phi|^{p-2}\nabla \phi) + (|\nabla u|^{q-2}\nabla u - |\nabla \phi|^{q-2}\nabla \phi) \Big] \nabla \zeta$$

$$+ \frac{1}{2} \Big[(u^{p-1} - \phi^{p-1}) + (u^{q-1} - \phi^{q-1}) \Big] \zeta \Bigg) \, dx$$

$$\geq \frac{1}{2} \int\limits_{\{|x|>R\,:\,u(x)>\phi(x)\}} (u^{p-1} - \phi^{p-1})(u - \phi) \, dx \geq 0,$$

where we have used (4.8). Consequently,

$$(u^{p-1} - \phi^{p-1})(u - \phi) = 0 \quad \text{ a.e. in } \{|x| > R : u(x) > \phi(x)\}.$$

Since u and ϕ are continuous in $\mathbb{R}^N$, we deduce that the set $\{|x| > R : u(x) > \phi(x)\}$ is empty. As a result, $u(x) \leq \phi(x)$ for all $x \in \mathbb{R}^N$. The proof of the theorem is now complete.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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References

[1] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, Pure Appl. Math. (Amst.), vol. 140, Elsevier/Academic Press, Amsterdam, 2003, xiv+305 pp.
[2] C.O. Alves, M. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, J. Differ. Equ. 257 (2014) 4133–4164.
[3] C.O. Alves, M. Yang, Multiplicity and concentration of solutions for a quasilinear Choquard equation, J. Math. Phys. 55 (6) (2014) 061502.
[4] C.O. Alves, M. Yang, Investigating the multiplicity and concentration behaviour of solutions for a quasilinear Choquard equation via penalization method, Proc. R. Soc. Edinb., Sect. A 146 (2016) 23–58.
[5] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[6] V. Ambrosio, Regularity and Pohozaev identity for the Choquard equation involving the p–Laplacian operator, Appl. Math. Lett. 145 (2023) 108742.
[7] V. Ambrosio, The nonlinear (p,q)-Schrödinger equation with a general nonlinearity: existence and concentration, J. Math. Pures Appl. (9) 178 (2023) 141–184.
[8] V. Ambrosio, Multiple concentrating solutions for a fractional (p,q)-Choquard equation, Adv. Nonlinear Stud. 24 (2) (2024) 510.
[9] V. Ambrosio, Nonlinear scalar field (p_1, p_2)-Laplacian equations in $\mathbb{R}^N$: existence and multiplicity, submitted for publication.
[10] V. Ambrosio, T. Isernia, A multiplicity result for a (p,q)-Schrödinger-Kirchhoff type equation, Ann. Mat. Pura Appl. (4) 201 (2) (2022) 943–984.
[11] T. Bartsch, Z. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Commun. Partial Differ. Equ. 20 (1995) 1725–1741.

[12] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (4) (1983) 313–345.
[13] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. 19 (6) (1992) 581–597.
[14] H. Brézis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. (9) 58 (2) (1979) 137–151.
[15] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Am. Math. Soc. 88 (1983) 486–490.
[16] L. Cherfils, V. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian, Commun. Pure Appl. Anal. 4 (1) (2005) 9–22.
[17] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957) 25–43.
[18] M. Degiovanni, A. Musesti, M. Squassina, On the regularity of solutions in the Pucci-Serrin identity, Calc. Var. Partial Differ. Equ. 18 (3) (2003) 317–334.
[19] G.M. Figueiredo, Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^N$, J. Math. Anal. Appl. 378 (2011) 507–518.
[20] C. He, G. Li, The existence of a nontrivial solution to the $p\&q$-Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$, Nonlinear Anal. 68 (5) (2008) 1100–1119.
[21] C. He, G. Li, The regularity of weak solutions to nonlinear scalar field elliptic equations containing $p\&q$-Laplacians, Ann. Acad. Sci. Fenn., Math. 33 (2) (2008) 337–371.
[22] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, Nonlinear Anal. 28 (10) (1997) 1633–1659.
[23] L. Jeanjean, K. Tanaka, A remark on least energy solutions in $\mathbb{R}^N$, Proc. Am. Math. Soc. 131 (8) (2003) 2399–2408.
[24] G. Li, G. Zhang, Multiple solutions for the $p\&q$-Laplacian problem with critical exponent, Acta Math. Sci. Ser. B Engl. Ed. 29 (4) (2009) 903–918.
[25] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1977) 93–105.
[26] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997, xxii+346 pp.
[27] P.-L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980) 1063–1072.
[28] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455–467.
[29] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. 501 (1) (2021) 124408.
[30] G. Mingione, V.D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, J. Math. Anal. Appl. 501 (1) (2021) 125197.
[31] I.M. Moroz, R. Penrose, P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Class. Quantum Gravity 15 (9) (1998) 2733–2742.
[32] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2) (2013) 153–184.
[33] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Am. Math. Soc. 367 (9) (2015) 6557–6579.
[34] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (1) (2017) 773–813.
[35] J. Moser, A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations, Commun. Pure Appl. Math. 13 (1960) 457–468.
[36] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, On a class of parametric $(p, 2)$-equations, Appl. Math. Optim. 75 (2) (2017) 193–228.
[37] S. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.
[38] A. Pomponio, T. Watanabe, Some quasilinear elliptic equations involving multiple p-Laplacians, Indiana Univ. Math. J. 67 (6) (2018) 2199–2224.
[39] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[40] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, in: Journées d'Analyse Non Linéaire, Proc. Conf., Besançon, 1977, in: Lecture Notes in Math., vol. 665, Springer, Berlin, 1978, pp. 205–227.
[41] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Commun. Pure Appl. Math. 20 (1967) 721–747.
[42] J. Van Schaftingen, Approximation of symmetrizations and symmetry of critical points, Topol. Methods Nonlinear Anal. 28 (1) (2006) 61–85.

[43] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996, x+162 pp.
[44] M. Willem, Functional Analysis. Fundamentals and Applications, Cornerstones, Birkhäuser/Springer, New York, 2013, xiv+213 pp.
[45] X. Xie, T. Wang, W. Zhang, Existence of solutions for the (p, q)-Laplacian equation with nonlocal Choquard reaction, Appl. Math. Lett. 135 (2023) 108418.
[46] W. Zhang, J. Zhang, V.D. Rădulescu, Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction, J. Differ. Equ. 347 (2023) 56–103.
[47] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Izv. Akad. Nauk SSSR, Ser. Mat. 50 (4) (1986) 675–710; English translation in Math. USSR, Izv. 29 (1) (1987) 33–66.
[48] V.V. Zhikov, On Lavrentiev's phenomenon, Russ. J. Math. Phys. 3 (2) (1995) 249–269.