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Abstract

In this work, nonlinear free vibrations of fully geometrically exact Timoshenko-Ehrenfest beams are investigated. First, the exact strong form of the Timonshenko-Ehrenfest beam, considering the geometrical nonlinearity, is derived, and the required formulations are obtained. Since the strong forms of governing equations are highly nonlinear, a nonlinear finite element analysis (FEA) is employed to obtain the weak form. The FEA is utilized to compute natural frequencies and mode shapes; the direct scheme is adopted to solve the eigenvalue problem which is obtained by eliminating nonlinear terms. Then, each eigenvector is normalized, and the nonlinear stiffness matrix is derived and the nonlinear free vibration analysis is carried out. A recursive procedure is adopted to proceed until the convergence criterion is satisfied. Finally, the applicability of the proposed formulation is provided with some examples and results are compared with those available in the literature.

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1. Introduction

Beams are slender structural elements that are subjected to shear and axial forces and bending and torsional moments. Due to their simple shapes and resistance to different types of loading, they have been widely used in mechanical, civil, structural and aerospace engineering (Aria et al., 2019, Karami et al., 2020, Alambeigi et al., 2020, Dadgar-Rad and Firouzi, 2021, Żur et al., 2023). During the past century, several authors studied beam behaviors from different points of view. It is not the aim of this paper or even possible to mention all investigations on beams. However, some selective papers with reference to this paper are discussed below.

Nonlinear vibration of simply-supported beams using elliptic integrals was studied by Woinowsky-Krieger (1950). Later, Srinivasan (1966) studied free and forced vibration of beams and plates undergoing moderately large amplitude using the average method of Ritz. On the other hand, Mei (1973) proposed a finite element formulation for free vibration of beams and plates. By contrast, Manolis and Beskos (1980) investigated the dynamic response of beams to applied thermal loads by the Laplace transform whereas Sato (1983) studied free vibration of beam with sudden change in the cross-section. Lee et al. (1990) studied free and force vibration of non-uniform Euler-Bernoulli beam. Free vibration of simply supported Euler-Bernoulli beam based on Rayleigh-Ritz method was analyzed by Singh et al. (1990). Interestingly, Wickert (1992) studied free vibration of axially moving tensioned beam over sub- and supercritical transport speed ranges. Around the same period, Shen and Chu (1992) determined the vibration of uniform beam containing a single edge fatigue crack. Ribeiro and Petyt (1999) derived a hierarchical FEM and harmonic balance method to investigate the nonlinear free vibration of geometrically nonlinear beams. Zhong and Guo (2003) studied nonlinear vibration of a simply supported Timoshenko beam using differential quadrature method. Ding and Chen (2011) studied nonlinear vibration of axially moving beams using fast Fourier transform. Ghayesh and Amabili (2013a) studied the nonlinear forced vibrations and stability of an axially moving Timoshenko beam with an intra-span spring-support. Ghayesh and Amabili (2013b) studied nonlinear planar dynamics of an axially moving

Timoshenko beam using Galerkin method. Nonlinear forced dynamics of an axially moving Timoshenko beam with internal resonance was studied by Ghayesh and Amabili (2013c). Lenci and Rega (2016a) obtained the exact equations of motion of a planar, initially straight beam within the large displacement framework, by considering geometrical nonlinearities and linear elastic behaviour of the material. To do so, the kinematics of the beam in nonlinear deformation was derived, and then the strong form for the beam was derived without specific approximations. Studying nonlinear free oscillations of a straight planar Timoshenko-Ehrenfest beam analytically by means of the asymptotic development method was done by Lenci and Rega (2016b). Lenci et al. (2016) studied hardening and softening behaviours of planar beams with exact equation of motion. Ding et al. (2018) derived natural frequencies and mode shapes of asymmetric beam with an approximate analytical and a numerical method. Lenci and Clementi (2018) studied nonlinear oscillations of a Euler–Bernoulli beam hinged at one end and having a roller support sliding on a inclined line on the other end. Aria et al. (2019) studied thermal vibration of a cracked nanobeam employing FE analysis. Karami et al. (2020) studied dynamics of two-dimensional Timoshenko nanobeam using nonlocal strain gradient theory. Free and forced vibration of sandwich beam considering porous core on Vlasov's foundation was investigated by Alambeigi et al. (2020). Utzeri et al. (2021) investigated nonlinear dynamic behaviour of a cantilever composite beam with analytical, numerical and experimental approaches. Experimental vibration responses of beams with nonlinear boundary conditions were investigated by Balasubramanian et al. (2021). Nonlinear vibration of imperfect hyperelastic beams with porosity was studied by Khaniki et al. (2021). Firouzi and Kazemi (2023) studied stability of Timoshenko beam using time-modulated axial force. Nonlinear vibrations and viscoelastivity of a self-healing composite cantilever beam were investigated by Amabili et al. (2022).

A survey of the literature shows that, by and large, analytical approaches have been used to investigate nonlinear vibration of beams. Moreover, in case of numerical approaches, the variational methods were used to study the nonlinear vibration of the beams by finite element method. Accordingly, the main objective of this paper is to investigate nonlinear vibrations of Timoshenko-Ehrenfest beams considering geometrical nonlinearity. To achieve this, at first, the exact equation of motion, or in other words, the strong form of the initially straight Timoshenko-Ehrenfest beam is derived. Afterwards, multiplying the test function and taking integrals over the whole body of the beam, the weak form of the solution is obtained. To solve the finite element formulation derived for geometrically exact Timoshenko-Ehrenfest beam, the direct scheme (DS) is utilized. To do so, firstly, the linear eigenvalue and eigenvector is acquired by eliminating nonlinear terms. Then, the eigenvector must be normalized to the arbitrary given amplitude for that specific problem, and finally, the nonlinear stiffness matrix is obtained. This recursive method goes on as long as the convergence criterion is satisfied.

The structure of the paper is organized as follows. Following this Introduction section, in Section 2, the strong form of an initially straight Timoshenko-Ehrenfest beam is derived. The finite element procedure and derivation of the weak form are discussed in Section 3. In Section 4, direct scheme (DS) is explained. Numerical examples and comparison with literature are provided in Section 5. Finally, concluding remarks are discussed in Section 6.

2. Derivation of the strong formulation for Timoshenko-Ehrenfest beams

In this section, the exact governing equations of motions for nonlinear deformation of Timoshenko-Ehrenfest beams are derived. A straight planar beam is considered, as shown in Fig. 1. It is noted that u and w are the axial and the transversal displacement of the beam, respectively.

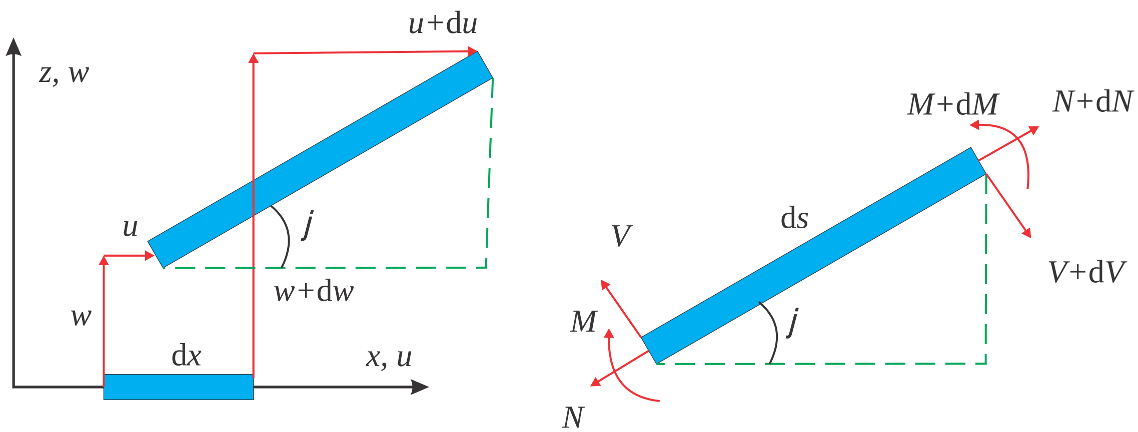


Fig. 1. Schematic of undeformed and deformed shapes of a small element of the beam, and the applied forces and moments

According to Fig. 1, the following relations can be obtained

$$s' = \sqrt{(1+u')^2 + w'^2}, \ \cos\varphi = \frac{1+u'}{s'}, \ \sin\varphi = \frac{w'}{s'} \tag{1}$$

where φ is the slope angle with respect to the axis of the beam, and the prime denotes derivative with respect to the beam axis x. Moreover, the strain measures are derived as follows:

$$e = s' - 1, \quad \gamma = \theta - \varphi, \quad k = \frac{\theta'}{s'}, \tag{2}$$

where e, γ, k are the elongation, shear strain and the curvature of the beam, respectively, and θ denotes the rotation of the cross-section. Furthermore, the equilibrium equations for u, w and θ are derived as follows:

$$\left(N\cos\varphi + V\sin\varphi\right)' = \rho A\ddot{u}, \ \left(N\sin\varphi - V\cos\varphi\right)' = \rho A\ddot{w}, \ M' - Vs' = \rho J\ddot{\theta}. \tag{3}$$

where the superscript dot denotes the time derivative, ρ is the mass density and J is the inertia moment. Moreover, N and V are axial and shear forces and M is bending moment having the following relations:

$$N = EAe, \ V = GA\gamma, \ M = EJk \tag{4}$$

Considering the fact that the generalized axial strain obtained as follows:

$$\varepsilon = u' + \frac{1}{2}u'^2 + \frac{1}{2}w'^2, \ \sqrt{(1+u')^2 + w'^2} = \sqrt{2\varepsilon + 1} \tag{5}$$

the following equations of motions are derived for large deformation of geometrically exact Timoshenko-Ehrenfest beams (see, Lenci and Rega, 2016b):

$$\left(EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{1+u'}{\sqrt{2\varepsilon+1}} + GA\left(\theta - \arctan\left(\frac{w'}{1+u'}\right)\right)\frac{w'}{\sqrt{2\varepsilon+1}}\right)' = \omega^2\rho A\ddot{u} \tag{6}$$

$$\left(EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{w'}{\sqrt{2\varepsilon+1}} - GA\left(\theta - \arctan\left(\frac{w'}{1+u'}\right)\right)\frac{1+u'}{\sqrt{2\varepsilon+1}}\right)' = \omega^2\rho A\ddot{w} \tag{7}$$

$$\left(EJ\frac{\theta'}{\sqrt{2\varepsilon+1}}\right)' - GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\sqrt{2\varepsilon+1} = \omega^2\rho J\ddot{\theta} \tag{8}$$

3) Derivation of the weak form by finite element method

In the previous section, the strong form of the large deformation of the geometrically exact Timoshenko-Ehrenfest beam is derived. In this section, the weak formulation is derived. For later use, some constant values are defined as follows:

$$K_0 = E\int y^0 dA = EA,\quad K_S = GA,\quad K_2 = E\int y^2 dA = EJ \tag{9}$$

To derive the weak formulations, the strong formulations of the beam, namely Eqs. (6)-(8), are multiplied by virtual generalized displacements and then integrated through the length of the beam. To do so, Eq. (6), which is the balance equation in axial direction, is multiplied by virtual axial displacement δu, and then integrated through x as follows:

$$\begin{aligned}&\int_{x_a}^{x_b}\left(\left(EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{1+u'}{\sqrt{2\varepsilon+1}} + GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\frac{w'}{\sqrt{2\varepsilon+1}}\right)'\delta u\right)\mathrm{d}x\\ &= \omega^2\int_{x_a}^{x_b}\rho A\ddot{u}\delta u\,\mathrm{d}x,\end{aligned} \tag{10}$$

where x_a and x_b are the positions of the two-noded element. Using integration by part, the following relation is obtained:

$$\begin{aligned}&\int_{x_a}^{x_b}\left(-\left(EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{1+u'}{\sqrt{2\varepsilon+1}} + GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\frac{w'}{\sqrt{2\varepsilon+1}}\right)\delta u'\right)\mathrm{d}x\\ &= \omega^2\int_{x_a}^{x_b}\rho A\ddot{u}\delta u\,\mathrm{d}x.\end{aligned} \tag{11}$$

Employing the introduced constants in the first of this section, Eq. (11) can be rewritten as:

$$\begin{aligned}&\int_{x_a}^{x_b} -K_0\left(\sqrt{2\varepsilon+1}-1\right)\frac{(1+u')}{\sqrt{2\varepsilon+1}}\delta u'\,\mathrm{d}x\\ &+\int_{x_a}^{x_b} -K_S\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\frac{1}{\sqrt{2\varepsilon+1}}w'\delta u'\,\mathrm{d}x = \int_{x_a}^{x_b}\omega^2\rho A\ddot{u}\delta u\,\mathrm{d}x.\end{aligned} \tag{12}$$

Moreover, Eq. (7) which is the balance equation in transverse direction is multiplied by virtual transverse displacement δw, and then integrated through x as follows:

$$\int_{x_a}^{x_b}\left(EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{w'}{\sqrt{2\varepsilon+1}}-GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\frac{1+u'}{\sqrt{2\varepsilon+1}}\right)'\delta w\,\mathrm{d}x \\ =\omega^2\int_{x_a}^{x_b}\rho A\ddot{w}\delta w\,\mathrm{d}x, \tag{13}$$

and employing integration by part, the following relation is obtained:

$$\int_{x_a}^{x_b}-EA\left(\sqrt{2\varepsilon+1}-1\right)\frac{w'\delta w'}{\sqrt{2\varepsilon+1}}\mathrm{d}x+\int_{x_a}^{x_b}GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\frac{1+u'}{\sqrt{2\varepsilon+1}}\delta w'\,\mathrm{d}x \\ =\omega^2\int_{x_a}^{x_b}\rho A\ddot{w}\delta w\,\mathrm{d}x. \tag{14}$$

Using the defined constants, Eq. (14) may be rearranged as:

$$\int_{x_a}^{x_b}-K_0\left(\sqrt{2\varepsilon+1}-1\right)\frac{w'\delta w'}{\sqrt{2\varepsilon+1}}\mathrm{d}x+\int_{x_a}^{x_b}K_S\frac{1+u'}{\sqrt{2\varepsilon+1}}\theta\delta w'\mathrm{d}x \\ +\int_{x_a}^{x_b}-K_S\arctan\left(\frac{w'}{1+u'}\right)\frac{1+u'}{\sqrt{2\varepsilon+1}}\delta w'\mathrm{d}x=\omega^2\int_{x_a}^{x_b}\rho A\ddot{w}\delta w\,\mathrm{d}x. \tag{15}$$

Finally, Eq. (8) which is the balance equation for rotation is multiplied by virtual rotation $\delta\theta$, and then integrated through x as follows:

$$\int_{x_a}^{x_b}\left(\left(EJ\frac{\theta'}{\sqrt{2\varepsilon+1}}\right)'-GA\left(\theta-\arctan\left(\frac{w'}{1+u'}\right)\right)\sqrt{2\varepsilon+1}\right)\delta\theta\,\mathrm{d}x=\omega^2\int_{x_a}^{x_b}\rho J\ddot{\theta}\delta\,\theta\mathrm{d}x \tag{16}$$

Taking integration by part, and using the defined constants lead to the following relation:

$$\int_{x_a}^{x_b}-K_2\frac{\theta'\delta\theta'}{\sqrt{2\varepsilon+1}}\mathrm{d}x+\int_{x_a}^{x_b}-K_S\sqrt{2\varepsilon+1}\,\theta\delta\theta\,\mathrm{d}x \\ +\int_{x_a}^{x_b}K_S\arctan\left(\frac{w'}{1+u'}\right)\sqrt{2\varepsilon+1}\,\delta\theta\,\mathrm{d}x=\omega^2\int_{x_a}^{x_b}\rho J\ddot{\theta}\delta\theta\,\mathrm{d}x. \tag{17}$$

In this paper, two-noded beam element is considered. Accordingly, the position, axial and transversal displacements, and also rotation at each typical element are approximated via the following relations:

$$x(\xi)=\psi_1(\xi)x_1^e+\psi_2(\xi)x_2^e \tag{18}$$

$$u^e(\xi)=\psi_{u_1}(\xi)u_1^e+\psi_{u_2}(\xi)u_2^e, \tag{19}$$

$$w^e(\xi)=\psi_{w_1}(\xi)w_1^e+\psi_{w_2}(\xi)w_2^e, \tag{20}$$

$$\theta^e(\xi) = \psi_{\theta_1}(\xi)\theta_1^e + \psi_{\theta_2}(\xi)\theta_2^e, \tag{21}$$

where the superscript e denotes the typical element, and $\psi(\xi)$ are the interpolation functions. Moreover, the ξ is the natural coordinate of which domain is $\xi \in [-1,1]$. Considering isoparametric elements and the same interpolations for displacements and rotation, the shape functions are defined as follows:

$$\begin{aligned} \psi_1(\xi) &= \psi_{u_1}(\xi) = \psi_{w_1}(\xi) = \psi_{\theta_1}(\xi) = \frac{1}{2}(1-\xi), \\ \psi_2(\xi) &= \psi_{u_2}(\xi) = \psi_{w_2}(\xi) = \psi_{\theta_2}(\xi) = \frac{1}{2}(1+\xi) \end{aligned} \tag{22}$$

By substituting the approximations into Eqs. (12), (15) and (17), the following components for stiffness matrix are derived:

$$\begin{aligned} K_{uu} &= \int_{x_e}^{x_e+h_e} -K_0\left(\sqrt{2\varepsilon+1}-1\right)\frac{1}{\sqrt{2\varepsilon+1}}\psi'_{ui}\psi'_{uj}\,\mathrm{d}x \\ &\quad + \frac{1}{u_1^e+u_2^e}\int_{x_e}^{x_e+h_e} -K_0\left(\sqrt{2\varepsilon+1}-1\right)\frac{1}{\sqrt{2\varepsilon+1}}\psi'_{ui}\,\mathrm{d}x, \end{aligned} \tag{23}$$

$$K_{uw} = \int_{x_e}^{x_e+h_e} -K_S\left(\theta - \arctan\left(\frac{w'}{1+u'}\right)\right)\frac{1}{\sqrt{2\varepsilon+1}}\psi'_{ui}\psi'_{wj}\,\mathrm{d}x, \tag{24}$$

$$\begin{aligned} K_{ww} &= \int_{x_e}^{x_e+h_e} -K_0\left(\sqrt{2\varepsilon+1}-1\right)\frac{1}{\sqrt{2\varepsilon+1}}\psi'_{wi}\psi'_{wj}\,dx \\ &\quad + \frac{1}{w_1^e+w_2^e}\int_{x_e}^{x_e+h_e} -K_S\arctan\left(\frac{w'}{1+u'}\right)\frac{1+u'}{\sqrt{2\varepsilon+1}}\psi'_{wi}\,\mathrm{d}x, \end{aligned} \tag{25}$$

$$K_{w\theta} = \int_{x_e}^{x_e+h_e} K_S\frac{1+u'}{\sqrt{2\varepsilon+1}}\psi'_{wi}\psi_{\theta j}\,\mathrm{d}x, \tag{26}$$

$$K_{\theta w} = \frac{1}{w_1^e+w_2^e}\int_{x_e}^{x_e+h_e} K_S\arctan\left(\frac{w'}{1+u'}\right)\sqrt{2\varepsilon+1}\,\psi_{\theta i}\,\mathrm{d}x, \tag{27}$$

$$K_{\theta\theta} = \int_{x_e}^{x_e+h_e}\left(-K_2\frac{1}{\sqrt{2\varepsilon+1}}\psi'_{\theta i}\psi'_{\theta j} - K_S\sqrt{2\varepsilon+1}\,\psi_{\theta i}\psi_{\theta j}\right)\mathrm{d}x, \tag{28}$$

$$K_{u\theta} = 0, \quad K_{wu} = 0, \quad K_{\theta u} = 0. \tag{29}$$

It should be noted that in the above relations, the indices i, j are get the values $i = 1,2$ and $j = 1,2$. Moreover, h_e is the length of a typical element. By applying the same procedure for the RHS of the previous relations, the components for mass matrix are obtained as:

$$M_{uu}=\int_{x_e}^{x_e+h_e}\rho A\psi_{ui}\psi_{uj}\,\mathrm{d}x,\;\; M_{ww}=\int_{x_e}^{x_e+h_e}\rho A\psi_{wi}\psi_{wj}\,\mathrm{d}x,\;\; M_{\theta\theta}=\int_{x_e}^{x_e+h_e}\rho J\psi_{\theta i}\psi_{\theta j}\,\mathrm{d}x. \tag{30}$$

The explicit forms for the components of stiffness and mass matrices are derived in Appendix. Accordingly, the final relation at each element is derived as follows:

$$\mathbf{M}^e\ddot{\mathbf{U}}^e=\mathbf{K}^e\left(\mathbf{U}^e\right)\mathbf{U}^e, \tag{31}$$

where $\mathbf{U}^e$, $\mathbf{K}^e\left(\mathbf{U}^e\right)$, $\mathbf{M}^e$ are the displacement vector, stiffness matrix and mass matrix, respectively, and given by:

$$\mathbf{U}^e=\begin{bmatrix}u_1^e & u_2^e & w_1^e & w_2^e & \theta_1^e & \theta_2^e\end{bmatrix}^{\mathrm{T}} \tag{32}$$

$$\mathbf{K}^e\left(\mathbf{U}^e\right)=\begin{bmatrix}K_{uu}^e & K_{uw}^e & K_{u\theta}^e\\ K_{wu}^e & K_{ww}^e & K_{w\theta}^e\\ K_{\theta u}^e & K_{\theta w}^e & K_{\theta\theta}^e\end{bmatrix},\qquad \mathbf{M}^e=\begin{bmatrix}M_{uu}^e & 0 & 0\\ 0 & M_{ww}^e & 0\\ 0 & 0 & M_{\theta\theta}^e\end{bmatrix}. \tag{33}$$

By assembling these matrices over all elements, one obtains:

$$\mathbf{M}\ddot{\mathbf{U}}=\mathbf{K}\left(\mathbf{U}\right)\mathbf{U}, \tag{34}$$

where

$$\mathbf{M}=\mathcal{A}_{e=1}^{m}\mathbf{M}^e,\quad \mathbf{K}=\mathcal{A}_{e=1}^{m}\mathbf{K}^e,\quad \mathbf{U}=\mathcal{A}_{e=1}^{m}\mathbf{U}^e, \tag{35}$$

where $\mathcal{A}_{e=1}^{m}$ is assembly operator over all elements and m is the number of elements.

4) Direct Scheme

In previous sections, the strong and weak formulations for nonlinear vibrations of geometrically exact Timoshenko-Ehrenfest beams are derived. For free vibrations, the equations of motion can be written as in Eq.(34). Eq.(34) gives an eigenvalue problem when the non-linear terms are neglected. The direct scheme (DS) is used to solve Eq.(34). The procedure of this method is as follows:

a) The stiffness matrix $\mathbf{K}\left(\mathbf{U}\right)$ is evaluated in the first step by neglecting all the non-linear terms, which yields a linear stiffness matrix $\mathbf{K}_{linear}$. Using $\mathbf{K}_{linear}$ and $\mathbf{M}$, the natural frequency when linear theory is used, ω_0, associated to each eigenvector is obtained.

b) For any specified value of the vibration amplitude ratio a/r of the beam which is the ratio between the normalized mode shape and the radius of gyration r, the eigenvector is normalized with respect to the point of the beam where the amplitude is the largest.

c) Using the normalized eigenvector, the nonlinear terms in the stiffness matrix are obtained and the corresponding $\mathbf{K}(\mathbf{U})$ is evaluated.

d) Using the nonlinear $\mathbf{K}(\mathbf{U})$ and $\mathbf{M}$ and treating the problem as a linear eigenvalue problem, the new frequency parameter ω and eigenvector are obtained,

e) Steps from (b) to (d) are repeated till a convergence in frequency parameter is achieved to a prescribed accuracy.

5) Numerical results

To investigate the performance and the applicability of the formulation derived in previous sections, some examples are provided in this section. Numerical results include nonlinear vibrations of simply-supported, clamped-clamped, clamped-simply supported beams as well as cantilever and simply-supported beams with additional intermediate supports. While the initial cases are used for the validation of the proposed method, the last sub-sections introduce more complex problems which are easily solved with the proposed approach.

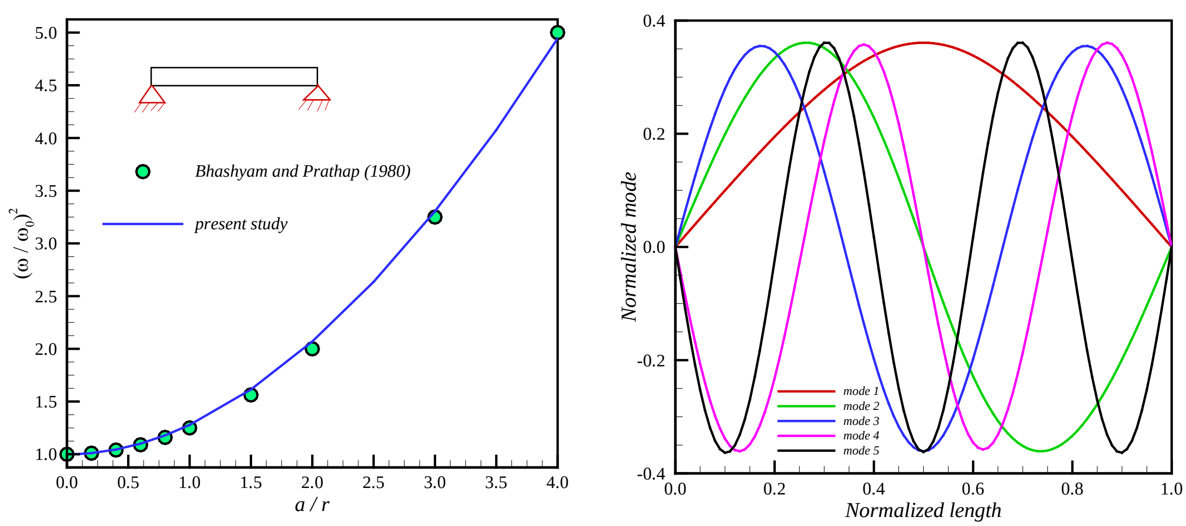
5.1. Nonlinear vibration analysis of a simply-supported beam

In this example, vibration of a Timoshenko-Ehrenfest beam with simply-supported boundary conditions in both ends is investigated (see, Fig. 2, a). The ratio of natural frequencies $(\omega/\omega_0)^2$ for different values of central amplitude ratio of the beam from $a/r \in \{0.2, 4\}$ is calculated. As can be observed in the Table 1, the results are in very good agreement

Table 1. The ratio of natural frequency $(\omega/\omega_0)^2$ of simply-supported beam for different values of central amplitude for the fundamental mode.

a/r	present study	Sarma and Varadan (1984)	Bhashyam and Prathap (1980)	Sarma and Varadan (1985)

0.2	1.0113	1.0099	1.0100	1.0100
0.4	1.0451	1.0399	1.0400	1.0400
0.6	1.1011	1.0899	1.0900	1.0900
0.8	1.1789	1.1600	1.1600	1.1600
1.0	1.2779	1.2500	1.2500	1.2500
1.5	1.6142	1.5621	1.5625	1.5624
2.0	2.0695	1.9999	2.0000	2.0000
2.5	2.6352	2.5616	-	2.5625
3.0	3.3052	3.2501	3.2500	3.2500
3.5	4.0756	4.0627	-	4.0625
4.0	4.9439	4.9966	5.0000	5.0000



(a) (b)

Fig. 2. (a) The backbone curves $(\omega/\omega_0)^2$ versus amplitude ratio a/r for simply-supported Timoshenko-Ehrenfest beam, (b) the first five mode shapes of simply-supported beam with amplitude ratio $a/r=4$

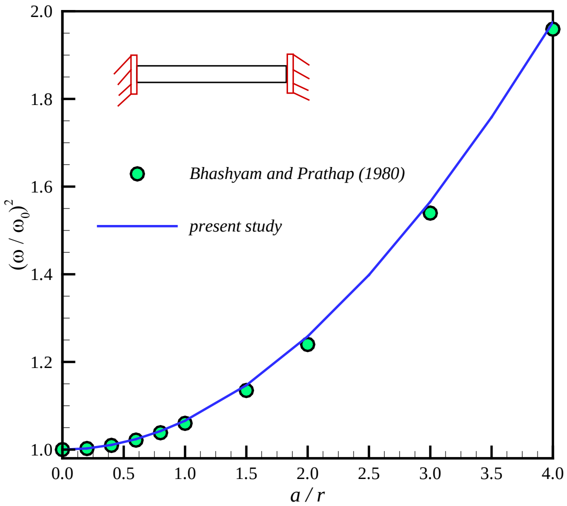
with the results reported Sarma and Vardan (1984), Bhashyam and Prathap (1980) and Sarma and Varadan (1985). Moreover, the curve of frequency ratio $(\omega/\omega_0)^2$ is provided in Fig. 2 (a) and compared with Bhashyam and Prathap (1980). Eventually, the first five mode shapes for Timoshenko-Ehrenfest beam with simply-supported boundary condition in both sides are portrayed in Fig. 2 (b).

5.2. Nonlinear vibration analysis of a clamped-clamped beam

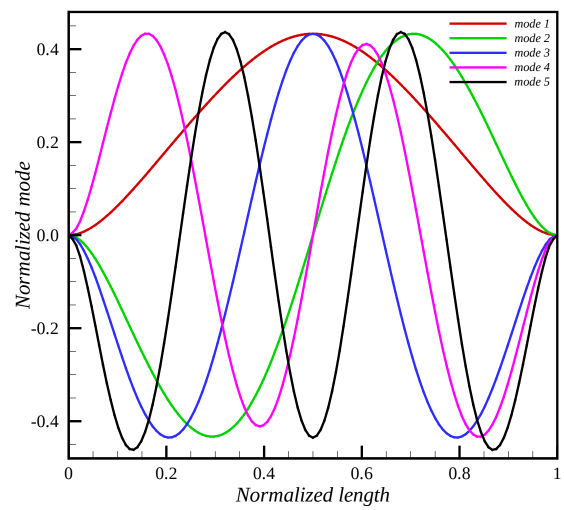
In the second example, vibration of a clamped-clamped Timoshenko-Ehrenfest beam is studied (see, Fig. 3, a). The ratio of natural frequencies $(\omega/\omega_0)^2$ for different values of central amplitude ratio of the beam from $a/r \in \{0.2, 4\}$ is calculated. As can be observed in the Table 2, the results are in very good agreement with the results reported Sarma and Vardan (1984) and Bhashyam and Prathap (1980). Moreover, the curve of frequency ratio $(\omega/\omega_0)^2$ is provided in Fig. 2 (a) and compared with Bhashyam and Prathap (1980). Finally, the first five mode shapes for Timoshenko-Ehrenfest beam with clamped boundary condition in both sides are displayed in Fig. 3 (b).

Table 2. The ratio of natural frequency $(\omega/\omega_0)^2$ of clamped-clamped beam for different values of central amplitude for the fundamental mode

a/r	present study	Sarma and Varadan (1984)	Bhashyam and Prathap (1980)
0.2	1.0026	1.0024	1.0024
0.4	1.0103	1.0096	1.0096
0.6	1.0238	1.0214	1.0216
0.8	1.0422	1.0383	1.0384
1.0	1.0658	1.0598	1.0599
1.5	1.1468	1.1344	1.1349
2.0	1.2582	1.2383	1.2398
2.5	1.3984	1.3714	-
3.0	1.5655	1.5327	1.5395
3.5	1.7582	1.7226	-
4.0	1.9748	1.9406	1.9591



(a)



(b)

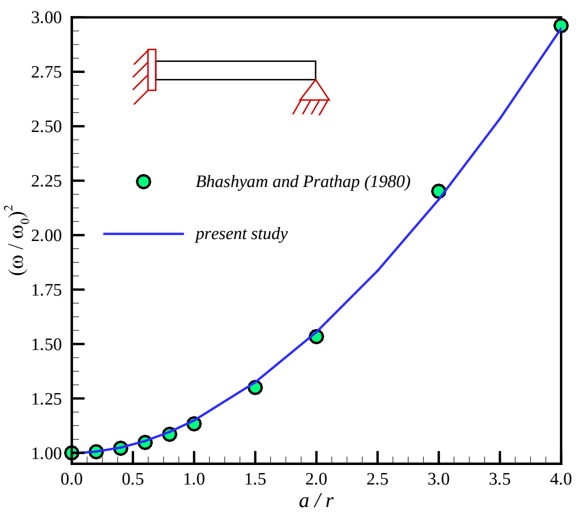
Fig. 3. (a) The backbone curves $(\omega/\omega_0)^2$ versus amplitude ratio a/r for clamped-clamped Timoshenko-Ehrenfest beam, (b) the first five mode shapes clamped-clamped beam with amplitude ratio $a/r=4$

5.3. Nonlinear vibration analysis of a clamped-simply supported beam

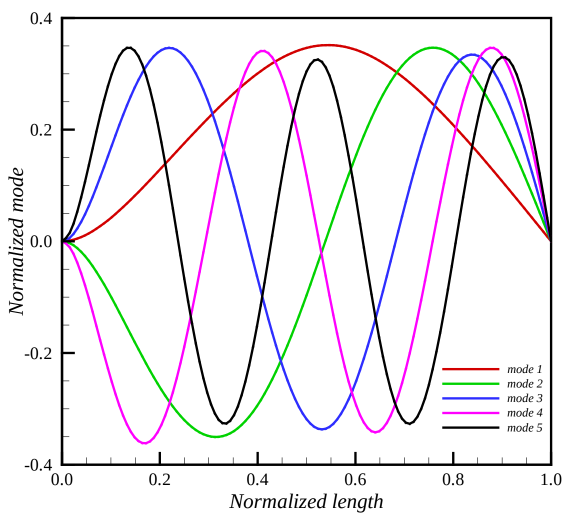
For this example, vibration analysis of a clamped-simply supported Timoshenko-Ehrenfest beam is considered (see, Fig. 4, a). The ratio of natural frequencies $(\omega/\omega_0)^2$ for different values of maximum amplitude ratio of the beam from $a/r \in \{0.2, 4\}$ is calculated. According to the Table 3, the results are in very good agreement with the results reported Sarma and Vardan (1984) and Bhashyam and Prathap (1980). Besides, the curve of frequency ratio $(\omega/\omega_0)^2$ is shown in Fig. 4 (a) and compared with the results from Bhashyam and Prathap (1980). Eventually, the first five mode shapes for Timoshenko-Ehrenfest beam with clamped-simply supported boundary conditions are portrayed in Fig. 4 (b).

Table 3. The ratio of natural frequency $(\omega/\omega_0)^2$ of clamped-supported beam for different values of amplitude ratio for the fundamental mode

a/r	present study	Sarma and Varadan (1984)	Bhashyam and Prathap (1980)
0.2	1.0062	1.0053	1.0053
0.4	1.0245	1.0021	1.0214
0.6	1.0548	1.0479	1.0481
0.8	1.0965	1.0850	1.0854
1.0	1.1490	1.1324	1.1335
1.5	1.3243	1.2948	1.3004
2.0	1.5551	1.5177	1.5340
2.5	1.8361	1.7973	-
3.0	2.1637	2.1338	2.2015
3.5	2.5345	2.5226	-
4.0	2.9489	-	2.9619



(a)



(b)

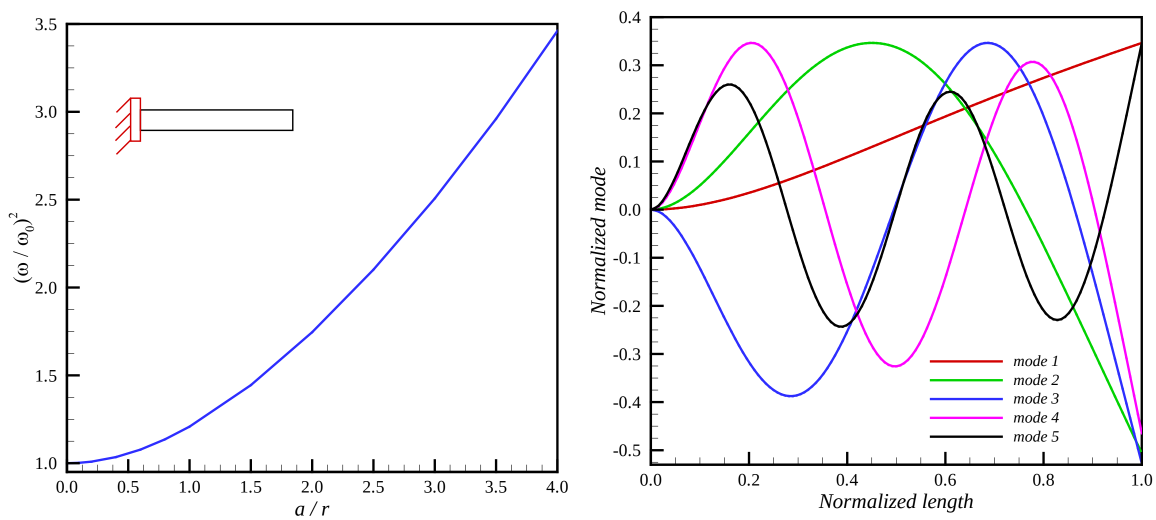
Fig. 4. (a) The backbone curves $(\omega/\omega_0)^2$ versus amplitude ratio a/r for clamped-simply supported Timoshenko-Ehrenfest beam, (b) the first five mode shapes clamped-simply supported beam with amplitude ratio $a/r = 4$

5.4. Nonlinear vibration analysis of a cantilever

In this example, vibration analysis of a cantilever is investigated (see, Fig. 5, a). The ratio of natural frequencies $(\omega/\omega_0)^2$ for different values of amplitude ratio of the beam from $a/r \in \{0.1, 4\}$ is calculated and provided in Table 4. Besides, the curve of frequency ratio $(\omega/\omega_0)^2$ against the ratio of maximum amplitude is shown in Fig. 5 (a). Furthermore, the first five mode shapes for a cantilever Timoshenko-Ehrenfest beam are portrayed in Fig. 5 (b).

Table 4. The ratio of natural frequency $(\omega/\omega_0)^2$ of clamped-clamped beam for different values of amplitude ratio for the fundamental mode

a/r	present study	a/r	present study
0.1	1.0022	1.5	1.4442
0.2	1.0088	2.0	1.7457
0.4	1.0348	2.5	2.1015
0.6	1.0775	3.0	2.5068
0.8	1.1356	3.5	2.9598
1.0	1.2082	4.0	3.4603



(a) (b)

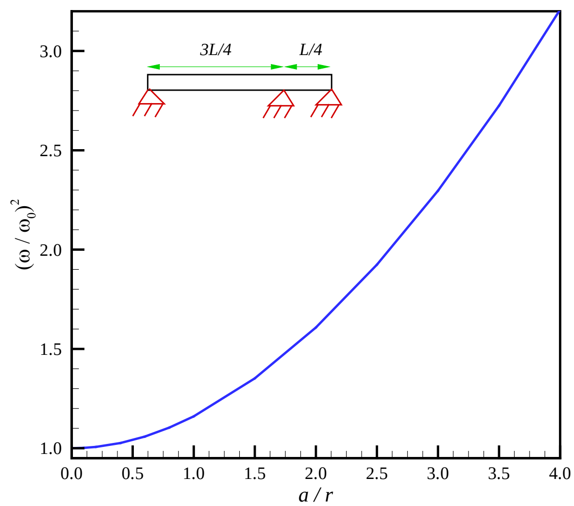
Fig. 5. (a) The backbone curve $(\omega/\omega_0)^2$ versus amplitude ratio a/r for a cantilever Timoshenko-Ehrenfest beam, (b) the first five mode shapes for cantilever with amplitude ratio $a/r = 4$

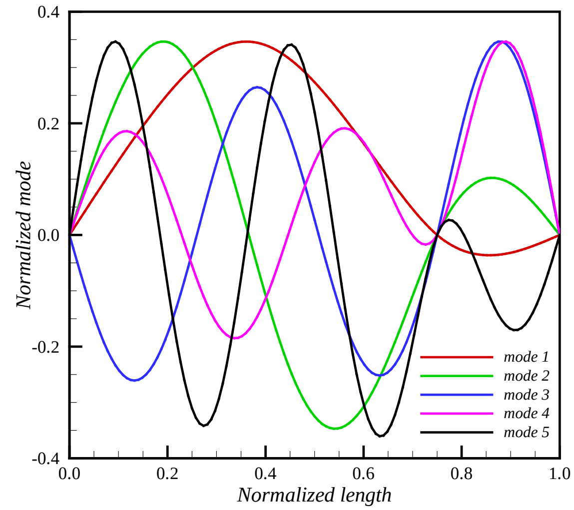
5.5. Nonlinear vibration analysis of a simply supported beam with an additional support

The vibration analysis of a Timoshenko-Ehrenfest beam with simply supported boundary conditions at both ends is considered. Besides, an additional intermediate support is added at $3L/4$, as shown in Fig. 6(a). The ratio of natural frequency $(\omega/\omega_0)^2$ for different values of amplitude ratio of the beam from $a/r \in \{0.2, 4\}$ is calculated and provided in Table 5. Furthermore, the curve of the frequency ratio $(\omega/\omega_0)^2$ versus the maximum amplitude ratio is shown in Fig. 6(a). It is interesting to compare Fig. 6(a) to Fig. 2(a) obtained for a simply supported beam. This comparison returns that the hardening-type nonlinearity is reduced in presence of the additional intermediate support. This is associated to the fact that the fundamental mode shape of the beam with intermediate support is closer to the second mode shape of the simply supported beam than to its fundamental mode shape. Finally, the first five modes are displayed in Fig. 6(b).

Table 5. The ratio of natural frequency $(\omega/\omega_0)^2$ of clamped-clamped beam for different values of amplitude ratio for the fundamental mode

a/r	present study	a/r	present study
0.1	1.0017	1.5	1.3517
0.2	1.0066	2.0	1.6077
0.4	1.0263	2.5	1.9240
0.6	1.0588	3.0	2.2964
0.8	1.1037	3.5	2.7248
1.0	1.1606	4.0	3.2084





(a) (b)

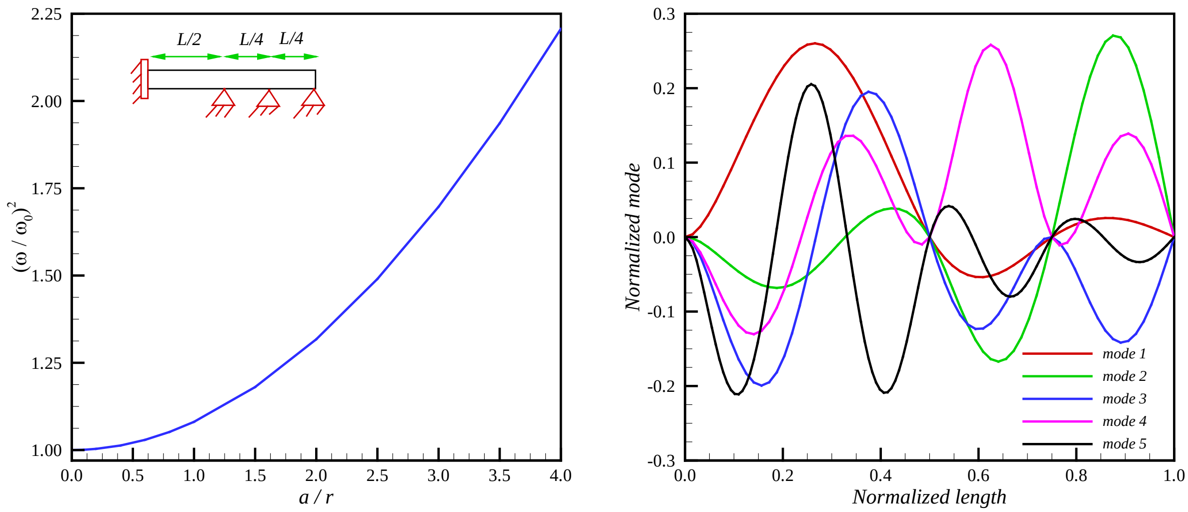
Fig. 6. (a) The backbone curve $(\omega/\omega_0)^2$ versus amplitude ratio a/r for the Timoshenko-Ehrenfest beam with an intermediate support at 3*L*/4; (b) the first five mode shapes of the simply-supported beam with an intermediate support at $3L/4$ for amplitude ratio $a/r=4$

5.6. Nonlinear vibration analysis of a clamped-simply supported beam with two additional supports

In this example, the vibration analysis of a Timoshenko-Ehrenfest beam with clamped-supported boundary conditions at both ends is considered. Moreover, two additional intermediate supports are added at $L/2$ and $3L/4$, as shown in Fig. 7(a). The ratio of natural frequency $(\omega/\omega_0)^2$ for different values of maximum amplitude ratio of the beam from $a/r \in \{0.2, 4\}$ is calculated and provided in Table 6. Furthermore, the curve of frequency ratio $(\omega/\omega_0)^2$ against the ratio of maximum amplitude is shown in Fig. 7(a). This shows a reduced hardening-type nonlinearity with respect to the one observed in Fig. 4(a) for a clamped-simply supported beam without intermediate supports. In fact, the mode shape of the fundamental mode of the beam with intermediate support is closer to the third mode of the beam without the intermediate supports. Finally, the first five modes are displayed in Fig. 7(b).

Table 6. The ratio of natural frequency $(\omega/\omega_0)^2$ of clamped-clamped beam for different values of amplitude ratio for the fundamental mode

a/r	present study	a/r	present study
0.1	1.0008	1.5	1.1804
0.2	1.0033	2.0	1.3174
0.4	1.0130	2.5	1.4902
0.6	1.0292	3.0	1.6970
0.8	1.0518	3.5	1.9362
1.0	1.0808	4.0	2.2066



(a) (b)

Fig. 7. (a) The backbone curves $(\omega/\omega_0)^2$ versus amplitude ratio a/r for the clamped-simply supported Timoshenko-Ehrenfest beam with two intermediate supports, (b) the first five mode shapes for beam with amplitude ratio $a/r = 4$

5.7. Comparison of frequency ratio for higher modes

In this example, a comparison for frequency ratio $(\omega/\omega_0)^2$ of the first and second modes of a simply supported beam is performed. To do so, following Rakowski and Guminiak (2015), a simply supported beam with the properties $\nu = 0.3$ and $L/r = 100$ is considered. For the first mode, the maximum amplitude is at the middle of the beam, and for the second mode, the maximum amplitude is at $x = L/4$. The backbone curves for the first and second modes are shown in Fig. 8 (a) and (b), respectively. As can be observed, the results from this study are in good agreement with those reported in Rakowski and Guminiak (2015).

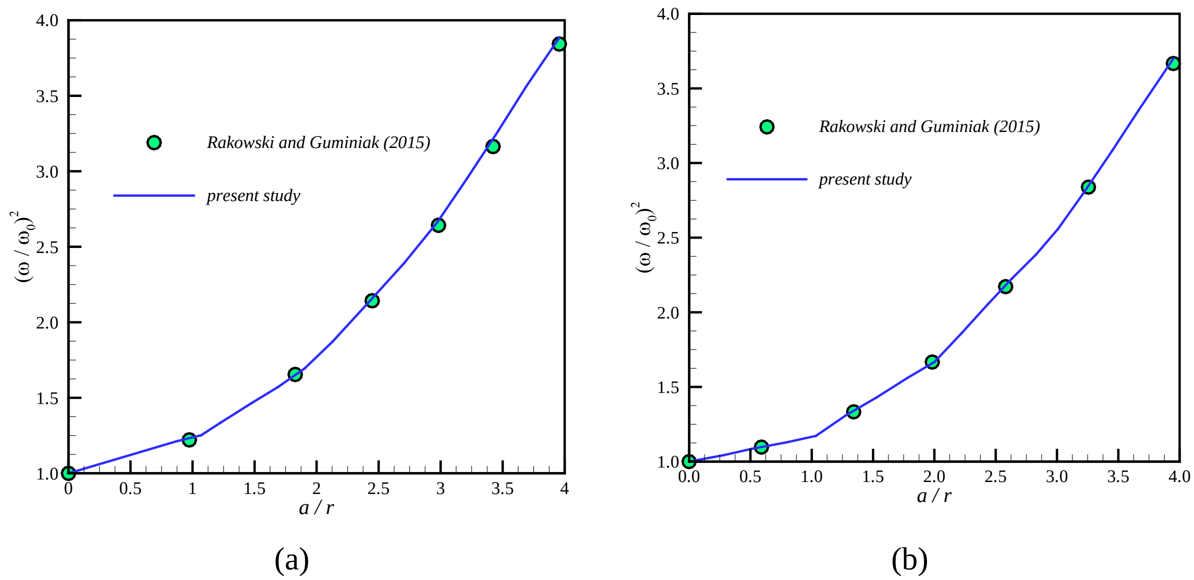


Fig. 8. The backbone curves for the slenderness ratio $L/r = 100$ for two modes in simply supported Timoshenko-Ehrenfest beam

6. Conclusion

In this study, nonlinear vibrations of geometrically exact Timoshenko-Ehrenfest beams are investigated. Equations of motions for geometrically nonlinear Timoshenko-Ehrenfest beam are derived exactly, with no approximations or simplification. Then, the weak form and finite element formulation is derived. To solve the eigenvalue problem, the direct scheme (DS) is employed. Finally, several examples are provided and accurate results, compared to those available in the literature, are obtained. Moreover, due to use of the finite element method, the present model is not restricted to a specific boundary condition, and nonlinear free vibrations of different cases, including a cantilever beam, a Timoshenko-Ehrenfest beam on three supports, and a clamped-supported beam with two additional supports have been investigated. This study shows that for straight perfect beams, nonlinearity leads to a hardening system. Besides, the FEM approach for studying nonlinear vibrations enabled to study problems hard to study analytically; in particular, those presented in sections 5.5 and 5.6 where (i) a simply-supported beam with one additional support, and (ii) a clamped-supported beam with two additional supports equally spaced were investigated.

Data Availability Statement: Data can be made available upon reasonable request.

Conflict of interest: The authors declare that they have no conflict of interest.

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Appendix

It is known that in FE derivation, it is convenient to work in natural coordinates. The relation between two different coordinates is as follows:

$$dx = \frac{dx}{d\xi} d\xi = J d\xi, \qquad \frac{d(*)}{dx} = \frac{d(*)}{d\xi}\frac{d\xi}{dx} = J^{-1}\frac{d(*)}{d\xi} \tag{A-1}$$

The arrays of the stiffness and mass matrices are derived as follows:

$$\left(K_{uu}\right)_{11} = \frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \tag{A-2}$$

$$\left(K_{uu}\right)_{12} = \left(K_{uu}\right)_{21} = -\frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \tag{A-3}$$

$$\left(K_{uu}\right)_{22} = \frac{K_0}{h_e\left(u_1^e - u_1^e\right)^2} - \left(u_1^e - u_1^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \tag{A-4}$$

$$\left(K_{uw}\right)_{11} = -\frac{K_S\left(\theta_1^e + \theta_2^e + 2\arctan\left(\frac{w_1^e - w_2^e}{h_e - u_1^e + u_2^e}\right)\right)}{2\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \tag{A-5}$$

$$\left(K_{uw}\right)_{12} = \left(K_{uw}\right)_{21} = \frac{K_S\left(\theta_1^e + \theta_2^e + 2\arctan\left(\frac{w_1^e - w_2^e}{h_e - u_1^e + u_2^e}\right)\right)}{2\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \tag{A-6}$$

$$\left(K_{uw}\right)_{22} = -\frac{K_S\left(\theta_1^e + \theta_2^e + 2\arctan\left(\frac{w_1^e - w_2^e}{h_e - u_1^e + u_2^e}\right)\right)}{2\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \tag{A-7}$$

$$\left(K_u\right)_{11} = -\frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e} - \frac{1}{2} \quad \text{(A-8)}$$

$$\left(K_u\right)_{12} = -\frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e} - \frac{1}{2} \quad \text{(A-9)}$$

$$\left(K_u\right)_{21} = \frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e} - \frac{1}{2} \quad \text{(A-10)}$$

$$\left(K_u\right)_{22} = \frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e} - \frac{1}{2} \quad \text{(A-11)}$$

$$\left(K_{ww}\right)_{11} = \frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \quad \text{(A-12)}$$

$$\left(K_{ww}\right)_{12} = \left(K_{ww}\right)_{21} = -\frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \quad \text{(A-13)}$$

$$\left(K_{ww}\right)_{22} = \frac{K_0}{h_e\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2h_e^2} - \frac{1}{2h_e} \quad \text{(A-14)}$$

$$\left(K_{w\theta}\right)_{11} = \frac{K_S h_e\left(u_1^e - u_2^e - h_e\right)}{2\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2} \quad \text{(A-15)}$$

$$\left(K_{w\theta}\right)_{12} = \frac{K_S h_e\left(u_1^e - u_2^e - h_e\right)}{2\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2} \quad \text{(A-16)}$$

$$\left(K_{w\theta}\right)_{21} = -\frac{K_S h_e\left(u_1^e - u_2^e - h_e\right)}{2\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2} \quad \text{(A-17)}$$

$$\left(K_{w\theta}\right)_{22} = -\frac{K_S h_e\left(u_1^e - u_2^e - h_e\right)}{2\left(u_1^e - u_2^e\right)^2} - \left(u_1^e - u_2^e\right) + \frac{\sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}}{2} \quad \text{(A-18)}$$

$$\left(K_w\right)_{11} = -\frac{K_S \arctan\left(\dfrac{w_1^e - w_2^e}{\left(h_e - u_1^e + u_2^e\right)^2}\right)}{\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \qquad \text{(A-19)}$$

$$\left(K_w\right)_{12} = -\frac{K_S \arctan\left(\dfrac{w_1^e - w_2^e}{\left(h_e - u_1^e + u_2^e\right)^2}\right)}{\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \qquad \text{(A-20)}$$

$$\left(K_w\right)_{21} = \frac{K_S \arctan\left(\dfrac{w_1^e - w_2^e}{\left(h_e - u_1^e + u_2^e\right)^2}\right)}{\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \qquad \text{(A-21)}$$

$$\left(K_w\right)_{22} = \frac{K_S \arctan\left(\dfrac{w_1^e - w_2^e}{\left(h_e - u_1^e + u_2^e\right)^2}\right)}{\sqrt{h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2}} \qquad \text{(A-22)}$$

$$\left(K_{\theta\theta}\right)_{11} = -\frac{K_S\left(h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2 + 3\dfrac{K_2}{K_S}\right)}{3\left(\dfrac{\left(u_1^e - u_2^e\right)^2}{h_e} - 2\left(u_1^e - u_2^e\right) + \sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}\right)} \qquad \text{(A-23)}$$

$$\left(K_{\theta\theta}\right)_{12} = \left(K_{\theta\theta}\right)_{21} = \frac{-K_S\left(h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2 - 6\dfrac{K_2}{K_S}\right)}{6\left(\dfrac{\left(u_1^e - u_2^e\right)^2}{h_e} - 2\left(u_1^e - u_2^e\right) + \sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}\right)} \qquad \text{(A-24)}$$

$$\left(K_{\theta\theta}\right)_{22} = -\frac{K_S\left(h_e^2 - 2h_e\left(u_1^e - u_2^e\right) + \left(u_1^e - u_2^e\right)^2 + \left(w_1^e - w_2^e\right)^2 + 3\dfrac{K_2}{K_S}\right)}{3\left(\dfrac{\left(u_1^e - u_2^e\right)^2}{h_e} - 2\left(u_1^e - u_2^e\right) + \sqrt{\left(w_1^e - w_2^e\right)^2 + h_e^2}\right)} \qquad \text{(A-25)}$$

$$\left(K_\theta\right)_{11}=\frac{1}{2}K_S\arctan\left(\frac{w_1^e-w_2^e}{u_1^e-u_2^e-h_e}\right)\left(\frac{\left(u_1^e-u_2^e\right)^2}{h_e}-2\left(u_1^e-u_2^e\right)+\sqrt{\left(w_1^e-w_2^e\right)^2+h_e^2}\right) \tag{A-26}$$

$$\left(K_\theta\right)_{12}=\left(K_\theta\right)_{21}=\frac{1}{2}K_S\arctan\left(\frac{w_1^e-w_2^e}{u_1^e-u_2^e-h_e}\right)\left(\frac{\left(u_1^e-u_2^e\right)^2}{h_e}-2\left(u_1^e-u_2^e\right)+\sqrt{\left(w_1^e-w_2^e\right)^2+h_e^2}\right) \tag{A-27}$$

$$\left(K_\theta\right)_{12}=\frac{1}{2}K_S\arctan\left(\frac{w_1^e-w_2^e}{u_1^e-u_2^e-h_e}\right)\left(\frac{\left(u_1^e-u_2^e\right)^2}{h_e}-2\left(u_1^e-u_2^e\right)+\sqrt{\left(w_1^e-w_2^e\right)^2+h_e^2}\right) \tag{A-28}$$

$$\left(K_\theta\right)_{22}=\frac{1}{2}K_S\arctan\left(\frac{w_1^e-w_2^e}{u_1^e-u_2^e-h_e}\right)\left(\frac{\left(u_1^e-u_2^e\right)^2}{h_e}-2\left(u_1^e-u_2^e\right)+\sqrt{\left(w_1^e-w_2^e\right)^2+h_e^2}\right) \tag{A-29}$$

$$\begin{aligned}
&\left(K_{W\theta}\right)_{11}=\left(K_{W\theta}\right)_{12}=\left(K_{W\theta}\right)_{21}=\left(K_{W\theta}\right)_{22}=0\\
&\left(K_{UW}\right)_{11}=\left(K_{UW}\right)_{12}=\left(K_{UW}\right)_{21}=\left(K_{UW}\right)_{22}=0\\
&\left(K_{\theta W}\right)_{11}=\left(K_{\theta W}\right)_{12}=\left(K_{\theta W}\right)_{21}=\left(K_{\theta W}\right)_{22}=0\\
&\left(K_{\theta U}\right)_{11}=\left(K_{\theta U}\right)_{12}=\left(K_{\theta U}\right)_{21}=\left(K_{\theta U}\right)_{22}=0
\end{aligned} \tag{A-30}$$

$$\left(M_{uu}\right)_{11}=\left(M_{uu}\right)_{22}=\frac{h_e m_0}{3},\ \left(M_{uu}\right)_{12}=\left(M_{uu}\right)_{21}=\frac{h_e m_0}{6} \tag{A-31}$$

$$\left(M_{ww}\right)_{11}=\left(M_{ww}\right)_{22}=\frac{h_e m_0}{3},\ \left(M_{ww}\right)_{12}=\left(M_{ww}\right)_{21}=\frac{h_e m_0}{6} \tag{A-32}$$

$$\left(M_{\theta\theta}\right)_{11}=\left(M_{\theta\theta}\right)_{22}=\frac{h_e m_2}{3},\ \left(M_{\theta\theta}\right)_{12}=\left(M_{\theta\theta}\right)_{21}=\frac{h_e m_2}{6} \tag{A-33}$$