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# NON-RIDGE-CHORDAL COMPLEXES WHOSE CLIQUE COMPLEX HAS SHELLABLE ALEXANDER DUAL

BRUNO BENEDETTI AND DAVIDE BOLOGNINI

ABSTRACT. A recent conjecture that appeared in three papers by Bigdeli–Faridi, Dochtermann, and Nikseresht, is that every simplicial complex whose clique complex has shellable Alexander dual, is ridge-chordal. This strengthens the long-standing Simon’s conjecture that the  $k$ -skeleton of the simplex is extendably shellable, for any  $k$ . We show that the stronger conjecture has a negative answer, by exhibiting an infinite family of counterexamples.

## 1. INTRODUCTION

Shellability is a property satisfied by two important families of objects in combinatorics, namely, polytope boundaries [27] and order complexes of geometric lattices [9]. Moreover, skeleta of shellable complexes are themselves shellable [12]. *Extendable shellability* is the stronger demand that any shelling of any full-dimensional subcomplex may be continued into a shelling of the whole complex. This property is less understood than shellability, and much less common. It is easy to construct polytopes that are not extendably shellable [27]. In 1994 Simon conjectured that, for any integer  $0 \leq d \leq n$ , the  $d$ -skeleton of the  $n$ -simplex is extendably shellable [23, Conjecture 4.2.1]. For  $d \leq 2$  this was soon proven by Björner and Eriksson [11], but for  $3 \leq d \leq n - 3$  the conjecture remains open.

Recently Bigdeli et al. [8] and Dochtermann et al. [18] established Simon’s conjecture for  $d \geq n - 2$ , showing also that shellability is equivalent to extendable shellability for  $d$ -complexes with up to  $d + 3$  vertices [18]. Their approach is based on a higher-dimensional extension of the graph-theoretic notion of chordality, called *ridge-chordality*, which we recall below. Given a  $d$ -dimensional pure simplicial complex  $\Delta$ , any  $(d - 1)$ -dimensional face of it is called a *ridge*. “*Deleting above a ridge*” of  $\Delta$  means to consider the simplicial complex whose facets are the facets of  $\Delta$  not containing that ridge. A *clique* of  $\Delta$  is any subset  $V \subseteq [n]$  such that all subsets of  $V$  of size  $d + 1$  appear among the facets of  $\Delta$ . For example, if  $\Delta$  is the graph  $\{12, 23, 13, 14\}$ , then 1, 12 and 123 are cliques, whereas 124 and 1234 are not. A pure  $d$ -dimensional simplicial complex  $\Delta$  is called *ridge-chordal* if  $\Delta = \emptyset$  or if it can be reduced to the empty set by repeatedly deleting above a ridge  $r$  such that the vertices of the star of  $r$  form a clique [7]. One can see that “ridge-chordal 1-complexes” are precisely the graphs admitting a perfect elimination ordering, i.e. graphs in which every minimal vertex cut is a clique; by Dirac’s theorem, these are precisely the “chordal graphs”, the graphs where every cycle of length at least four has a chord [14].

Now, let  $\text{Cl}(\Delta)$  be the “clique complex” of  $\Delta$ , i.e., the simplicial complex whose faces are the cliques of  $\Delta$ . This  $\text{Cl}(\Delta)$  is a simplicial complex of dimension at least  $d$ , with the same  $d$ -faces of  $\Delta$  and the same  $(d - 1)$ -faces of the  $n$ -simplex. The following conjecture appeared naturally, in several recent works:

**Conjecture A** ([6, Question 6.3], [15, Conjecture 4.8], [21, Statement A]).

If the Alexander dual of  $\text{Cl}(\Delta)$  is shellable, then  $\Delta$  is ridge-chordal.

There are three reasons why Conjecture A is natural and of interest:

- (1) As explained by Bigdeli et al. [8, Corollary 3.7] and [21, Corollary 4.16], Conjecture A directly implies Simon’s conjecture, cf. Remark 6.
- (2) The conjecture is true if one slightly strengthens the assumption “shellable” into “vertex-decomposable”. This fact is proven in the pure case by Nikseresht [21, Theorem 3.10], and in full-generality by Bigdeli–Faridi [6, Theorem 5.2]; see also Remark 5 below. Also, Conjecture A holds for  $\dim \Delta = 1$ .
- (3) Some partial converse holds: If  $\Delta$  is ridge-chordal, then the Alexander dual of  $\text{Cl}(\Delta)$  is Cohen–Macaulay over any field [7, Theorem 3.2], although not necessarily shellable or constructible [7, Example 3.14].

The purpose of this short note is to strongly disprove Conjecture A:

**Theorem A.** For any  $k \geq 2$  there is a constructible 2-dimensional complex  $\Delta_k$  that is not ridge-chordal, such that the Alexander dual of  $\text{Cl}(\Delta_k)$  is pure  $(5k - 2)$ -dimensional, shellable, and even 4-decomposable.

Theorem A provides a non-trivial class of complexes that are 4- but not 0-decomposable (cf. Remark 5) in arbitrarily high dimension. This infinite family does not disprove Simon’s conjecture, because the shelling of the Alexander dual of  $\text{Cl}(\Delta_k)$ , which is  $(5k - 2)$ -dimensional on  $5k + 2$  vertices, does extend to a shelling of the  $(5k - 2)$ -skeleton of the  $(5k + 1)$ -simplex, as we will see in Remark 8.

## 2. CONSTRUCTION OF THE COUNTEREXAMPLES

Recall that the link and the deletion of a face  $\sigma \in \Delta$  are defined respectively by

$$\text{link}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \subseteq F \supseteq \tau \text{ for some facet } F\} \quad \text{and} \quad \text{del}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \not\subseteq \tau\}.$$

We say that a face  $\sigma$  in a pure simplicial complex  $\Delta$  is *shedding* if  $\text{del}_\Delta(\sigma)$  is pure of dimension  $\dim \Delta$ . An equivalent formulation (see for instance [26, Definition 3.1]) is the following:  $\sigma$  is shedding if and only if for every face  $F \in \Delta$  such that  $\sigma \subseteq F$  and for every  $v \in \sigma$ , there exists  $w \notin F$  such that  $(F \setminus \{v\}) \cup \{w\} \in \Delta$ . A pure simplicial complex  $\Delta$  is *k-decomposable* if  $\Delta$  is a simplex or if there exists a shedding face  $\sigma \in \Delta$  with  $\dim \sigma \leq k$  such that  $\text{link}_\Delta(\sigma)$  and  $\text{del}_\Delta(\sigma)$  are both *k-decomposable* [22]. It is easy to see that if  $\Delta$  is *k-decomposable* then it is also *t-decomposable*, for every  $k \leq t \leq \dim \Delta$ . The notion of *k-decomposable* interpolates between vertex-decomposable complexes (which are the same as 0-decomposable complexes) and shellable complexes (which are the same as *d-decomposable* complexes, where *d* is their dimension).

We start with a Lemma that is implicit in the work of Bigdeli–Faridi [6]. Recall that a *free face* in a simplicial complex  $\Delta$  is a face strictly contained in only one facet of  $\Delta$ .

**Lemma 1.** *Let  $r$  be a ridge of a pure  $d$ -dimensional simplicial complex  $\Delta$ , with  $d \geq 1$ . Let  $S$  be the set of vertices of  $\text{Star}(r, \Delta)$ . Then  $S \in \text{Cl}(\Delta) \iff r$  is a free face in  $\text{Cl}(\Delta)$ .*

*Proof.*  $\implies$ : If  $r$  lies in two facets  $F_1$  and  $F_2$  of  $\text{Cl}(\Delta)$ , then  $F_i = r \cup S_i$  for some  $S_i \subseteq [n]$ . Since  $F_1, F_2 \in \text{Cl}(\Delta)$ , for every  $s \in S_1 \cup S_2$  we have  $r \cup \{s\} \in \Delta$ . So  $r \cup (S_1 \cup S_2) \subseteq S$  is a clique of  $\Delta$ . Since  $r \cup S_1$  and  $r \cup S_2$  are both facets of  $\text{Cl}(\Delta)$ , we have  $S_1 = S_2$ , whence  $F_1 = F_2$ .

$\impliedby$ : Let  $F$  be the unique facet of  $\text{Cl}(\Delta)$  that contains  $r$ . Were there a vertex  $s$  of  $S$  outside  $F$ , we would have  $r \cup \{s\} \in \Delta \subseteq \text{Cl}(\Delta)$ ; so there would be  $G \in \text{Cl}(\Delta)$ ,  $G \neq F$ , such that  $r \cup \{s\} \subseteq G$ , a contradiction. Hence  $S \subseteq F$  and  $S \in \text{Cl}(\Delta)$ .  $\square$

**Lemma 2.** *Let  $\Delta$  be a pure simplicial complex. If  $\Delta$  is ridge-chordal and  $\dim \Delta = \dim \text{Cl}(\Delta)$ , then  $\Delta$  has at least one free ridge.*

*Proof.* If  $\Delta$  is ridge-chordal, then it must have a ridge  $r$  such that the vertices of  $\text{Star}(r, \Delta)$  form a clique. By Lemma 1, this  $r$  is a free face of  $\text{Cl}(\Delta)$ . But since  $\dim \Delta = \dim \text{Cl}(\Delta)$ , the set of free ridges of  $\Delta$  coincides with the set of  $(\dim \Delta - 1)$ -dimensional free faces of  $\text{Cl}(\Delta)$ , because any ridge we add when passing from  $\Delta$  to  $\text{Cl}(\Delta)$  belongs to no face of dimension equal to  $\dim \Delta$ .  $\square$

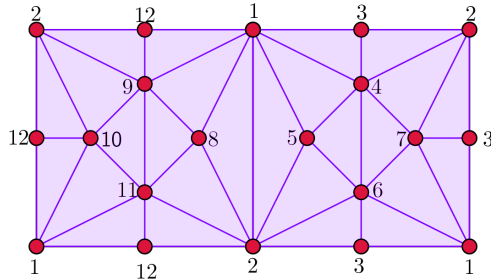


FIGURE 1. The constructible contractible 2-complex  $\Delta_2^2$  without free edges, constructed by Barmak in [4, Example 11.2.9], is not ridge-chordal: See Proposition 3 below.

A pure  $d$ -dimensional complex  $\Delta$  with  $N$  facets is *constructible* if either  $d(N - 1) = 0$ , or if  $\Delta$  splits as  $\Delta = \Delta_1 \cup \Delta_2$ , with  $\Delta_1, \Delta_2$  constructible  $d$ -dimensional complexes and  $\Delta_1 \cap \Delta_2$  constructible  $(d - 1)$ -dimensional. All shellable complexes are constructible, but the converse is false, cf. e.g. [5, Prop. 6.7].

**Proposition 3.** *For any integers  $d, k \geq 2$  there is a constructible, contractible  $d$ -dimensional complex on  $k2^d + k + d$  vertices that is not ridge-chordal.*

*Proof.* For any  $d \geq 2$ , there exists a shellable contractible simplicial  $d$ -complex  $C_d$  on  $2^d + d + 1$  vertices that has only one free ridge [2]. Let  $\Delta_k^d$  be the  $d$ -complex obtained by glueing together  $k$  copies of the complex  $C_d$  via the identification of their free ridges; the case  $d = k = 2$  is illustrated in Figure 1, and appeared also in Barmak's book [4, Example 11.2.9]. By definition,  $\Delta_k^d$  is constructible and has  $k(2^d + d + 1) - (k - 1)d = k2^d + k + d$  vertices. By van Kampen's theorem,  $\Delta_k^d$  is contractible. We claim that  $\dim \Delta_k^d = \dim \text{Cl}(\Delta_k^d)$ . In fact, were there a face in  $\text{Cl}(\Delta_k^d)$  of dimension  $t > d$ , then  $\Delta_k^d$  would contain an induced subcomplex  $S$  on  $t + 1$  vertices with the same  $d$ -skeleton of the  $t$ -simplex. So  $S$  would have nontrivial  $d$ -th homology, against the contractibility of  $\Delta_k^d$ . This proves the claim. But since  $\Delta_k^d$  has no free ridge, it is neither ridge-chordal (by Lemma 2) nor shellable (because all shellable contractible complexes are collapsible, cf. [16, Lemma 17]).  $\square$

All minimal non-faces of  $\text{Cl}(\Delta_k^d)$  have dimension  $d$ . So the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is pure  $(k2^d + k - 2)$ -dimensional, with  $k2^d + k + d$  vertices and  $\binom{k2^d + d + k}{d + 1} - f_d(\Delta_k^d)$  facets. To disprove Conjecture A, it remains to find values of  $d$  and  $k$  for which the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is shellable.

**Lemma 4.** *Let  $\Delta$  be a pure simplicial complex on  $[n]$ . Suppose that the minimal non-faces  $N_1, \dots, N_t$  of  $\Delta$  have the property that  $N_j \cap N_h = \emptyset$  for every  $j \neq h$ . Then  $\Delta$  is vertex-decomposable.*

*Proof.* Let  $m := \max\{|N_i|\}_{1 \leq i \leq t}$  and  $V := [n] \setminus \bigcup_{i=1}^t N_i$ . If  $m = 1$ , then  $|N_i| = 1$  for every  $1 \leq i \leq t$ . So

$$\Delta = \begin{cases} \{\emptyset\} & \text{if } V = \emptyset \\ \text{a simplex} & \text{if } V \neq \emptyset. \end{cases}$$

Either way,  $\Delta$  is vertex-decomposable and we are done. Now suppose  $m > 1$  and denote by  $\partial N_i$  the boundary of a simplex on the vertices of  $N_i$ . Then

$$\Delta = \begin{cases} \partial N_1 * \cdots * \partial N_t & \text{if } V = \emptyset \\ V * \partial N_1 * \cdots * \partial N_t & \text{if } V \neq \emptyset, \end{cases}$$

where  $*$  denotes the join of simplicial complexes on disjoint sets of vertices. Either way,  $\Delta$  is the join of vertex-decomposable complexes, hence vertex-decomposable.  $\square$

**Proof of Theorem A.** Let  $k \geq 2$  and let  $A_k$  be the Alexander dual of  $\text{Cl}(\Delta_k^2)$ . Since all minimal non-faces of  $\text{Cl}(\Delta_k^2)$  have dimension 2, this  $A_k$  is pure  $(5k - 2)$ -dimensional, with  $n := 5k + 2$  vertices and  $\binom{5k+2}{3} - 13k$  facets. Let  $\gamma_j$  be the set of vertices in the  $j$ -th copy of  $C_2$  that do not belong to the free face. Then  $[n] \setminus \gamma_j$  is not in  $\text{Cl}(\Delta_k^2)$ , because  $\dim([n] \setminus \gamma_j) = 5(k - 1) + 1 > 2 = \dim \text{Cl}(\Delta_k^2)$ . So  $\gamma_j \in A_k$ , for all  $1 \leq j \leq k$ . Define

$$D_0^k := A_k, \quad D_j^k := \text{del}_{D_{j-1}^k}(\gamma_j), \quad \text{and } L_j^k := \text{link}_{D_{j-1}^k}(\gamma_j), \quad \text{for } 1 \leq j \leq k.$$

If  $j > 1$  and  $t \geq j$ , we have  $\gamma_t \in D_{j-1}^k$ , because  $\gamma_h \not\subseteq \gamma_t$ , for every  $h \leq j - 1$ . Moreover, if  $k > 2$ ,  $\gamma_{j-1} \cup \gamma_j \in D_{j-2}^k$ , i.e.  $\gamma_{j-1} \in \text{link}_{D_{j-2}^k}(\gamma_j)$ , because  $\dim([n] \setminus (\gamma_{j-1} \cup \gamma_j)) = 5(k - 2) + 1 > 2 = \dim \text{Cl}(\Delta_k^2)$ .

We are going to show that  $A_k$  is 4-decomposable by induction on  $k \geq 2$ . Let  $k = 2$ . We checked using [13] that  $D_1^2$  and  $D_2^2$  are pure 8-dimensional. Moreover, we checked that  $L_1^2 \simeq L_2^2 \simeq A_1$ , where  $A_1$  is the Alexander dual of  $\text{Cl}(C_2)$ . The reader may verify that a shelling for such 3-complex is

$$\begin{aligned} & [4, 5, 6, 7], [3, 5, 6, 7], [2, 4, 6, 7], [1, 4, 6, 7], [1, 3, 6, 7], [1, 2, 6, 7], [3, 4, 5, 7], [1, 3, 5, 7], \\ & [1, 2, 5, 7], [2, 3, 5, 7], [2, 3, 4, 7], [1, 2, 4, 7], [3, 4, 5, 6], [2, 3, 4, 6], [2, 3, 5, 6], [1, 2, 5, 6], \\ & [1, 3, 4, 6], [1, 2, 4, 6], [1, 2, 3, 6], [1, 3, 4, 5], [1, 2, 4, 5], [1, 2, 3, 4]. \end{aligned}$$

Since  $D_2^2$  is vertex-decomposable, it follows that  $A_2$  is 4-decomposable.

Now let  $k > 2$ . Notice that  $\text{link}_{A_k}(\gamma_j) \simeq A_{k-1}$ , for every  $j$ , where ‘ $\simeq$ ’ stands for ‘combinatorially equivalent’. In particular,  $L_1^k \simeq A_{k-1}$ . In general, we have  $L_j^k \simeq D_{j-1}^{k-1}$ . We proceed by induction on  $j$ . Let  $j > 1$ . We have

$$L_j^k = \text{link}_{D_{j-1}^k}(\gamma_j) = \text{link}_{\text{del}_{D_{j-2}^k}(\gamma_{j-1})}(\gamma_j) = \text{del}_{\text{link}_{D_{j-2}^k}(\gamma_{j-1})}(\gamma_{j-1}) \simeq \text{del}_{D_{j-2}^{k-1}}(\gamma_{j-1}) = D_{j-1}^{k-1},$$

where the combinatorial equivalence is ensured by  $\text{link}_{D_{j-2}^k}(\gamma_j) \simeq L_{j-1}^k \simeq D_{j-2}^{k-1}$ . Moreover, the third equality holds because, for every  $G \in \Delta$  and  $F \in \text{link}_{\Delta}(G)$ , we have  $\text{link}_{\text{del}_{\Delta}(F)}(G) = \text{del}_{\text{link}_{\Delta}(G)}(F)$ . We have to verify that for  $j = 1, 2, 3$ ,  $\gamma_j$  is a shedding face of  $D_{j-1}^k$ . Here is a proof:

- Let  $F = [n] \setminus S$  be a facet of  $A_k = D_0^k$  containing  $\gamma_1$ . Let  $w \in \gamma_1$ . We claim that there exists  $s \in S$  such that  $\{s, w\} \notin \Delta_k^2$ . In fact,  $S \cap \gamma_j \neq \emptyset$  for some  $j \geq 2$ , otherwise  $\bigcup_{j=2}^k \gamma_j \subseteq F$ . Let  $r$  be the free ridge of  $C_2$ . Hence  $S \subseteq r \cup \gamma_1$  and  $S \cap \gamma_1 \neq \emptyset$ , a contradiction. Let  $v \in S \setminus \{s\}$  and we have  $(F \setminus \{w\}) \cup \{v\} \in A_k$ , because  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ .

- Let  $F = [n] \setminus S$  be a facet in  $D_1^k$  containing  $\gamma_2$ . Let  $w \in \gamma_2$ . Notice that  $S \cap \gamma_1 \neq \emptyset$ . Let  $s \in S \cap \gamma_1$  and consider  $v \in S \setminus \{s\}$ . We have  $(F \setminus \{w\}) \cup \{v\} \in D_1^k$ . In fact,  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ , because  $\{s, w\} \notin \Delta_k^2$ , and  $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_1 \neq \emptyset$ .
- Let  $F = [n] \setminus S$  be a facet in  $D_2^k$  containing  $\gamma_3$ . Let  $w \in \gamma_3$ . Notice that  $S \cap \gamma_1 \neq \emptyset$  and  $S \cap \gamma_2 \neq \emptyset$ . Let  $s_i \in S \cap \gamma_i$ , for  $i = 1, 2$ , and consider  $v \in S \setminus \{s_1, s_2\}$ . We have  $(F \setminus \{w\}) \cup \{v\} \in D_2^k$ . In fact,  $(S \setminus \{v\}) \cup \{w\} \notin \text{Cl}(\Delta_k^2)$ , because  $\{s_1, s_2\} \notin \Delta_k^2$ , and  $[(S \setminus \{v\}) \cup \{w\}] \cap \gamma_i \neq \emptyset$ , for  $i = 1, 2$ .

Now we are ready to conclude.

Since  $L_j^k \simeq D_{j-1}^{k-1}$ , the complexes  $L_j^k$  are 4-decomposable for  $1 \leq j \leq 3$ , by the inductive assumption. The unique minimal non-faces of  $D_3^k$  are  $\{\gamma_1, \gamma_2, \gamma_3\}$ , because the set of facets of  $D_3^k$  is  $\{[n] \setminus S \in A_k : |S| = 3, |S \cap \gamma_j| = 1, j = 1, 2, 3\}$ . Since  $\{\gamma_1, \gamma_2, \gamma_3\}$  are disjoint, then  $D_3^k$  is vertex-decomposable by Lemma 4. Hence  $A_k$  is 4-decomposable, as desired.  $\square$

**Remark 5.** By the work of Bidgeli, Faridi [6] and Nikseresht [21] there cannot be any 0-decomposable counterexample to Conjecture A. To see this, recall that the  $d$ -closure of a pure  $d$ -dimensional simplicial complex  $\Delta$  (see [6, Definition 2.1]) is exactly the clique complex  $\text{Cl}(\Delta)$ . Hence, by [6, Proposition 2.7] and [6, Theorem 3.4], the following properties are equivalent:

- $\Delta$  is ridge-chordal;
- $\text{Cl}(\Delta)$  is  $d$ -chordal, in the sense of Bigdeli-Faridi [6, Definition 2.6];
- $\text{Cl}(\Delta)$  is  $d$ -collapsible, in the sense of Wegner [25].

Now, let  $\Delta$  be a complex such that the Alexander dual of  $\text{Cl}(\Delta)$  is 0-decomposable. By [6, Theorem 5.2], the complex  $\text{Cl}(\Delta)$  is  $d$ -chordal; so by the equivalence above,  $\Delta$  is ridge-chordal and Conjecture A holds. En passant, this also explains why Conjecture A is equivalent to [6, Question 6.3]. Our complex  $\Delta_2^2$  of Figure 1 is not ridge-chordal, so in particular  $\text{Cl}(\Delta_2^2)$  is not 2-chordal.

**Remark 6.** In the literature, the problems we discussed are often phrased in terms of “clutters”. Let  $d \geq 1$  be an integer. A  $d$ -uniform clutter  $\mathcal{C}$  is the collection of the facets of a pure  $(d-1)$ -dimensional simplicial complex  $\Gamma_{\mathcal{C}}$ . Denote by  $I(\mathcal{C})$  the *edge ideal* of  $\mathcal{C}$ . Let  $\bar{\mathcal{C}}$  be the clutter with vertices  $1, \dots, n$  whose edges are the  $(d-1)$ -dimensional non-faces of  $\Gamma_{\mathcal{C}}$ . It is easy to see that the edge ideal of  $\bar{\mathcal{C}}$  is the Stanley–Reisner ideal of  $\text{Cl}(\Gamma_{\mathcal{C}})$ . Moreover, the ridge-chordality of  $\Gamma_{\mathcal{C}}$  is equivalent to the chordality of  $\mathcal{C}$ , as defined in [7]. With this terminology, Conjecture A can be rephrased as

“For  $d \geq 2$ , if  $\mathcal{C}$  is a  $d$ -uniform clutter such that  $I(\bar{\mathcal{C}})$  has linear quotients, then  $\mathcal{C}$  is chordal.”

Theorem A, forgetting the constructibility and the 4-decomposability claims, could be then stated as

“Infinitely many 3-uniform clutters  $\mathcal{C}$  such that  $I(\bar{\mathcal{C}})$  has linear quotients, are not chordal.”

**Remark 7.** Ridge-chordality was introduced in [7] with the goal to extend Fröberg’s characterization of the squarefree monomial ideals with 2-linear resolutions [20]. This notion was also implicit in [3, Section 6.2] and [17]. Several other higher-dimensional extensions of graph chordality exist in the literature: see for instance [1], [19], [24], [26]. A weakening of ridge-chordality is the demand that  $I(\bar{\Delta})$  have a linear resolution over any field [7, Theorem 3.2], where  $\bar{\Delta}$  is the complex whose facets are the  $d$ -dimensional non-faces of  $\Delta$ . As shown by [6, Example 4.8] or by our complex  $\Delta_2^2$  of Figure 1, some complexes satisfying this property are not ridge-chordal. En passant, this clarifies what is new in Proposition 3: examples of constructible and even shellable non-ridge-chordal complexes were previously known, but they are not contractible, see for instance [10, Exercise 7.37, pag. 277]. Examples of contractible non-ridge-chordal complexes were also known, like [6, Example 4.8], but they are not constructible.

**Remark 8.** Let  $\Delta$  be a pure  $d$ -complex on  $n + 1$  vertices such that  $\dim \Delta = \dim \text{Cl}(\Delta)$ . We claim that if the Alexander dual of  $\text{Cl}(\Delta)$  is shellable, then the shelling extends to the  $(n - d - 1)$ -skeleton of the  $n$ -simplex. In fact, all the minimal non-faces of  $\text{Cl}(\Delta)$  have cardinality  $d + 1$ . Hence the Alexander dual  $A$  of  $\text{Cl}(\Delta)$  has dimension  $k - 1$ , where  $k = n - d$ . Moreover, the  $(k - 2)$ -skeleton of  $A$  is the  $(k - 2)$ -skeleton of the  $n$ -simplex. By contradiction, let  $N$  be a minimal non-face of  $A$ , with  $|N| < k$ . Then  $[n + 1] \setminus N$  is a facet of  $\text{Cl}(\Delta)$  of cardinality  $|[n + 1] - N| = n + 1 - |N| > n + 1 - k = d + 1$  and  $\dim \text{Cl}(\Delta) > d$ . This implies that all the missing facets of  $A$  of dimension  $k - 1$  can be attached along their whole boundary to extend the shelling.

### 3. OPEN PROBLEMS

We conclude proposing two questions:

**Question 9.** Is it true that the Alexander dual of  $\text{Cl}(\Delta_k^d)$  is  $2^d$ -decomposable?

**Question 10.** If both  $\Delta$  and the Alexander dual of  $\text{Cl}(\Delta)$  are shellable, is it true that  $\Delta$  is ridge-chordal?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FL 33146

*Email address:* [bruno@math.miami.edu](mailto:bruno@math.miami.edu)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA, BOLOGNA, ITALIA, 40126

*Email address:* [davide.bolognini.cast@gmail.com](mailto:davide.bolognini.cast@gmail.com)