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(Article begins on next page)

CONCENTRATION PHENOMENON FOR A FRACTIONAL SCHRÖDINGER EQUATION WITH DISCONTINUOUS NONLINEARITY

VINCENZO AMBROSIO

*Dedicated to Prof. Vicențiu D. Rădulescu on the occasion of his 65th birthday,
with great affection and esteem.*

ABSTRACT. We study the following fractional Schrödinger equation with discontinuous nonlinearity:

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = H(u - \beta)f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon, \beta > 0$, $s \in (0, 1)$, $N > 2s$, H is the Heaviside function, $(-\Delta)^s$ is the fractional Laplacian operator, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous potential satisfying del Pino-Felmer type assumptions and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a superlinear continuous nonlinearity with subcritical growth at infinity. By using a penalization method and nonsmooth analysis, we investigate the existence and concentration of solutions for the above problem.

1. INTRODUCTION

In this paper, we deal with the following nonlinear fractional Schrödinger equation with discontinuous nonlinearity:

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = H(u - \beta)f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $\varepsilon, \beta > 0$, $s \in (0, 1)$, $N > 2s$, H is the Heaviside function, namely

$$H(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

$V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous potential satisfying the following del Pino-Felmer type conditions [20]:

(V₁) there exists $V_1 > 0$ such that $V_1 := \inf_{x \in \mathbb{R}^N} V(x)$,

(V₂) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 := \inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x), \quad 0 \in M := \{x \in \Lambda : V(x) = V_0\},$$

and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nonlinearity such that $f(t) = 0$ for $t \leq 0$ and

(f₁) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$,

(f₂) $\limsup_{t \rightarrow \infty} \frac{f(t)}{t^p} = 0$, for some $p \in (1, 2_s^* - 1)$, where $2_s^* := \frac{2N}{N-2s}$ is the fractional critical exponent,

(f₃) there exists $\theta \in (2, p+1)$ such that

$$0 < \theta F(t) \leq t f(t) \quad \text{for all } t > 0,$$

where $F(t) := \int_0^t f(\tau) d\tau$,

(f₄) $t \mapsto \frac{f(t)}{t}$ is increasing in $(0, \infty)$.

The operator $(-\Delta)^s$ is the so-called fractional Laplacian which may be defined as

$$\mathcal{F}((-\Delta)^s u)(k) := |k|^{2s} \mathcal{F}u(k), \quad k \in \mathbb{R}^N,$$

where

$$\mathcal{F}u(k) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ik \cdot x} u(x) dx, \quad k \in \mathbb{R}^N,$$

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stands for the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^N)$, or equivalently through singular integrals by

$$(-\Delta)^s u(x) := C_{N,s} \lim_{\zeta \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\zeta(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N, \quad \text{with } C_{N,s} := \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-s).$$

We recall that fractional and nonlocal operators gained a lot of attention both from the mathematical research community and in view of concrete real-world applications, because these operators appear in many different fields, such as, phase transitions, crystal dislocation, anomalous diffusion, flame propagation, conservation laws, geophysical fluid dynamics, American options in finance, signal processing and network communication systems; see [21, 34] for more details and applications.

In these last years a great interest has been devoted to the following singularly perturbed fractional Schrödinger equation:

$$\varepsilon^{2s} (-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

for which several existence and multiplicity results have been established by various authors; see for instance [3, 6–8, 19, 24, 30]. In particular, under assumptions (V_1) – (V_2) and (f_1) – (f_4) , in [3] the authors studied the existence and the concentration phenomenon as $\varepsilon \rightarrow 0$ for (1.2) by adapting the local mountain pass approach in [20]. Later, by combining penalization method, Nehari manifold method and Lusternik-Schnirelman category theory, the author [6] proved a multiplicity result to (1.2) for $\varepsilon > 0$ small enough, by considering subcritical, critical and supercritical nonlinearities. We note that in these works only continuous nonlinearities were involved and suitable variational methods for C^1 -functionals were used (see [8] for more details).

As $s \rightarrow 1$, equation (1.2) reduces to the classical nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

When the nonlinearity f is continuous, some classical existence, multiplicity and concentration results to (1.3) can be found in [11, 20, 26, 32, 35, 36]. On the other hand, many authors dealt with nonlinear Schrödinger equations with discontinuous nonlinearities. For instance, in [28] the authors accomplished the existence of solutions for a class of general elliptic equations in \mathbb{R}^N involving a discontinuous nonlinearity with subcritical growth and a coercive potential. In [1] the authors obtained a multiplicity result for a discontinuous problem with critical growth by means of appropriate variational techniques for locally Lipschitz continuous functionals. Motivated by [28, 32, 35], the authors in [4] studied the existence and the concentration of solutions to

$$\begin{cases} -\Delta u + V(\varepsilon x)u = H(u - \beta)u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.4)$$

where $\varepsilon, \beta > 0$ are small parameters, $N \geq 3$, $p \in (1, \frac{N+2}{N-2})$, $V \in C(\mathbb{R}^N)$ satisfies the following condition due to Rabinowitz [32]:

$$\liminf_{|x| \rightarrow \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) > 0. \quad (1.5)$$

The results in [4] has been extended in [2] when the potential V in (1.4) fulfills (V_1) – (V_2) with $V_1 = V_0$. We stress that nonlinear partial differential equations with discontinuous nonlinearities can be used to manage free boundary problems arising in mathematical physics, such as, the obstacle problem, the seepage surface problem and the Elenbaas equation; see [14–16]. For more details on elliptic problems with discontinuous nonlinearities, we refer to [5, 9, 10, 13, 22, 25, 27, 33], in which many convenient techniques, such as, variational methods for non differentiable functionals, lower and upper solutions, global branching and the theory of multivalued mappings, are developed to attack the problems under consideration.

Differently from the local case $s = 1$, in nonlocal fractional framework only few papers analyzed nonlinear problems with discontinuous nonlinearities. For instance, in [37] the authors proved existence and multiplicity results for a critical p -Laplacian Kirchhoff type problem with discontinuous nonlinearity. In [12] the author obtained existence and multiplicity results for a fractional elliptic problem in a bounded domain with a subcritical discontinuous nonlinearity. In [23] the existence and behavior of solutions for a fractional elliptic problem in a bounded domain and involving critical and discontinuous nonlinearity were investigated.

Particularly motivated by [2–4, 6–8], in this paper we focus on the existence and concentration of solutions to (1.1). In our context, we say that a function u is a weak solution to (1.1) if $u \in H^s(\mathbb{R}^N)$ and there exists

$\rho \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that

$$\rho(x) \in [\underline{f}_H(u(x)), \bar{f}_H(u(x))] \text{ a.e. in } \mathbb{R}^N,$$

where $f_H(t) := H(t - \beta)f(t)$,

$$\underline{f}_H(t) := \lim_{\delta \downarrow 0} \text{ess inf}_{|t-\tau| < \delta} f_H(\tau), \quad \bar{f}_H(t) := \lim_{\delta \downarrow 0} \text{ess sup}_{|t-\tau| < \delta} f_H(\tau),$$

and it holds

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x) uv dx = \int_{\mathbb{R}^N} \rho v dx \quad \text{for all } v \in X_\varepsilon.$$

Remark 1.1. From the definitions of f_H , \underline{f}_H and \bar{f}_H , we have

$$[\underline{f}_H(t), \bar{f}_H(t)] = \begin{cases} \{0\} & \text{if } t < \beta, \\ [0, f(\beta)] & \text{if } t = \beta, \\ \{f(t)\} & \text{if } t > \beta. \end{cases}$$

The main result of this paper can be stated as follows.

Theorem 1.1. Assume that (V_1) – (V_2) and (f_1) – (f_4) hold. Then, there exist $\varepsilon^*, \beta^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$ and $\beta \in (0, \beta^*)$, (1.1) admits a weak solution $u_{\varepsilon, \beta}$. Moreover, if $\eta_{\varepsilon, \beta} \in \mathbb{R}^N$ denotes a maximum point of $u_{\varepsilon, \beta}$, then

$$\lim_{(\varepsilon, \beta) \rightarrow (0, 0)} V(\varepsilon \eta_{\varepsilon, \beta}) = V_0,$$

and there exists a constant $C > 0$ such that

$$0 < u_{\varepsilon, \beta}(x) \leq \frac{C}{1 + |x - \eta_{\varepsilon, \beta}|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N.$$

The proof of Theorem 1.1 is obtained by using some variational arguments inspired by [2, 3, 8]. Nevertheless, we prefer to underline some crucial differences with respect to these works:

- in contrast with [3, 8], we do not exploit variational methods for C^1 -functionals because the modified energy functional $J_{\varepsilon, \beta}$ associated with (1.1) is only locally Lipschitz continuous, and thus we will apply suitable variational techniques for non-differentiable functionals (see section 3);
- contrary to [2], we suppose that $V_1 \leq V_0$ and we consider more general nonlinearities, and so, some accurate estimates will be needed to cover compactness (see Lemma 4.1);
- with respect to [2], we have to deal with the nonlocality of $(-\Delta)^s$, therefore a more refined analysis will be performed to overcome some technical difficulties (see Lemma 3.3 and Lemma 4.2);
- differently from [2], in order to verify the crucial identity $\max_{t \geq 0} J_{\varepsilon, \beta}(tu_{\varepsilon, \beta}) = J_{\varepsilon, \beta}(u_{\varepsilon, \beta})$, which is important to get some useful estimates involving the mountain pass level, we use the fact that the set $\{x \in \Lambda_\varepsilon : u_{\varepsilon, \beta}(x) > \beta\}$ has positive Lebesgue measure for $\beta \in (0, a)$ (see Theorem 3.1 and Proposition 3.2);
- in [2] the weak solutions $u_{\varepsilon, \beta}$ of the modified problem associated with (1.4) belong to $W^{2, \frac{p+1}{p}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and satisfy

$$-\Delta u_{\varepsilon, \beta} + V(\varepsilon x)u_{\varepsilon, \beta} = \rho_{\varepsilon, \beta} \text{ a.e. in } \mathbb{R}^N,$$

where $\rho_{\varepsilon, \beta} \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ is such that $\rho_{\varepsilon, \beta}(x) \in [\underline{g}_H(\varepsilon x, u_{\varepsilon, \beta}(x)), \bar{g}_H(\varepsilon x, u_{\varepsilon, \beta}(x))]$ a.e. in \mathbb{R}^N . By using a contradiction argument and the Morrey-Stampacchia theorem, the authors proved that the set $\{x \in \mathbb{R}^N : u_{\varepsilon, \beta}(x) = \beta\}$ has null Lebesgue measure for $\beta > 0$ small, and so

$$-\Delta u_{\varepsilon, \beta} + V(\varepsilon x)u_{\varepsilon, \beta} = H(u_{\varepsilon, \beta}(x) - \beta)g(\varepsilon x, u_{\varepsilon, \beta}(x)) \text{ a.e. in } \mathbb{R}^N.$$

In our context, the solutions $u_{\varepsilon, \beta}$ of the modified problem related to (1.1) are in $H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for all $q \in [2, \infty]$ (see Theorem 3.1), but more regularity is needed so that the solutions verify

$$(-\Delta)^s u_{\varepsilon, \beta} + V(\varepsilon x)u_{\varepsilon, \beta} = \rho_{\varepsilon, \beta} \text{ a.e. in } \mathbb{R}^N.$$

Moreover, we do not have a fractional version of the Morrey-Stampacchia theorem.

Our arguments are sufficiently flexible and can be also adapted for the case $s = 1$, hence we are able to generalize the results in [2]. As far as we know, this is the first time that the penalization approach and the nonsmooth critical point theory are used to investigate the concentration phenomenon for a fractional Schrödinger equation with discontinuous nonlinearity.

The structure of the paper is the following. In Section 2 we recall some facts about fractional Sobolev spaces and nonsmooth critical point theory. In Section 3 we study the modified problem. Section 4 is devoted to the proof of Theorem 1.1.

Notations: The letters C, C_i will be repeatedly used to denote various positive constants whose exact values are irrelevant and can change from line to line. For $x \in \mathbb{R}^N$ and $R > 0$, we will denote by $B_R(x)$ the open ball in \mathbb{R}^N centered at $x \in \mathbb{R}^N$ with radius $r > 0$. When $x = 0$, we set $B_R := B_R(0)$. The notation $A^c := \mathbb{R}^N \setminus A$ stands for the complement of the set A in \mathbb{R}^N . If $A \subset \mathbb{R}^N$ is a measurable set, $|A|$ denotes the Lebesgue measure of A . We will use $\|u\|_{L^p(A)}$ for the L^p -norm of $u : \mathbb{R}^N \rightarrow \mathbb{R}$. For a generic real-valued function u , we set $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$.

2. PRELIMINARIES

In this section we start by collecting some useful facts about fractional Sobolev spaces; see [8, 21, 30] for further details. Let us consider the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : [u]_s := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \left([u]_s^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}}.$$

We recall that $C_c^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, $H^s(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2_s^*)$ and compactly in $L_{loc}^p(\mathbb{R}^N)$ for all $p \in [1, 2_s^*)$; see [8, 21]. We also have the following vanishing Lions-type result for $H^s(\mathbb{R}^N)$ (see Lemma 1.4.4 in [8]).

Lemma 2.1. [8] *Let $(u_n) \subset H^s(\mathbb{R}^N)$ be such that*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} u_n^2 dx = 0,$$

for some $R > 0$. Then $u_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2_s^)$.*

In order to study (1.1), for $\varepsilon > 0$, we introduce the space

$$X_\varepsilon := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x) u^2 dx < \infty \right\}$$

equipped with the norm

$$\|u\|_\varepsilon := \left([u]_s^2 + \int_{\mathbb{R}^N} V(\varepsilon x) u^2 dx \right)^{\frac{1}{2}}.$$

Hereafter, X_ε^* denotes the dual space of X_ε .

Next we present some basic facts about critical point theory for locally Lipschitz continuous functionals; see [16–18, 29] for more details.

Let X be a real Banach space with norm $\|\cdot\|_X$, X^* its dual space, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^* . Let $I : X \rightarrow \mathbb{R}$ be a locally Lipschitz continuous functional ($I \in \text{Lip}_{loc}(X, \mathbb{R})$, for short), that is, for every $u \in X$ we can find an open neighborhood $V := V_u \subset X$ of u and a constant $K := K_u > 0$ such that

$$|I(v_1) - I(v_2)| \leq K \|v_1 - v_2\|_X \quad \text{for all } v_1, v_2 \in V.$$

The generalized directional derivative of I at $u \in X$ in the direction $v \in X$ is defined as

$$I^0(u; v) := \limsup_{h \rightarrow 0, \sigma \downarrow 0} \frac{I(u + h + \sigma v) - I(u + h)}{\sigma}.$$

Hence, $I^0(u; \cdot)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$\partial I^0(u; z) := \{\mu \in X^* : I^0(u; v) \geq I^0(u; z) \langle \mu, v - z \rangle \quad \text{for all } v \in X\}.$$

The generalized gradient of I at $u \in X$ is the set

$$\partial I(u) := \{\mu \in X^* : \langle \mu, v \rangle \leq I^0(u; v) \quad \text{for all } v \in X\}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is simply the subdifferential of $I^0(u; \cdot)$ at $z = 0$. We also recall the following facts:

$$\begin{aligned} \partial I(u) &\subset X^* \text{ is convex, nonempty and weak}^*\text{-compact,} \\ \lambda(u) &:= \min\{\|\mu\|_{X^*} : \mu \in \partial I(u)\}, \\ \partial I(u) &= \{I'(u)\} \text{ if } I \in C^1(X, \mathbb{R}). \end{aligned}$$

A point $u_0 \in X$ is a critical point of I if $0 \in \partial I(u_0)$. A number $c \in \mathbb{R}$ is a critical value of I if there exists a critical point $u_0 \in X$ with $I(u_0) = c$. We say that I satisfies the nonsmooth Palais-Smale condition at the level $c \in \mathbb{R}$ (nonsmooth $(PS)_c$ -condition, for short), if for every sequence $(u_n) \subset X$ such that $I(u_n) \rightarrow c$ and $\lambda(u_n) \rightarrow 0$ has a strongly convergent subsequence. In the sequel, we will need the following nonsmooth version of the mountain pass theorem.

Theorem 2.1. [29, 33] *Let X be a real Banach space and $I \in Lip_{loc}(X, \mathbb{R})$ with $I(0) = 0$. Assume that there exist $\alpha, r > 0$ and $e \in X$ such that*

- (i) $I(u) \geq \alpha$ for all $u \in X$ with $\|u\|_X = r$,
- (ii) $I(e) < 0$ and $\|e\|_X > r$.

Let

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) \quad \text{and} \quad \Gamma := \{\gamma \in C^0([0, 1], X) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.$$

Then $c \geq \alpha$ and there is a sequence $(u_n) \subset X$ (named a nonsmooth $(PS)_c$ -sequence) fulfilling

$$I(u_n) \rightarrow c \quad \text{and} \quad \lambda(u_n) \rightarrow 0.$$

If, in addition, I satisfies the nonsmooth $(PS)_c$ -condition, then c is a critical value of I .

Finally, we recall the following useful result.

Proposition 2.1. [16, 18] *Let $\Psi(u) := \int_{\mathbb{R}^N} F_H(u) dx$ for $u \in L^{p+1}(\mathbb{R}^N)$, where $F_H(t) := \int_0^t f_H(\tau) d\tau$. Then, $\Psi \in Lip_{loc}(L^{p+1}(\mathbb{R}^N), \mathbb{R})$ and $\partial \Psi(u) \subset L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Moreover, if $\rho \in \partial \Psi(u)$, then*

$$\rho(x) \in [\underline{f}_H(u(x)), \bar{f}_H(u(x))] \quad \text{a.e. in } \mathbb{R}^N.$$

3. THE PENALIZATION APPROACH

To study (1.1), we will adapt to our case the penalization argument in [20]. Fix $k, a, \beta > 0$ such that $k > \frac{\theta}{\theta-2} > 1$, $\frac{f(a)}{a} = \frac{V_1}{k}$ and $\beta < a$. Define

$$\tilde{f}(t) := \begin{cases} f(t) & \text{for } t < a, \\ \frac{V_1}{k} t & \text{for } t \geq a, \end{cases}$$

and

$$g(x, t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x)) \tilde{f}(t) \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

where χ_Λ denotes the characteristic function of Λ . Set $G(x, t) := \int_0^t g(x, \tau) d\tau$. It is easy to check that $g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the following properties:

- (g₁) $g(\cdot, t) = 0$ for $t < 0$ and $\lim_{t \rightarrow 0} \frac{g(x, t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$,
- (g₂) $\limsup_{|t| \rightarrow \infty} \frac{g(x, t)}{|t|^p} < \infty$ uniformly in $x \in \mathbb{R}^N$,
- (g₃) (i) $0 < \theta G(x, t) \leq g(x, t)t$ for all $x \in \Lambda$ and $t > 0$,
(ii) $0 \leq 2G(x, t) \leq g(x, t)t \leq \frac{V_1}{k} t^2$ for all $x \in \Lambda^c$ and $t > 0$,
- (g₄) for each $x \in \mathbb{R}^N$ the function $t \mapsto \frac{g(x, t)}{t}$ is nondecreasing in $(0, \infty)$.

Let us introduce the following auxiliary problem:

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u = g_H(\varepsilon x, u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (3.1)$$

where $g_H(x, t) := H(t - \beta)g(x, t)$. We say that a function u is a weak solution to (3.1) whenever $u \in H^s(\mathbb{R}^N)$ and there exists $\rho \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that

$$\rho(x) \in [\underline{g}_H(\varepsilon x, u(x)), \bar{g}_H(\varepsilon x, u(x))] \quad \text{a.e. in } \mathbb{R}^N,$$

where

$$\underline{g}_H(x, t) := \lim_{\delta \downarrow 0} \text{ess inf}_{|t-\tau| < \delta} g_H(x, \tau), \quad \bar{g}_H(x, t) := \lim_{\delta \downarrow 0} \text{ess sup}_{|t-\tau| < \delta} g_H(x, \tau),$$

and it holds

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x)uv dx = \int_{\mathbb{R}^N} \rho v dx \quad \text{for all } v \in X_\varepsilon.$$

Note that if u is a solution of (3.1) satisfying $u(x) < a$ for all $x \in \Lambda_\varepsilon^c$, where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then u is a solution of (1.1). The energy functional associated with (3.1) is given by

$$J_{\varepsilon, \beta}(u) := \frac{1}{2} \|u\|_\varepsilon^2 - \int_{\mathbb{R}^N} G_H(\varepsilon x, u) dx \quad \text{for all } u \in X_\varepsilon,$$

where $G_H(x, t) := \int_0^t g_H(x, \tau) d\tau$. We put

$$Q_\varepsilon(u) := \frac{1}{2} \|u\|_\varepsilon^2 \quad \text{and} \quad \Psi_{\varepsilon, \beta}(u) := \int_{\mathbb{R}^N} G_H(\varepsilon x, u) dx,$$

so that $J_{\varepsilon, \beta}(u) = Q_\varepsilon(u) - \Psi_{\varepsilon, \beta}(u)$. Clearly, $Q_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$. On the other hand, since there exists a constant $C = C(\beta) > 0$ such that $g_H(x, t) \leq C|t|^p$ for $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, we obtain the following analog of Proposition 2.1 (see also Theorems 2.1 and 2.2 in [14]).

Proposition 3.1. $\Psi_{\varepsilon, \beta} \in Lip_{loc}(L^{p+1}(\mathbb{R}^N), \mathbb{R})$ and $\partial \Psi_{\varepsilon, \beta}(u) \subset L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Moreover, if $\rho \in \partial \Psi_{\varepsilon, \beta}(u)$, then

$$\rho(x) \in [\underline{g}_H(\varepsilon x, u(x)), \bar{g}_H(\varepsilon x, u(x))] \quad \text{a.e. in } \mathbb{R}^N.$$

If $\hat{\Psi}_{\varepsilon, \beta} = (\Psi_{\varepsilon, \beta})|_{H^s(\mathbb{R}^N)}$, then $\partial \hat{\Psi}_{\varepsilon, \beta}(u) = \partial \Psi_{\varepsilon, \beta}(u)$ for all $u \in H^s(\mathbb{R}^N)$.

By Proposition 3.1, $\Psi_{\varepsilon, \beta} \in Lip_{loc}(X_\varepsilon, \mathbb{R})$, and thus $J_{\varepsilon, \beta} \in Lip_{loc}(X_\varepsilon, \mathbb{R})$. In the next lemma, we prove that $J_{\varepsilon, \beta}$ satisfies the geometric assumptions of Theorem 2.1.

Lemma 3.1. *The functional $J_{\varepsilon, \beta}$ fulfills the following properties:*

- (i) $J_{\varepsilon, \beta}(0) = 0$,
- (ii) *there exist $\alpha, \rho > 0$ such that $J_{\varepsilon, \beta}(u) \geq \alpha$ for all $u \in X_\varepsilon$ with $\|u\|_\varepsilon = \rho$,*
- (iii) *there exists $e \in X_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $J_{\varepsilon, \beta}(e) < 0$.*

Proof. Condition (i) is obvious. By (g_1) and (g_2) , for all $\eta > 0$ there exists $C_\eta > 0$ such that

$$|g(x, t)| \leq \eta|t| + C_\eta|t|^{2_s^*-1} \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}, \quad (3.2)$$

and

$$|G(x, t)| \leq \frac{\eta}{2}|t|^2 + \frac{C_\eta}{2_s^*}|t|^{2_s^*} \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (3.3)$$

Take $\eta = \frac{V_1}{2}$. Using (3.3), (V_1) , $0 \leq H(t) \leq 1$, and the embedding $H^s(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \in [2, 2_s^*]$, we see that

$$\begin{aligned} J_{\varepsilon, \beta}(u) &\geq \frac{1}{2} \|u\|_\varepsilon^2 - \frac{V_1}{4} \|u\|_{L^2(\mathbb{R}^N)}^2 - C \|u\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*} \\ &\geq \frac{1}{4} \|u\|_\varepsilon^2 - C \|u\|_\varepsilon^{2_s^*}, \end{aligned}$$

from which we deduce that (ii) is satisfied. Finally, pick $v \in C_c^\infty(\mathbb{R}^N)$ such that $v \geq 0$, $v \not\equiv 0$, $K := \text{supp}(v) \subset \Lambda_\varepsilon$ and $|\{x \in K : v(x) > \beta\}| > 0$. Then, from the definition of g and (f_3) , we have, for all $t > 1$,

$$\begin{aligned} J_{\varepsilon,\beta}(tv) &= \frac{t^2}{2} \|v\|_\varepsilon^2 - \int_{\mathbb{R}^N} F_H(tv) dx \\ &\leq \frac{t^2}{2} \|v\|_\varepsilon^2 - C_1 t^\theta \int_{K \cap \{v > \beta\}} v^\theta dx + C_2 |K|, \end{aligned}$$

which combined with $\theta > 2$ implies that $J_{\varepsilon,\beta}(tv) \rightarrow -\infty$ as $t \rightarrow \infty$. Therefore, (iii) follows by taking $e := Tv$, with $T > 0$ large enough. \square

By combining Lemma 3.1 and Theorem 2.1, we can find a nonsmooth $(PS)_{c_{\varepsilon,\beta}}$ -sequence $(u_n) \subset X_\varepsilon$ such that

$$J_{\varepsilon,\beta}(u_n) \rightarrow c_{\varepsilon,\beta} \quad \text{and} \quad \lambda_{\varepsilon,\beta}(u_n) \rightarrow 0 \text{ in } X_\varepsilon^*,$$

as $n \rightarrow \infty$, where

$$c_{\varepsilon,\beta} := \inf_{\gamma \in \Gamma_{\varepsilon,\beta}} \max_{t \in [0,1]} J_{\varepsilon,\beta}(\gamma(t))$$

and

$$\Gamma_{\varepsilon,\beta} := \{\gamma \in C^0([0,1], X_\varepsilon) : \gamma(0) = 0, J_{\varepsilon,\beta}(\gamma(1)) < 0\}.$$

In what follows, we show that the boundedness of nonsmooth $(PS)_c$ -sequences of $J_{\varepsilon,\beta}$.

Lemma 3.2. *Let $c \in \mathbb{R}$ and $(u_n) \subset X_\varepsilon$ be a nonsmooth $(PS)_c$ -sequence of $J_{\varepsilon,\beta}$. Then, (u_n) is bounded in X_ε .*

Proof. By assumptions, we know that

$$J_{\varepsilon,\beta}(u_n) \rightarrow c \quad \text{and} \quad \lambda_{\varepsilon,\beta}(u_n) \rightarrow 0 \text{ in } X_\varepsilon^*, \quad (3.4)$$

as $n \rightarrow \infty$. Let us consider $(w_n) \subset X_\varepsilon^*$ satisfying

$$\lambda_{\varepsilon,\beta}(u_n) = \|w_n\|_{X_\varepsilon^*} = o_n(1)$$

and $w_n := Q'_\varepsilon(u_n) - \rho_n$, where $\rho_n \in \partial \Psi_{\varepsilon,\beta}(u_n)$. Since $(J_{\varepsilon,\beta}(u_n))$ is bounded in \mathbb{R} and

$$-\frac{1}{\theta} \langle w_n, u_n \rangle \leq \|w_n\|_{X_\varepsilon^*} \|u_n\|_\varepsilon,$$

there exists $M > 0$ such that

$$\begin{aligned} M + \|w_n\|_{X_\varepsilon^*} \|u_n\|_\varepsilon &\geq J_{\varepsilon,\beta}(u_n) - \frac{1}{\theta} \langle w_n, u_n \rangle + o_n(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\varepsilon^2 + \frac{1}{\theta} \langle \rho_n, u_n \rangle - \int_{\{u_n > \beta\}} G_H(\varepsilon x, u_n) dx. \end{aligned}$$

In view of Proposition 3.1, we know that

$$\frac{1}{\theta} \langle \rho_n, u_n \rangle = \frac{1}{\theta} \int_{\mathbb{R}^N} \rho_n u_n dx \geq \frac{1}{\theta} \int_{\mathbb{R}^N} \underline{g}_H(\varepsilon x, u_n) u_n dx.$$

On the other hand, from (g_3) we deduce that

$$\frac{1}{\theta} \int_{\mathbb{R}^N} \underline{g}_H(\varepsilon x, u_n) u_n dx \geq \int_{\{u_n > \beta\} \cap \Lambda_\varepsilon} G_H(\varepsilon x, u_n) dx + \frac{2}{\theta} \int_{\{u_n > \beta\} \cap \Lambda_\varepsilon^c} G_H(\varepsilon x, u_n) dx.$$

Consequently, using (g_3) -(ii), we get

$$\begin{aligned} M + \|w_n\|_{X_\varepsilon^*} \|u_n\|_\varepsilon &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\varepsilon^2 - \left(1 - \frac{2}{\theta}\right) \int_{\{u_n > \beta\} \cap \Lambda_\varepsilon^c} G_H(\varepsilon x, u_n) dx \\ &\geq \frac{(\theta - 2)}{2\theta} \|u_n\|_\varepsilon^2 - \frac{\theta - 2}{2k\theta} \int_{\Lambda_\varepsilon^c} V(\varepsilon x) u_n^2 dx, \end{aligned}$$

and thus

$$M + \|w_n\|_{X_\varepsilon^*} \|u_n\|_\varepsilon \geq \left(1 - \frac{1}{k}\right) \frac{(\theta - 2)}{2\theta} \|u_n\|_\varepsilon^2.$$

Since $k > 1$ and $\theta > 2$, we can conclude that (u_n) is bounded in X_ε . \square

Now we prove the tightness of the Palais-Smale sequences of $J_{\varepsilon,\beta}$. More precisely, we have the following result.

Lemma 3.3. *Let $c \in \mathbb{R}$ and $(u_n) \subset X_\varepsilon$ be a nonsmooth $(PS)_c$ -sequence of $J_{\varepsilon,\beta}$. Then, for all $\xi > 0$ there exists $R = R(\xi) > 0$ such that*

$$\limsup_{n \rightarrow \infty} \left[\int_{B_R^c} \int_{\mathbb{R}^n} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{B_R^c} V(\varepsilon x) u_n^2 dx \right] < \xi. \quad (3.5)$$

Proof. Let $(w_n) \subset X_\varepsilon^*$ be such that

$$\lambda_{\varepsilon,\beta}(u_n) = \|w_n\|_{X_\varepsilon^*} = o_n(1),$$

and $w_n := Q'_\varepsilon(u_n) - \rho_n$, where $(\rho_n) \subset \partial \Psi_{\varepsilon,\beta}(u_n)$. For $R > 0$, let $\eta_R \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \eta_R \leq 1$, $\eta_R(x) = 0$ if $x \in B_{\frac{R}{2}}$, $\eta_R(x) = 1$ if $x \in B_R^c$, and $\|\nabla \eta_R\|_{L^\infty(\mathbb{R}^N)} \leq C/R$ for some $C > 0$ independent of $R > 0$. Since $(u_n \eta_R)$ is bounded in X_ε , we have that

$$\langle Q'_\varepsilon(u_n), u_n \eta_R \rangle = \langle \rho_n, u_n \eta_R \rangle + o_n(1),$$

that is

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \eta_R dx \\ &= \int_{\mathbb{R}^N} \rho_n u_n \eta_R dx - \left(\iint_{\mathbb{R}^{2N}} u_n(y) \frac{(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx dy \right) + o_n(1). \end{aligned}$$

In light of Proposition 3.1,

$$\int_{\mathbb{R}^N} \rho_n u_n \eta_R dx \leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon x, u_n) u_n \eta_R dx,$$

and so

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \eta_R dx \\ & \leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon x, u_n) u_n \eta_R dx - \left(\iint_{\mathbb{R}^{2N}} u_n(y) \frac{(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx dy \right) + o_n(1). \end{aligned}$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset B_{\frac{R}{2}}$. Therefore, using (g_3) -(ii), we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \eta_R dx \\ & \leq \frac{1}{k} \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \eta_R dx - \left(\iint_{\mathbb{R}^{2N}} u_n(y) \frac{(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx dy \right) + o_n(1). \end{aligned}$$

Applying Hölder's inequality, we see that

$$\begin{aligned} \left| \left(\iint_{\mathbb{R}^{2N}} u_n(y) \frac{(u_n(x) - u_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx dy \right) \right| & \leq [u_n]_s \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ & \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ & \leq \frac{C}{R^s}, \end{aligned}$$

where we used

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^2 |\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} dx dy \\ &= \int_{\mathbb{R}^N} |u_n(y)|^2 \left(\int_{|x-y|>R} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} dx + \int_{|x-y|\leq R} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x - y|^{N+2s}} dx \right) dy \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^N} |u_n(y)|^2 \left(\int_{|x-y|>R} \frac{4\|\eta\|_{L^\infty(\mathbb{R}^N)}^2}{|x-y|^{N+2s}} dx + R^{-2} \int_{|x-y|\leq R} \frac{\|\nabla\eta\|_{L^\infty(\mathbb{R}^N)}^2}{|x-y|^{N+2s-2}} dx \right) dy \\
&\leq C \int_{\mathbb{R}^N} |u_n(y)|^2 dy \left(\int_R^\infty \frac{1}{r^{2s+1}} dr + R^{-2} \int_0^R \frac{1}{r^{2s-1}} dr \right) \\
&\leq \frac{C}{R^{2s}}.
\end{aligned}$$

Now, fix $\xi > 0$. Then we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dx dy + \left(1 - \frac{1}{k}\right) \int_{\mathbb{R}^N} V(\varepsilon x) u_n^2 \eta_R dx \\
&\leq \frac{C}{R^s} + o_n(1),
\end{aligned}$$

from which

$$\limsup_{n \rightarrow \infty} \left[\int_{B_R^c} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dx dy + \int_{B_R^c} V(\varepsilon x) u_n^2 dx \right] \leq \frac{C}{R^s} < \xi,$$

whenever $R > 0$ is sufficiently large. \square

At this point, we can show that the modified functional satisfies the following compactness condition.

Lemma 3.4. *$J_{\varepsilon,\beta}$ satisfies the nonsmooth $(PS)_c$ -condition at every level $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and $(u_n) \subset X_\varepsilon$ be a nonsmooth $(PS)_c$ -sequence. Then

$$o_n(1) = \langle w_n, u_n \rangle = \langle Q'_\varepsilon(u_n) - \rho_n, u_n \rangle,$$

as $n \rightarrow \infty$, where the sequences (w_n) and (ρ_n) are as in the proof of Lemma 3.3. In particular,

$$\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^N} \rho_n u_n dx + o_n(1). \quad (3.6)$$

By Lemma 3.2, we know that (u_n) is bounded in X_ε . Hence, up to a subsequence, we may suppose that $u_n \rightharpoonup u$ in X_ε . Now we prove that this convergence is actually strong. Since Proposition 3.1 yields

$$\rho_n(x) \in [\underline{g}_H(\varepsilon x, u_n(x)), \bar{g}_H(\varepsilon x, u_n(x))] \text{ a.e. in } \mathbb{R}^N,$$

and (u_n) is bounded in X_ε , we infer that (ρ_n) is bounded in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Then there exists $\rho \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that, up to a subsequence,

$$\rho_n \rightharpoonup \rho \text{ in } L^{\frac{p+1}{p}}(\mathbb{R}^N), \quad (3.7)$$

which combined with

$$\langle w_n, u \rangle = \langle Q'_\varepsilon(u_n) - \rho_n, u \rangle = o_n(1),$$

implies that

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} \rho u dx. \quad (3.8)$$

Using Lemma 3.3 and (V_1) , we can see that for all $\xi > 0$ there exists $R = R(\xi) > 0$ such that

$$\limsup_{n \rightarrow \infty} \int_{B_R^c} u_n^2 dx \leq \frac{1}{V_1} \limsup_{n \rightarrow \infty} \int_{B_R^c} V(\varepsilon x) u_n^2 dx < \frac{\xi}{V_1}.$$

Because $u \in L^2(\mathbb{R}^N)$, we may assume, increasing $R > 0$ if necessary, that

$$\int_{B_R^c} u^2 dx < \xi.$$

Therefore, by exploiting the fact that X_ε is compactly embedded in $L_{loc}^2(\mathbb{R}^N)$, we get

$$\limsup_{n \rightarrow \infty} \|u_n - u\|_{L^2(\mathbb{R}^N)}^2 = \limsup_{n \rightarrow \infty} \left[\|u_n - u\|_{L^2(B_R)}^2 + \|u_n - u\|_{L^2(B_R^c)}^2 \right]$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(B_R)}^2 + \limsup_{n \rightarrow \infty} \|u_n - u\|_{L^2(B_R^c)}^2 \\
&\leq C\xi.
\end{aligned}$$

By the arbitrariness of $\xi > 0$, we deduce that $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$. By interpolation on L^q -spaces and the boundedness of (u_n) in $L^{2^*}(\mathbb{R}^N)$, we conclude that $u_n \rightarrow u$ in $L^q(\mathbb{R}^N)$ for all $q \in [2, 2^*)$. In particular,

$$u_n \rightarrow u \text{ in } L^{p+1}(\mathbb{R}^N). \quad (3.9)$$

Combining (3.7) and (3.9), we obtain

$$\int_{\mathbb{R}^N} (\rho_n u_n - \rho u) dx = \int_{\mathbb{R}^N} \rho_n (u_n - u) dx + \int_{\mathbb{R}^N} (\rho_n - \rho) u dx \rightarrow 0. \quad (3.10)$$

From (3.6), (3.8) and (3.10), we derive that $\|u_n\|_\varepsilon^2 \rightarrow \|u\|_\varepsilon^2$ as $n \rightarrow \infty$. Recalling that X_ε is a Hilbert space, we conclude that $u_n \rightarrow u$ in X_ε as $n \rightarrow \infty$. \square

We are ready to give the following existence result for (3.1).

Theorem 3.1. *For all $\varepsilon > 0$ and $\beta > 0$ there exists $u_{\varepsilon, \beta} \in X_\varepsilon \setminus \{0\}$ such that*

$$J_{\varepsilon, \beta}(u_{\varepsilon, \beta}) = c_{\varepsilon, \beta} \quad \text{and} \quad 0 \in \partial J_{\varepsilon, \beta}(u_{\varepsilon, \beta}). \quad (3.11)$$

Moreover, there exists $\beta^ > 0$ such that, if $\beta \in (0, \beta^*)$, then the set $A_{\varepsilon, \beta} := \{x \in \Lambda_\varepsilon : u_{\varepsilon, \beta}(x) > \beta\}$ has positive Lebesgue measure for all $\varepsilon > 0$.*

Proof. In light of Lemmas 3.1 and 3.4, we can apply Theorem 2.1 to see that for all $\varepsilon, \beta > 0$ there exists $u_{\varepsilon, \beta} \in X_\varepsilon \setminus \{0\}$ such that (3.11) holds. Hence, there is $\rho_{\varepsilon, \beta} \in \partial \Psi(u_{\varepsilon, \beta})$ such that

$$\rho_{\varepsilon, \beta}(x) \in [\underline{g}_H(\varepsilon x, u_{\varepsilon, \beta}(x)), \bar{g}_H(\varepsilon x, u_{\varepsilon, \beta}(x))] \text{ a.e. in } \mathbb{R}^N, \quad (3.12)$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{(u_{\varepsilon, \beta}(x) - u_{\varepsilon, \beta}(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon x) u_{\varepsilon, \beta} v dx = \int_{\mathbb{R}^N} \rho_{\varepsilon, \beta} v dx \quad \text{for all } v \in X_\varepsilon. \quad (3.13)$$

We claim that $u_{\varepsilon, \beta} \geq 0$ in \mathbb{R}^N . Indeed, choosing $v = u_{\varepsilon, \beta}^- = \min\{u_{\varepsilon, \beta}, 0\}$ in (3.13), and observing that $|x^- - y^-|^2 \leq (x - y)(x^- - y^-)$ for all $x, y \in \mathbb{R}$, we have

$$[u_{\varepsilon, \beta}^-]_s^2 + \int_{\mathbb{R}^N} V(\varepsilon x) (u_{\varepsilon, \beta}^-)^2 dx \leq 0,$$

from which we get the assertion. By using a Moser iteration argument (see Lemma 7.2.9 in [8]), we can prove that $u_{\varepsilon, \beta} \in L^q(\mathbb{R}^N) \cap C_{loc}^{0, \alpha}(\mathbb{R}^N)$ for all $q \in [2, \infty]$. From the strong maximum principle (see Theorem 1.3.5 in [8]), we infer that $u_{\varepsilon, \beta} > 0$ in \mathbb{R}^N . Finally, we show that $A_{\varepsilon, \beta}$ has positive Lebesgue measure whenever $\beta \in (0, a)$. Arguing by contradiction, suppose that $|A_{\varepsilon, \beta}| = 0$. Then $u_{\varepsilon, \beta}(x) \leq \beta$ for a.e. $x \in \Lambda_\varepsilon$. Since $\beta < a$ and $g(x, t)t \leq \frac{V_1}{k} t^2$ for $(x, t) \in (\Lambda_\varepsilon \times [0, a]) \cup (\Lambda_\varepsilon^c \times [0, \infty))$, it follows from (3.12), (3.13) and (V_1) that

$$\begin{aligned}
\|u_{\varepsilon, \beta}\|_\varepsilon^2 &= \int_{\mathbb{R}^N} \rho_{\varepsilon, \beta} u_{\varepsilon, \beta} dx \\
&\leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon x, u_{\varepsilon, \beta}) u_{\varepsilon, \beta} dx \\
&\leq \frac{1}{k} \int_{\mathbb{R}^N} V(\varepsilon x) u_{\varepsilon, \beta}^2 dx,
\end{aligned}$$

from which

$$\left(1 - \frac{1}{k}\right) \|u_{\varepsilon, \beta}\|_\varepsilon^2 \leq 0.$$

Recalling that $k > 1$, we reach $u_{\varepsilon, \beta} = 0$ a.e. in \mathbb{R}^N which is a contradiction because $J_{\varepsilon, \beta}(u_{\varepsilon, \beta}) = c_{\varepsilon, \beta} > 0$. \square

The result below plays a fundamental role in proving that the solution in Theorem 3.1 is a solution to (1.1) for $\beta > 0$ small enough.

Proposition 3.2. *For $\beta \in (0, \beta^*)$, let $u_{\varepsilon, \beta}$ be the solution obtained in Theorem 3.1. Then*

$$\max_{t \geq 0} J_{\varepsilon, \beta}(tu_{\varepsilon, \beta}) = J_{\varepsilon, \beta}(u_{\varepsilon, \beta}).$$

Proof. Let $h : [0, \infty) \rightarrow \mathbb{R}$ be the locally Lipschitz continuous function defined as

$$h(t) := J_{\varepsilon, \beta}(tu_{\varepsilon, \beta}), \quad t \geq 0.$$

It is easy to check that there exist $\delta, \alpha, t_0 > 0$ such that $h(t) > 0$ for all $t \in (0, \delta)$ and $h(t) < 0$ for all $t \in [t_0, \infty)$. Let $t_* > 0$ be such that

$$h(t_*) := \max_{t \geq 0} h(t).$$

We first show that $t_* = 1$. Since h is a locally Lipschitz continuous function, it follows that h is differentiable a.e. and thus the set I of points in which h' does not exist is such that $|I| = 0$. We aim to prove that:

(a) $h'(t) > 0$ for all $t \in (0, 1) \cap I^c$,

(b) $h'(t) < 0$ for all $t \in (1, \infty) \cap I^c$.

These relations imply that h has a local maximum at $t = 1$ (indeed, it is the global maximum) and, in particular, $t = 1$ is the unique point in which the global maximum is achieved. We will verify only (a) because the proof of (b) can be done in a similar manner. For simplicity, we denote $u_{\varepsilon, \beta}$ by u . Recalling the chain rule for locally Lipschitz continuous functions, we know that there exists $w \in \partial J_{\varepsilon, \beta}(tu)$ such that $h'(t) = \langle w, u \rangle$, or equivalently, there exists $\rho \in \partial \Psi_{\varepsilon, \beta}(tu)$ such that

$$h'(t) = t \|u\|_{\varepsilon}^2 - \int_{\mathbb{R}^N} \rho u \, dx.$$

On the other hand, there exists $\tilde{\rho} \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that

$$\tilde{\rho}(x) \in [\underline{g}_H(\varepsilon x, u), \bar{g}_H(\varepsilon x, u)] \text{ a.e. in } \mathbb{R}^N,$$

and it holds

$$\|u\|_{\varepsilon}^2 = \int_{\mathbb{R}^N} \tilde{\rho} u \, dx.$$

Hence,

$$h'(t) = t \int_{\mathbb{R}^N} \tilde{\rho} u \, dx - \int_{\mathbb{R}^N} \rho u \, dx.$$

From Proposition 3.1, we get

$$\rho(x) \in [\underline{g}_H(\varepsilon x, tu), \bar{g}_H(\varepsilon x, tu)] \text{ a.e. in } \mathbb{R}^N,$$

so that

$$\begin{aligned} h'(t) &\geq t \int_{\mathbb{R}^N} \underline{g}_H(\varepsilon x, u) u \, dx - \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon x, tu) u \, dx \\ &= t \left(\int_{\mathbb{R}^N} \underline{g}_H(\varepsilon x, u) u \, dx - \int_{\mathbb{R}^N} \frac{\bar{g}_H(\varepsilon x, tu)}{t} u \, dx \right). \end{aligned}$$

By definitions of \underline{g}_H and \bar{g}_H , we can see that

$$\int_{\mathbb{R}^N} \underline{g}_H(\varepsilon x, u) u \, dx \geq \int_{\{u > \beta\}} g(\varepsilon x, u) u \, dx,$$

and

$$\int_{\mathbb{R}^N} \bar{g}_H(\varepsilon x, tu) u \, dx \leq \int_{\{tu \geq \beta\}} g(\varepsilon x, tu) u \, dx.$$

Then we have

$$h'(t) \geq t \left[\int_{\{u > \beta\}} \left(g(\varepsilon x, u) u - \frac{g(\varepsilon x, tu)}{t} u \right) dx \right] \quad \text{for all } t \in (0, 1).$$

Using $|A_{\varepsilon,\beta}| > 0$ and (g_4) , we deduce that

$$\int_{\{u>\beta\}} \left(g(\varepsilon x, u)u - \frac{g(\varepsilon x, tu)}{t}u \right) dx \geq \int_{A_{\varepsilon,\beta}} \left(g(\varepsilon x, u)u - \frac{g(\varepsilon x, tu)}{t}u \right) dx > 0 \quad \text{for all } t \in (0, 1),$$

and this implies that (a) is valid. The proof of proposition is complete. \square

Now, let ω be a ground state solution to

$$\begin{cases} (-\Delta)^s u + V_0 u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (3.14)$$

that is, if $L_\mu : Y_\mu \rightarrow \mathbb{R}$ is given by

$$L_\mu(u) := \frac{1}{2} \|u\|_{Y_\mu}^2 - \int_{\mathbb{R}^N} F(u) dx, \quad \mu > 0,$$

where $Y_\mu := H^s(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_{Y_\mu} := \left([u]_s^2 + \mu \|u\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}},$$

then $\omega \in Y_{V_0}$ satisfies $L_{V_0}(\omega) = d_{V_0}$ and $L'_{V_0}(\omega) = 0$, where

$$d_{V_0} := \inf_{u \in \mathcal{M}_{V_0}} L_{V_0}(u),$$

and

$$\mathcal{M}_{V_0} := \{u \in Y_{V_0} \setminus \{0\} : \langle L'_{V_0}(u), u \rangle = 0\}.$$

is the Nehari manifold associated with L_{V_0} . The existence of ω is guaranteed by Lemma 6.3.10 in [8]. Let us prove the following interesting relation between $c_{\varepsilon,\beta}$ and d_{V_0} .

Lemma 3.5. *It holds*

$$\limsup_{(\varepsilon,\beta) \rightarrow (0,0)} c_{\varepsilon,\beta} \leq d_{V_0}.$$

Proof. Suppose $\Lambda \Subset B_2$. For $R > 1$, define $\phi_R(x) := \phi(\frac{x}{R})$ and $\omega_R(x) := \phi_R(x)\omega(x)$, where $\phi \in C_c^\infty(\mathbb{R}^N)$ is such that $0 \leq \phi \leq 1$, $\text{supp}(\phi) \subset \Lambda$, $\phi = 1$ in B_1 , $\phi = 0$ in B_2^c . Let $t_R > 0$ be such that $t_R \omega_R \in \mathcal{M}_{V_0}$. Since $\omega \in \mathcal{M}_{V_0}$, it is easy to check that $t_R \rightarrow 1$ as $R \rightarrow \infty$. In particular, we can see that $t_R \omega_R \rightarrow \omega$ in $H^s(\mathbb{R}^N)$ and $L_{V_0}(t_R \omega_R) \rightarrow L_{V_0}(\omega) = d_{V_0}$, as $R \rightarrow \infty$. A direct computation shows that there exists $t_* > 0$ such that $J_{\varepsilon,\beta}(t_* t_R \omega_R) < 0$ uniformly for $\varepsilon, \beta > 0$ small. Set $\hat{\gamma}(t) := t t_* t_R \omega_R$ for $t \in [0, 1]$. Clearly, $\hat{\gamma} \in \Gamma_{\varepsilon,\beta}$. Using the definition of $c_{\varepsilon,\beta}$, we obtain that

$$c_{\varepsilon,\beta} \leq \max_{t \in [0,1]} J_{\varepsilon,\beta}(\hat{\gamma}(t)) \leq \max_{t \geq 0} J_{\varepsilon,\beta}(\hat{\gamma}(t)) = J_{\varepsilon,\beta}(\hat{t} t_R \omega_R)$$

for some $\hat{t} = \hat{t}(\varepsilon, \beta, R) > 0$. In particular, fixed $R > 0$, there exist $K_1, K_2 > 0$ such that $K_1 \leq \hat{t} \leq K_2$ whenever $\varepsilon, \beta > 0$ are small enough. Since $V(0) = V_0$, given $\eta > 0$ there exists $\varepsilon_0 > 0$ such that

$$0 < V(\varepsilon x) - V_0 < \eta \quad \text{for all } \varepsilon \in (0, \varepsilon_0), x \in B_{2R}.$$

Therefore,

$$\int_{\mathbb{R}^N} V(\varepsilon x) t_R^2 \omega_R^2 dx \leq (V_0 + \eta) t_R^2 \int_{\mathbb{R}^N} \omega_R^2 dx,$$

and thus

$$\begin{aligned} c_{\varepsilon,\beta} &\leq J_{\varepsilon,\beta}(\hat{t} t_R \omega_R) \leq L_{V_0}(\hat{t} t_R \omega_R) + \frac{\hat{t}^2}{2} \eta t_R^2 \int_{\mathbb{R}^N} \omega_R^2 dx + \int_{B_{2R} \cap \{\hat{t} t_R \omega_R \leq \beta\}} F(\hat{t} t_R \omega_R) dx + \int_{B_{2R} \cap \{\hat{t} t_R \omega_R > \beta\}} F(\beta) dx \\ &\leq L_{V_0}(\hat{t} t_R \omega_R) + C\eta + \int_{B_{2R} \cap \{\hat{t} t_R \omega_R \leq \beta\}} F(\hat{t} t_R \omega_R) dx + CF(\beta). \end{aligned}$$

Consequently,

$$\limsup_{(\varepsilon,\beta) \rightarrow (0,0)} c_{\varepsilon,\beta} \leq d_{V_0}.$$

\square

4. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. The idea is to show that the solution produced in Theorem 3.1 verifies the estimate $u_{\varepsilon,\beta}(x) < a$ for $x \in \Lambda_\varepsilon^c$ whenever $\varepsilon, \beta > 0$ are small enough. We start by proving the following crucial compactness result.

Lemma 4.1. *Let $\varepsilon_n \rightarrow 0$, $\beta_n \rightarrow 0$ and $(u_n) := (u_{\varepsilon_n, \beta_n}) \subset X_{\varepsilon_n}$ be such that $J_{\varepsilon_n, \beta_n}(u_n) = c_{\varepsilon_n, \beta_n}$ and $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$ has a convergent subsequence in $H^s(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \rightarrow y_0$ for some $y_0 \in M$.*

Proof. For simplicity, we denote $g_{H_n}(x, t) := H(t - \beta_n)g(x, t)$ by $g_H(x, t)$. From $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$, (g_1) and (g_2) , we derive that

$$\|u_n\|_{\varepsilon_n}^2 \leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon_n x, u_n) u_n dx \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n dx \leq \frac{1}{2} \|u_n\|_{\varepsilon_n}^2 + C \|u_n\|_{\varepsilon_n}^{p+1},$$

so there is $\sigma > 0$ such that

$$\|u_n\|_{\varepsilon_n} \geq \sigma \quad \text{for all } n \in \mathbb{N}. \quad (4.1)$$

Using $J_{\varepsilon_n, \beta_n}(u_n) = c_{\varepsilon_n, \beta_n}$, $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$ and Lemma 3.5, we can argue as in the proof of Lemma 3.2 to deduce that (u_n) is bounded in X_{ε_n} . Furthermore, we can find a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \alpha > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(\tilde{y}_n)} u_n^2 dx \geq \alpha. \quad (4.2)$$

Indeed, if the above limit does not hold, we can use Lemma 2.1 to see that

$$u_n \rightarrow 0 \text{ in } L^q(\mathbb{R}^N) \quad \text{for all } q \in (2, 2_s^*). \quad (4.3)$$

Since $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$, we have

$$\|u_n\|_{\varepsilon_n}^2 \leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon_n x, u_n) u_n dx.$$

By (g_1) and (g_2) , we know that for all $\xi > 0$ there is $C_\xi > 0$ such that

$$\|u_n\|_{\varepsilon_n}^2 \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n dx \leq \xi \|u_n\|_{L^2(\mathbb{R}^N)}^2 + C_\xi \|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1},$$

which combined with (4.3) yields $\|u_n\|_{\varepsilon_n} \rightarrow 0$ as $n \rightarrow \infty$. This fact contradicts (4.1). Hence, (4.2) is satisfied. Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Then, (\tilde{u}_n) is bounded in $H^s(\mathbb{R}^N)$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^s(\mathbb{R}^N). \quad (4.4)$$

Moreover, $\tilde{u} \neq 0$ because of

$$\int_{B_R} \tilde{u}^2 dx \geq \alpha. \quad (4.5)$$

Put $y_n := \varepsilon_n \tilde{y}_n$ and we show that (y_n) is bounded in \mathbb{R}^N . To this end, it is enough to prove the following claim:

Claim 1 $\lim_{n \rightarrow \infty} \text{dist}(y_n, \bar{\Lambda}) = 0$.

If the above claim does not hold, there is a $\delta > 0$ and a subsequence of (y_n) , still denoted by itself, such that

$$\text{dist}(y_n, \bar{\Lambda}) \geq \delta \quad \text{for all } n \in \mathbb{N}.$$

Then we can find $r > 0$ such that $B_r(y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Since $\tilde{u} \geq 0$ and $C_c^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, there exists $(\psi_j) \subset C_c^\infty(\mathbb{R}^N)$ such that $\psi_j \geq 0$ and $\psi_j \rightarrow \tilde{u}$ in $H^s(\mathbb{R}^N)$. Fix $j \in \mathbb{N}$. Then,

$$\iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\psi_j(x - \tilde{y}_n) - \psi_j(y - \tilde{y}_n))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon_n x) u_n \psi_j(x - \tilde{y}_n) dx = \int_{\mathbb{R}^N} \rho_n \psi_j(x - \tilde{y}_n) dx, \quad (4.6)$$

where $(\rho_n) \subset \partial \Psi_{\varepsilon_n, \beta_n}(u_n)$ is such that

$$\rho_n(x) \in [\underline{g}_H(\varepsilon_n x, u_n(x)), \bar{g}_H(\varepsilon_n x, u_n(x))] \text{ a.e. in } \mathbb{R}^N. \quad (4.7)$$

From the boundedness of (u_n) in X_ε , we deduce that (ρ_n) is bounded in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and thus we may suppose, up to a subsequence, that $\rho_n \rightharpoonup \rho$ in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Recalling that $u_n, \psi_j \geq 0$, and using (4.7) and the definitions of \bar{g}_H and g , we have

$$\begin{aligned} \int_{\mathbb{R}^N} \rho_n \psi_j(x - \tilde{y}_n) dx &\leq \int_{\mathbb{R}^N} \bar{g}_H(\varepsilon_n x, u_n) \psi_j(x - \tilde{y}_n) dx \\ &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) \psi_j(x - \tilde{y}_n) dx \\ &= \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \tilde{u}_n) \psi_j dx \\ &= \int_{B_{\frac{r}{\varepsilon_n}}} g(\varepsilon_n x + y_n, \tilde{u}_n) \psi_j dx + \int_{\mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_n}}} g(\varepsilon_n x + y_n, \tilde{u}_n) \psi_j dx \\ &\leq \frac{V_1}{k} \int_{B_{\frac{r}{\varepsilon_n}}} \tilde{u}_n \psi_j dx + \int_{\mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_n}}} f(\tilde{u}_n) \psi_j dx, \end{aligned}$$

which combined with (4.6) and (V_1) implies that

$$\iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy + A \int_{\mathbb{R}^N} \tilde{u}_n \psi_j dx \leq \int_{\mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_n}}} f(\tilde{u}_n) \psi_j dx, \quad (4.8)$$

where $A := V_1(1 - \frac{1}{k})$. By (4.4), ψ_j has compact support in \mathbb{R}^N and $\varepsilon_n \rightarrow 0$, we can see that, as $n \rightarrow \infty$,

$$\iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy \rightarrow \iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy,$$

and

$$\int_{\mathbb{R}^N \setminus B_{\frac{r}{\varepsilon_n}}} f(\tilde{u}_n) \psi_j dx \rightarrow 0.$$

The above limits and (4.8) yield

$$\iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy + A \int_{\mathbb{R}^N} \tilde{u} \psi_j dx \leq 0,$$

and sending $j \rightarrow \infty$ we arrive at

$$\|\tilde{u}\|_{Y_A}^2 = [\tilde{u}]_s^2 + A \|\tilde{u}\|_{L^2(\mathbb{R}^N)}^2 \leq 0,$$

which contradicts (4.5). Hence, there exists a subsequence of (y_n) such that $y_n \rightarrow y_0 \in \bar{\Lambda}$.

Claim 2 $y_0 \in \Lambda$.

From (4.6), (4.7) and the definitions of \bar{g}_H and g , we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \tilde{u}_n \psi_j dx \leq \int_{\mathbb{R}^N} f(\tilde{u}_n) \psi_j dx.$$

Letting $n \rightarrow \infty$, we find

$$\iint_{\mathbb{R}^{2N}} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(y_0) \tilde{u} \psi_j dx \leq \int_{\mathbb{R}^N} f(\tilde{u}) \psi_j dx,$$

and passing to the limit as $j \rightarrow \infty$ we have

$$[\tilde{u}]_s^2 + \int_{\mathbb{R}^N} V(y_0) \tilde{u}^2 dx \leq \int_{\mathbb{R}^N} f(\tilde{u}) \tilde{u} dx.$$

Then there exists $\tau \in (0, 1)$ such that $\tau \tilde{u} \in \mathcal{M}_{V(y_0)}$. Therefore, using Lemma 3.5 and Proposition 3.2, we get

$$d_{V(y_0)} \leq L_{V(y_0)}(\tau \tilde{u}) \leq \liminf_{n \rightarrow \infty} J_{\varepsilon_n, \beta_n}(u_n) = \liminf_{n \rightarrow \infty} c_{\varepsilon_n, \beta_n} \leq d_{V_0},$$

from which we derive that $V(y_0) \leq V(0) = V_0$. Since $V_0 = \inf_{\bar{\Lambda}} V$, we obtain that $V(y_0) = V_0$. Thanks to (V_2) , we can infer that $y_0 \notin \partial \Lambda$, that is, $y_0 \in \Lambda$.

Claim 3 $\tilde{u}_n \rightarrow \tilde{u}$ in $H^s(\mathbb{R}^N)$ as $n \rightarrow \infty$.

Taking the limit as $n \rightarrow \infty$ in the weak formulation of $(-\Delta)^s \tilde{u}_n + V_n(x) \tilde{u}_n = h_n$ in \mathbb{R}^N , where $V_n(x) := V(\varepsilon_n x + y_n)$ and

$$h_n(x) := \rho_n(x + \tilde{y}_n) \in [g_H(\varepsilon_n x + y_n, \tilde{u}_n(x)), \bar{g}_H(\varepsilon_n x + y_n, \tilde{u}_n(x))] \text{ a.e. in } \mathbb{R}^N,$$

we deduce that \tilde{u} is a positive solution to $(-\Delta)^s \tilde{u} + V_0 \tilde{u} = f(\tilde{u})$ in \mathbb{R}^N .

Let us define

$$\tilde{\Lambda}_n := \frac{\Lambda - y_n}{\varepsilon_n},$$

and consider

$$\begin{aligned} \tilde{\chi}_n^1(x) &:= \begin{cases} 1 & \text{if } x \in \tilde{\Lambda}_n, \\ 0 & \text{if } x \in \tilde{\Lambda}_n^c, \end{cases} \\ \tilde{\chi}_n^2(x) &:= 1 - \tilde{\chi}_n^1(x). \end{aligned}$$

Let us introduce the following nonnegative functions on \mathbb{R}^N :

$$\begin{aligned} \kappa_n^1(x) &:= \left(\frac{1}{2} - \frac{1}{\theta} \right) V(\varepsilon_n x + y_n) \tilde{u}_n^2(x) \tilde{\chi}_n^1(x), \\ \kappa^1(x) &:= \left(\frac{1}{2} - \frac{1}{\theta} \right) V_0 \tilde{u}^2(x), \\ \kappa_n^2(x) &:= \left[\left(\frac{1}{2} - \frac{1}{\theta} \right) V(\varepsilon_n x + y_n) \tilde{u}_n^2(x) + \frac{1}{\theta} g_H(\varepsilon_n x + y_n, \tilde{u}_n(x)) \tilde{u}_n(x) - G_H(\varepsilon_n x + y_n, \tilde{u}_n(x)) \right] \tilde{\chi}_n^2(x), \\ &= \left[\left(\frac{1}{2} - \frac{1}{\theta} \right) V(\varepsilon_n x + y_n) \tilde{u}_n^2(x) \right. \\ &\quad \left. + \left(\frac{1}{\theta} g(\varepsilon_n x + y_n, \tilde{u}_n(x)) \tilde{u}_n(x) - [G(\varepsilon_n x + y_n, \tilde{u}_n(x)) - G(\varepsilon_n x + y_n, \beta_n)] \right) \chi_{\{\tilde{u}_n > \beta_n\}}(x) \right] \tilde{\chi}_n^2(x), \\ \kappa_n^3(x) &:= \left[\frac{1}{\theta} g_H(\varepsilon_n x + y_n, \tilde{u}_n(x)) \tilde{u}_n(x) - G_H(\varepsilon_n x + y_n, \tilde{u}_n(x)) \right] \tilde{\chi}_n^1(x), \\ &= \left[\frac{1}{\theta} f(\tilde{u}_n(x)) \tilde{u}_n(x) - [F(\tilde{u}_n(x)) - F(\beta_n)] \right] \chi_{\{\tilde{u}_n > \beta_n\}}(x) \tilde{\chi}_n^1(x), \\ \kappa^3(x) &:= \frac{1}{\theta} f(\tilde{u}(x)) \tilde{u}(x) - F(\tilde{u}(x)). \end{aligned}$$

By (4.4) and Claim 2, we know that $\tilde{u}_n(x) \rightarrow \tilde{u}(x)$ a.e. in \mathbb{R}^N and $y_n \rightarrow y_0 \in \Lambda$. Accordingly,

$$\tilde{\chi}_n^1(x) \rightarrow 1, \kappa_n^1(x) \rightarrow \kappa^1(x), \kappa_n^2(x) \rightarrow 0 \text{ and } \kappa_n^3(x) \rightarrow \kappa^3(x) \text{ a.e. } x \in \mathbb{R}^N.$$

Let $w_n := Q'(u_n) - \rho_n$ be such that $\lambda_{\varepsilon_n, \beta_n}(u_n) = \|w_n\|_{X_{\varepsilon_n}^*} = o_n(1)$. Then, using Lemma 3.5, Fatou's lemma, (4.7) and a change of variable, we obtain that

$$\begin{aligned} d_{V_0} &\geq \limsup_{n \rightarrow \infty} c_{\varepsilon_n, \beta_n} = \limsup_{n \rightarrow \infty} \left(J_{\varepsilon_n, \beta_n}(u_n) - \frac{1}{\theta} \langle w_n, u_n \rangle \right) \\ &\geq \limsup_{n \rightarrow \infty} \left[\left(\frac{1}{2} - \frac{1}{\theta} \right) [\tilde{u}_n]_s^2 + \int_{\mathbb{R}^N} (\kappa_n^1 + \kappa_n^2 + \kappa_n^3) dx \right] \\ &\geq \liminf_{n \rightarrow \infty} \left[\left(\frac{1}{2} - \frac{1}{\theta} \right) [\tilde{u}_n]_s^2 + \int_{\mathbb{R}^N} (\kappa_n^1 + \kappa_n^2 + \kappa_n^3) dx \right] \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) [\tilde{u}]_s^2 + \int_{\mathbb{R}^N} (\kappa^1 + \kappa^3) dx \geq d_{V_0}. \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} [\tilde{u}_n]_s^2 = [\tilde{u}]_s^2, \tag{4.9}$$

and

$$\kappa_n^1 \rightarrow \kappa^1, \kappa_n^2 \rightarrow 0 \text{ and } \kappa_n^3 \rightarrow \kappa^3 \text{ in } L^1(\mathbb{R}^N).$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \tilde{u}_n^2 dx = \int_{\mathbb{R}^N} V_0 \tilde{u}^2 dx,$$

from which we deduce that

$$\lim_{n \rightarrow \infty} \|\tilde{u}_n\|_{L^2(\mathbb{R}^N)}^2 = \|\tilde{u}\|_{L^2(\mathbb{R}^N)}^2. \quad (4.10)$$

Combining (4.9) with (4.10), and recalling that $H^s(\mathbb{R}^N)$ is a Hilbert space, we conclude that $\tilde{u}_n \rightarrow \tilde{u}$ in $H^s(\mathbb{R}^N)$. \square

Next we develop a Moser iteration argument [31] to prove a very useful L^∞ -estimate which plays a fundamental role in the study of behavior of the maximum points of the solutions.

Lemma 4.2. *Let (\tilde{u}_n) be the sequence given in Lemma 4.1. Then $\tilde{u}_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that*

$$\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} \leq C \quad \text{for all } n \in \mathbb{N}.$$

Moreover, $\tilde{u}_n(x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in $n \in \mathbb{N}$.

Proof. We proceed as in [7] (see also [3, 8]). Let $\gamma > 1$ and $T > 0$, and we introduce the following function

$$\varphi(t) := \varphi_{T,\gamma}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t^\gamma & \text{if } 0 < t < T, \\ \gamma T^{\gamma-1}(t-T) + T^\gamma & \text{if } t \geq T. \end{cases}$$

Since φ is convex, Lipschitz continuous and $\varphi(0) = 0$, we can see that for each $u \in H^s(\mathbb{R}^N)$, $\varphi(u) \in H^s(\mathbb{R}^N)$ and $(-\Delta)^s \varphi(u) \leq \varphi'(u)(-\Delta)^s u$ in weak sense. Then, using the fractional Sobolev inequality (see Theorem 1.1.8 in [8]), (V1), $\tilde{u}_n \geq 0$ in \mathbb{R}^N , and the growth assumptions on g , we have

$$\begin{aligned} \|\varphi(\tilde{u}_n)\|_{L^{2_s^*}(\mathbb{R}^N)}^2 &\leq S_*^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi(\tilde{u}_n)|^2 dx \\ &= S_*^{-1} \int_{\mathbb{R}^N} \varphi(\tilde{u}_n)(-\Delta)^s \varphi(\tilde{u}_n) dx \\ &\leq S_*^{-1} \int_{\mathbb{R}^N} \varphi(\tilde{u}_n) \varphi'(\tilde{u}_n)(-\Delta)^s \tilde{u}_n dx \\ &= S_*^{-1} \int_{\mathbb{R}^N} \varphi(\tilde{u}_n) \varphi'(\tilde{u}_n)[-V_n + h_n] dx \\ &\leq C S_*^{-1} \int_{\mathbb{R}^N} \varphi(\tilde{u}_n) \varphi'(\tilde{u}_n)(1 + \tilde{u}_n^{2_s^*-1}) dx \\ &= C S_*^{-1} \left(\int_{\mathbb{R}^N} \varphi(\tilde{u}_n) \varphi'(\tilde{u}_n) dx + \int_{\mathbb{R}^N} \varphi(\tilde{u}_n) \varphi'(\tilde{u}_n) \tilde{u}_n^{2_s^*-1} dx \right), \end{aligned}$$

where $S_* := S_*(N, s) > 0$ denotes the best constant in the fractional Sobolev inequality, and $C > 0$ is a constant independent of γ and n . In light of $\varphi(\tilde{u}_n) \varphi'(\tilde{u}_n) \leq \gamma \tilde{u}_n^{2\gamma-1}$ and $\tilde{u}_n \varphi'(\tilde{u}_n) \leq \gamma \varphi(\tilde{u}_n)$, we get

$$\|\varphi(\tilde{u}_n)\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq C \gamma \left(\int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma-1} dx + \int_{\mathbb{R}^N} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx \right), \quad (4.11)$$

where $C > 0$ is a constant independent of γ and n . We stress that the last integral in (4.11) is well defined for every $T > 0$ in the definition of φ . Indeed,

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx &= \int_{\{\tilde{u}_n \leq T\}} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx + \int_{\{\tilde{u}_n > T\}} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx \\ &\leq T^{2\gamma-2} \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx + C \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx < \infty, \end{aligned}$$

where we have used that $\gamma > 1$ and that $\phi(t)$ is linear when $t \geq T$. Now we take γ in (4.11) such that $2\gamma - 1 = 2_s^*$, and we name it γ_1 , that is,

$$\gamma_1 := \frac{2_s^* + 1}{2}. \quad (4.12)$$

Let $R > 0$ to be fixed later. Applying Hölder's inequality in the last integral in (4.11), we see that

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx &= \int_{\{\tilde{u}_n \leq R\}} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx + \int_{\{\tilde{u}_n > R\}} (\varphi(\tilde{u}_n))^2 \tilde{u}_n^{2_s^*-2} dx \\ &\leq \int_{\{\tilde{u}_n \leq R\}} \frac{(\varphi(\tilde{u}_n))^2}{\tilde{u}_n} R^{2_s^*-1} dx + \left(\int_{\mathbb{R}^N} (\varphi(\tilde{u}_n))^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \left(\int_{\{\tilde{u}_n > R\}} \tilde{u}_n^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}}. \end{aligned} \quad (4.13)$$

Since (\tilde{u}_n) strongly converges in $H^s(\mathbb{R}^N)$ (by Lemma 4.1), we can take R sufficiently large such that

$$\left(\int_{\{\tilde{u}_n > R\}} \tilde{u}_n^{2_s^*} dx \right)^{\frac{2_s^*-2}{2_s^*}} \leq \frac{1}{2C\gamma_1},$$

where C is the same constant appearing in (4.11). This together with (4.11), (4.12) and (4.13), yields

$$\|\varphi(\tilde{u}_n)\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq 2C\gamma_1 \left(\int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^N} \frac{\varphi(\tilde{u}_n)^2}{\tilde{u}_n} dx \right). \quad (4.14)$$

From $\varphi(\tilde{u}_n) \leq \tilde{u}_n^{\gamma_1}$ and (4.12), and letting $T \rightarrow \infty$ in (4.14), we obtain

$$\left(\int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*\gamma_1} dx \right)^{\frac{2}{2_s^*}} \leq 2C\gamma_1 \left(\int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx + R^{2_s^*-1} \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx \right),$$

which combined the boundedness of (\tilde{u}_n) in $L^{2_s^*}(\mathbb{R}^N)$ implies

$$\|\tilde{u}_n\|_{L^{2_s^*\gamma_1}(\mathbb{R}^N)} \leq C' \quad \text{for all } n \in \mathbb{N}. \quad (4.15)$$

Now we suppose $\gamma > \gamma_1$. Thus, using $\varphi(\tilde{u}_n) \leq \tilde{u}_n^\gamma$ on the right hand side of (4.11) and sending $T \rightarrow \infty$, we deduce that

$$\left(\int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*\gamma} dx \right)^{\frac{2}{2_s^*}} \leq C\gamma \left(\int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma-1} dx + \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma+2_s^*-2} dx \right). \quad (4.16)$$

Set

$$a_1 := \frac{2_s^*(2_s^*-1)}{2(\gamma-1)} \quad \text{and} \quad a_2 := 2\gamma-1-a_1.$$

Notice that, since $\gamma > \gamma_1$, then $0 < a_1 < 2_s^*$ and $a_2 > 0$. Applying the Young inequality with exponents $r = \frac{2_s^*}{a_1}$ and $r' = \frac{2_s^*}{2_s^*-a_1}$, we see that

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma-1} dx &\leq \frac{a_1}{2_s^*} \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx + \frac{2_s^*-a_1}{2_s^*} \int_{\mathbb{R}^N} \tilde{u}_n^{\frac{2_s^*a_2}{2_s^*-a_1}} dx \\ &\leq \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*} dx + \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma+2_s^*-2} dx \\ &\leq C \left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma+2_s^*-2} dx \right), \end{aligned} \quad (4.17)$$

with $C > 0$ independent of γ and $n \in \mathbb{N}$. Combining (4.16) and (4.17), we achieve

$$\left(\int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*\gamma} dx \right)^{\frac{2}{2_s^*}} \leq C\gamma \left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma+2_s^*-2} dx \right),$$

with C changing from line to line, but remaining independent of γ and $n \in \mathbb{N}$. Therefore,

$$\left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^*\gamma} dx \right)^{\frac{1}{2_s^*(\gamma-1)}} \leq (C\gamma)^{\frac{1}{2(\gamma-1)}} \left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2\gamma+2_s^*-2} dx \right)^{\frac{1}{2(\gamma-1)}}. \quad (4.18)$$

For $m \in \mathbb{N}$, we define γ_m inductively so that $2\gamma_{m+1} + 2_s^* - 2 = 2_s^*\gamma_m$, that is

$$\gamma_{m+1} = \left(\frac{2_s^*}{2} \right)^m (\gamma_1 - 1) + 1.$$

Thus, from (4.18), we arrive at

$$\left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^* \gamma_{m+1}} dx\right)^{\frac{1}{2_s^* (\gamma_{m+1}-1)}} \leq (C \gamma_{m+1})^{\frac{1}{2(\gamma_{m+1}-1)}} \left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^* \gamma_m} dx\right)^{\frac{1}{2_s^* (\gamma_m-1)}}. \quad (4.19)$$

Setting

$$A_{m,n} := \left(1 + \int_{\mathbb{R}^N} \tilde{u}_n^{2_s^* \gamma_m} dx\right)^{\frac{1}{2_s^* (\gamma_m-1)}}$$

and

$$C_{m+1} := C \gamma_{m+1},$$

we can find a constant $C_0 > 0$ independent of m such that

$$A_{m+1,n} \leq \prod_{j=2}^{m+1} C_j^{\frac{1}{2(\gamma_j-1)}} A_{1,n} \leq C_0 A_{1,n} \quad \text{for all } m, n \in \mathbb{N}.$$

Since (4.15) implies that, for some $A_0 > 0$, $A_{1,n} \leq A_0$ for all $n \in \mathbb{N}$, we deduce that $A_{m+1,n} \leq C_0 A_0$ for all $m, n \in \mathbb{N}$. Consequently, letting $m \rightarrow \infty$,

$$\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} \leq C \quad \text{for all } n \in \mathbb{N}.$$

Now, we observe that \tilde{u}_n is a subsolution to

$$(-\Delta)^s \tilde{u}_n + V_1 \tilde{u}_n = h_n \quad \text{in } \mathbb{R}^N.$$

Since $\tilde{u}_n \rightarrow \tilde{u}$ in $H^s(\mathbb{R}^N)$ (by Lemma 4.1) and $\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$, it follows from interpolation on the L^r spaces and the growth assumptions on g that $h_n \rightarrow h := f(\tilde{u})$ in $L^q(\mathbb{R}^N)$ for all $q \in [2, \infty)$, and that $\|h_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Let z_n be the unique solution of

$$(-\Delta)^s z_n + V_1 z_n = h_n \quad \text{in } \mathbb{R}^N.$$

Then,

$$z_n(x) = (\mathcal{K} * h_n)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x-y) h_n(y) dy, \quad (4.20)$$

where the kernel $\mathcal{K}(x) := (2\pi)^{-\frac{N}{2}} \mathcal{F}^{-1}((|\xi|^{2s} + V_1)^{-1})$ satisfies the following properties (see [24]):

- (b₁) \mathcal{K} is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$;
- (b₂) there exists $C > 0$ such that $\mathcal{K}(x) \leq \frac{C}{|x|^{N+2s}}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
- (b₃) $\mathcal{K} \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{N-2s})$.

Next we show that $z_n(x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in $n \in \mathbb{N}$. For each fixed $\delta \in (0, 1)$, it holds

$$0 \leq z_n(x) = (\mathcal{K} * h_n)(x) = \int_{B_{\frac{1}{\delta}}^c(x)} \mathcal{K}(x-y) h_n(y) dy + \int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y) h_n(y) dy. \quad (4.21)$$

From (b₂) we derive that

$$\int_{B_{\frac{1}{\delta}}^c(x)} \mathcal{K}(x-y) h_n(y) dy \leq C \|h_n\|_{L^\infty(\mathbb{R}^N)} \int_{B_{\frac{1}{\delta}}^c(x)} \frac{dy}{|x-y|^{N+2s}} \leq C \delta^{2s} \int_{|\xi| \geq 1} \frac{d\xi}{|\xi|^{N+2s}} \leq C \delta^{2s}. \quad (4.22)$$

On the other hand,

$$\int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y) |h_n(y)| dy \leq \int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y) |h_n(y) - h(y)| dy + \int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y) |h(y)| dy.$$

Pick $q \in \left(1, \min\{\frac{N}{N-2s}, 2\}\right)$ so that $q' > 2$, where q' is such that $\frac{1}{q} + \frac{1}{q'} = 1$. From (b₃) and Hölder's inequality, we see that

$$\int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y) |h_n(y)| dy \leq \|\mathcal{K}\|_{L^q(\mathbb{R}^N)} \|h_n - h\|_{L^{q'}(\mathbb{R}^N)} + \|\mathcal{K}\|_{L^q(\mathbb{R}^N)} \|h\|_{L^{q'}(B_{\frac{1}{\delta}}(x))}.$$

Since $\|h_n - h\|_{L^{q'}(\mathbb{R}^N)} \rightarrow 0$ as $n \rightarrow \infty$ and $\|h\|_{L^{q'}(B_{\frac{1}{\delta}}(x))} \rightarrow 0$ as $|x| \rightarrow \infty$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{B_{\frac{1}{\delta}}(x)} \mathcal{K}(x-y)|h_n(y)| dy \leq C\delta \quad \text{for all } n \geq n_0, |x| \geq R. \quad (4.23)$$

Combining (4.22) and (4.23), we infer that

$$\int_{\mathbb{R}^N} \mathcal{K}(x-y)|h_n(y)| dy \leq C(\delta^{2s} + \delta) \quad \text{for all } n \geq n_0, |x| \geq R. \quad (4.24)$$

The same approach can be used to prove that for each $n \in \{1, \dots, n_0 - 1\}$, there is $R_n > 0$ such that

$$\int_{\mathbb{R}^N} \mathcal{K}(x-y)|h_n(y)| dy \leq C(\delta^{2s} + \delta) \quad \text{for all } |x| \geq R_n.$$

Hence, increasing R if necessary, we must have

$$\int_{\mathbb{R}^N} \mathcal{K}(x-y)|h_n(y)| dy \leq C(\delta^{2s} + \delta) \quad \text{for all } |x| \geq R, \text{ uniformly in } n \in \mathbb{N}.$$

Letting $\delta \rightarrow 0$ we reach the desired result for z_n . Since a comparison argument shows that $0 \leq \tilde{u}_n \leq z_n$ in \mathbb{R}^N , we obtain the assertion. \square

Proof of Theorem 1.1. We start by proving that there exist $\varepsilon_0 > 0$ and $\beta_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and $\beta \in (0, \beta_0)$ and every solution $u_{\varepsilon, \beta}$ of problem (3.1) (whose existence is given in Theorem 3.1), it holds

$$\|u_{\varepsilon, \beta}\|_{L^\infty(\Lambda_\varepsilon)} < a. \quad (4.25)$$

Suppose by contradiction that for some subsequences $\varepsilon_n \rightarrow 0$ and $\beta_n \rightarrow 0$, we can find $u_n := u_{\varepsilon_n, \beta_n}$ such that $J_{\varepsilon_n, \beta_n}(u_n) = c_{\varepsilon_n, \beta_n}$, $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$ and

$$\|u_n\|_{L^\infty(\Lambda_{\varepsilon_n})} \geq a. \quad (4.26)$$

In view of Lemma 4.1, we can produce $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\tilde{u}_n := u_n(\cdot + \tilde{y}_n) \rightarrow \tilde{u}$ in $H^s(\mathbb{R}^N)$ and $\varepsilon_n \tilde{y}_n \rightarrow y_0 \in M$. Let us choose $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$, and so

$$B_{\frac{r}{\varepsilon_n}}\left(\frac{y_0}{\varepsilon_n}\right) \subset \Lambda_{\varepsilon_n}.$$

Hence, for all $y \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$,

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n},$$

for all n sufficiently large. For these values of n , we get

$$\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(\tilde{y}_n). \quad (4.27)$$

Now, by Lemma 4.2, we know that

$$\tilde{u}_n(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \text{ uniformly in } n \in \mathbb{N}, \quad (4.28)$$

and thus there exists $R > 0$ such that

$$\tilde{u}_n(x) < a \quad \text{for all } |x| \geq R, n \in \mathbb{N}.$$

Therefore, $u_n(x) < a$ for $x \in B_R^c(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, by (4.27), there exists $n_0 \in \mathbb{N}$ such that

$$\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n) \quad \text{for all } n \geq n_0,$$

which yields $u_n(x) < a$ for $x \in \Lambda_{\varepsilon_n}^c$ and $n \geq n_0$. This fact is in contrast with (4.26), and thus (4.25) is true. By the definition of g , we deduce that $u_{\varepsilon, \beta}$ is a solution to (1.1) for all $\varepsilon \in (0, \varepsilon_0)$ and $\beta \in (0, \beta_0)$. Now we study the behavior of maximum points of (u_n) . Because of (g_1) , there exists $\tau \in (0, a)$ such that

$$g(\varepsilon x, t) \leq \frac{V_1}{2} t^2 \quad \text{for all } x \in \mathbb{R}^N, 0 \leq t \leq \tau. \quad (4.29)$$

As before, we can find $R > 0$ such that

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \tau. \quad (4.30)$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \tau. \quad (4.31)$$

Otherwise, if (4.31) is not true, then (4.30) shows that $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \tau$. This information combined with $0 \in \partial J_{\varepsilon_n, \beta_n}(u_n)$ and (4.29) yields

$$\|u_n\|_{Y_{V_1}}^2 \leq \|u_n\|_{\varepsilon_n}^2 \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n dx \leq \frac{V_1}{2} \|u_n\|_{L^2(\mathbb{R}^N)}^2$$

which gives $\|u_n\|_{Y_{V_1}} = 0$, that is a contradiction. Consequently, (4.31) is verified.

Let $p_n \in \mathbb{R}^N$ be a global maximum point of u_n . In view of (4.30) and (4.31), we can see that $p_n \in B_R(\tilde{y}_n)$. Therefore, $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R$. Thus, $\varepsilon_n p_n = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n \rightarrow y_0 \in M$, and exploiting the continuity of V , we arrive at

$$\lim_{n \rightarrow \infty} V(\varepsilon_n p_n) = V(y_0) = V_0.$$

Finally, we establish a decay estimate for u_n . It is well-known that there is a positive and continuous function w such that

$$0 < w(x) \leq \frac{C}{1 + |x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N, \quad (4.32)$$

and w satisfies in the classical sense

$$(-\Delta)^s w + \frac{V_1}{2} w = 0 \quad \text{in } \overline{B_{R_1}^c}, \quad (4.33)$$

for some $R_1 > 0$. On account of (g_2) and (4.28), there exists $R_2 > 0$ such that

$$\begin{aligned} (-\Delta)^s \tilde{u}_n + \frac{V_1}{2} \tilde{u}_n &= h_n - \left(V(\varepsilon_n \cdot + y_n) - \frac{V_1}{2} \right) \tilde{u}_n \\ &\leq g(\varepsilon_n \cdot + y_n, \tilde{u}_n) - \frac{V_1}{2} \tilde{u}_n \leq 0 \quad \text{in } \overline{B_{R_2}^c}. \end{aligned} \quad (4.34)$$

Let $R_3 := \min\{R_1, R_2\} > 0$, and define

$$c := \min_{\overline{B_{R_3}}} w > 0 \quad \text{and} \quad \tilde{w}_n := (d+1)w - c\tilde{u}_n, \quad (4.35)$$

where $d := \sup_{n \in \mathbb{N}} \|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} < \infty$. Let us note that

$$\tilde{w}_n \geq 0 \quad \text{in } \mathbb{R}^N. \quad (4.36)$$

Indeed, (4.33), (4.34) and (4.35) imply

$$\begin{aligned} \tilde{w}_n &\geq cd + w - cd = w > 0 \quad \text{in } \overline{B_{R_3}}, \\ (-\Delta)^s \tilde{w}_n + \frac{V_1}{2} \tilde{w}_n &\geq 0 \quad \text{in } \overline{B_{R_3}^c}, \end{aligned}$$

and using a comparison argument (see Lemma 1.3.8 in [8]) we obtain that (4.36) is satisfied. In light of (4.32) and (4.36), we get

$$0 < \tilde{u}_n(x) \leq \frac{\tilde{C}}{1 + |x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N, n \in \mathbb{N}.$$

The above inequality combined with $u_n(x) = \tilde{u}_n(x - \tilde{y}_n)$, $\|\tilde{u}_n\|_{L^\infty(\mathbb{R}^N)} \leq C$, $p_n = \tilde{y}_n + q_n$, $|q_n| < R$ and $\varepsilon_n \tilde{y}_n \rightarrow y_0 \in \mathbb{R}$, gives

$$0 < u_n(x) \leq \frac{C}{1 + |x - p_n|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N, n \in \mathbb{N}.$$

This completes the proof of Theorem 1.1. \square

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VINCENZO AMBROSIO
DIPARTIMENTO DI INGEGNERIA INDUSTRIALE E SCIENZE MATEMATICHE
UNIVERSITÀ POLITECNICA DELLE MARCHE
VIA BRECCE BIANCHE, 12
60131 ANCONA (ITALY)
E-mail address: v.ambrosio@staff.univpm.it