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# EXISTENCE OF HETEROCLINIC AND SADDLE TYPE SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS IN WHOLE $\mathbb{R}^2$

CLAUDIANOR O. ALVES, RENAN J. S. ISNERI, AND PIERO MONTECCHIARI

ABSTRACT. In this work, we use variational methods to prove the existence of heteroclinic and saddle type solutions for a class quasilinear elliptic equations of the form

$$-\Delta_{\Phi}u + A(x, y)V'(u) = 0 \quad \text{in } \mathbb{R}^2,$$

where  $\Phi : \mathbb{R} \rightarrow [0, +\infty)$  is a N-function,  $A : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a periodic positive function and  $V : \mathbb{R} \rightarrow \mathbb{R}$  is modeled on the Ginzburg Landau potential. In particular our main result includes the case of the potential  $V(t) = \Phi(|t^2 - 1|)$ , which reduces to the classical double well Ginzburg-Landau potential when  $\Phi(t) = |t|^2$ , that is, when we are working with the Laplacian operator.

## 1. INTRODUCTION

The problem of existence and classification of bounded solutions of stationary Allen Cahn type equations

$$-\Delta u + A(x)V'(u) = 0 \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{E_1}$$

has been widely studied in the last years, providing a rich amount of differently shaped families of solutions. The Allen-Cahn equation was introduced in 1979 by Allen and Cahn in [12] as a model for phase transitions in binary alloys. The standard model of  $V$  is the classical double well Ginzburg-Landau potential  $V(u) = (u^2 - 1)^2$ . The function  $u$  is a phase parameter describing pointwise the state of the material and the global minima of  $V$  represent energetically favorite pure phases. Different values of  $u$  depict mixed configurations and by transition solutions we mean entire solutions of  $(E_1)$  which are asymptotic in different directions to the pure phases of the systems. In the equation  $(E_1)$  the presence of the (positive) oscillatory factor  $A(x)$  models an inhomogeneous behavior of the system.

When  $A$  is a positive constant function (e.g.  $A(x) = 1$ ), a long standing problem is to characterize the set of the solutions  $u \in C^2(\mathbb{R}^n)$  of  $(E_1)$  satisfying  $|u(x)| \leq 1$  and  $\partial_{x_1}u(x) > 0$ . This problem was pointed out by De Giorgi in [25], where he conjectured that, when  $n \leq 8$  and  $V(s) = (s^2 - 1)^2$ , the whole set of these solutions reduces, modulo space roto-translations, to the unique solution  $q_+ \in C^2(\mathbb{R})$  of the one dimensional problem:

$$-\ddot{q}(x) + V'(q(x)) = 0, \quad q(0) = 0 \quad \text{and} \quad q(\pm\infty) = \pm 1.$$

The conjecture has been firstly proved in the planar case by Ghoussoub and Gui in [40] even for more general double well potential  $V$ . In the case  $n = 3$  it has been proved in [15] and, assuming  $u(x) \rightarrow \pm 1$  as  $x_1 \rightarrow \pm\infty$ , the same rigidity result has been obtained in dimension  $n \leq 8$  in [53], paper to which we refer also for an extensive bibliography on the argument. Del Pino, Kowalczyk and Wei showed in [28, 29] that the 1-D symmetry of these solutions is

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generally lost when  $n \geq 9$ . We refer also to [17, 19, 31], where a weaker version of the De Giorgi conjecture, known as Gibbons conjecture, has been obtained for all the dimensions  $n$  and in more general settings. These results show that when  $A$  is a positive constant and  $u$  is a bounded solution of  $(E_1)$  satisfying  $u(x) \rightarrow \pm 1$  as  $x_1 \rightarrow \pm\infty$  uniformly with respect to  $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$  then  $u(x) = q_+(x_1)$ .

This kind of heteroclinic type transition solutions persist when  $A$  is not constant. The heteroclinic type problem was first studied by variational methods for more general elliptic equations of the type

$$-\Delta u = g(x, y, u) \quad \text{in } x \in \mathbb{R}, y \in \Omega, u \in \mathbb{R}, \quad (E_2)$$

by Rabinowitz in [47], when  $\Omega$  is a bounded regular domain on  $\mathbb{R}^n$ . Assuming the nonlinearity  $g$  to be even and periodic in the variable  $x$ , Rabinowitz showed the existence of solutions of  $(E_2)$  in  $\mathbb{R} \times \Omega$  satisfying Dirichlet or Neumann boundary condition on  $\partial\Omega$  and being asymptotic as  $x \rightarrow \pm\infty$  to different minimal solutions  $u_{\pm}$ , periodic in the variable  $x$ . This result was generalized by Alves in [13] for different conditions on  $g$ , including the case in which  $g$  is only asymptotically periodic in the variable  $x$ . A related variational approach was used to study the heteroclinic type problem for equation  $(E_1)$  in the case in which  $A$  is periodic in its variable in [3, 48, 49], showing the existence of (minimal) solutions  $u(x)$  which are periodic in the variable  $(x_2, \dots, x_n)$  and such that  $u$  is asymptotic to different minima of the potential  $V$  as  $x_1 \rightarrow \pm\infty$ . Starting from the existence of this “basic” heteroclinic solutions, these papers show how the presence of a truly oscillatory factor  $A(x, y)$  gives generically the existence of complex classes of other heteroclinic type transition solutions in contrast with the above described rigidity results characterizing the autonomous case (see also [11, 18, 50]).

Another kind of transition solutions for  $(E_1)$  was introduced by Dang, Fife and Peletier in [24]. In the planar case  $n = 2$ , when  $V$  is an even double well potential and  $A$  is a positive constant, they showed by a sub-supersolution method that  $(E_1)$  has a unique bounded solution  $u \in C^2(\mathbb{R}^2)$  with the same sign as  $x_1x_2$ , odd in both the variables  $x_1$  and  $x_2$  and symmetric with respect to the diagonals  $x_2 = \pm x_1$ . Along any directions not parallel to the coordinate axes the *saddle* solution  $u$  is asymptotic to the minima of the potential  $V$  representing a phase transition with cross interface. Note that, even if it is related to minimal transition heteroclinic solutions, being asymptotic to  $q_+$  as  $x_2 \rightarrow +\infty$ , it no longer has minimal character (see [44, 54]). Many extensions for Allen-Cahn models have been considered. In the planar case we refer to [8] for a variational study of saddle type solutions with dihedral symmetries of order  $k$  (see also [43] for a global variational approach to the saddle problem) and to [30, 41] for a general study regarding  $k$ -end solutions. In higher dimension we mention [5, 6, 21, 22, 46] for the equations case and to [2, 7, 42] for the case of systems of autonomous Allen-Cahn equations.

The analogous for saddle solutions for  $(E_1)$  in the planar case, when  $A \in C(\mathbb{R}^2)$  is positive, even, periodic and symmetric with respect to the plane diagonal  $x_2 = x_1$ , i.e, when  $A$  satisfies

- (A<sub>1</sub>)  $A$  is a continuous function and  $A(x, y) > 0$  for each  $(x, y) \in \mathbb{R}^2$ ,
- (A<sub>2</sub>)  $A(x, y) = A(-x, y) = A(x, -y)$  for all  $(x, y) \in \mathbb{R}^2$ ,
- (A<sub>3</sub>)  $A(x, y) = A(x + 1, y) = A(x, y + 1)$  for any  $(x, y) \in \mathbb{R}^2$ ,
- (A<sub>4</sub>)  $A(x, y) = A(y, x)$  for all  $(x, y) \in \mathbb{R}^2$ ,

has been introduced in [9] where a variational procedure was introduced to find as in the autonomous case a solution  $u$  of  $(E_1)$  on  $\mathbb{R}^2$  which is odd with respect to both its variables, symmetric with respect to the diagonal, strictly positive on the first quadrant and is asymptotic to the minima of  $V$  along any directions not parallel to the coordinate axes. Moreover in [9] it is shown that, as  $y \rightarrow +\infty$  (uniformly w.r.t.  $x \in \mathbb{R}$ ), the solution  $u$  is asymptotic to the set of the  $x$ -odd minimal heteroclinic type solutions of  $(E_1)$  which are periodic in the variable  $y$  described above.

In the recent paper [14], motivated by results found in [8], we tackled the problem of existence of saddle solutions for the analogous of Allen Cahn model in the autonomous quasilinear setting. More precisely given an N-function  $\Phi : \mathbb{R} \rightarrow [0, +\infty)$  of the form

$$(1.1) \quad \Phi(t) = \int_0^{|t|} s\phi(s)ds$$

for a  $\phi \in C^1([0, +\infty), [0, +\infty))$  such that:

( $\phi_1$ ) there exist  $l, m \in \mathbb{R}$  such that  $1 < l \leq m$  and

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \quad \forall t > 0,$$

( $\phi_2$ )  $\phi(t) > 0$  and  $(\phi(t)t) > 0$  for any  $t > 0$ ,

( $\phi_3$ )  $\phi$  is non-decreasing,

( $\phi_4$ ) there exists  $\kappa > 0$  such that

$$\phi(|t|) + \phi'(|t|)|t| \leq \kappa\phi(|t|), \quad \forall t \in \mathbb{R},$$

( $\phi_5$ ) there is  $M > 0$  such that  $(\phi(t)t)' \geq M\phi(t)$  for all  $t > 0$ ,

and a potential  $V \in C^2(\mathbb{R}, \mathbb{R})$  verifying:

( $V_1$ )  $V(t) \geq 0$  for all  $t \in \mathbb{R}$  and  $V(t) = 0 \Leftrightarrow t = -1, 1$ ,

( $V_2$ )  $V(-t) = V(t)$  for any  $t \in \mathbb{R}$ ,

( $V_3$ ) there are  $\delta_1 \in (0, 1)$  and  $w_1, w_2 > 0$  such that

$$w_1\Phi(|t-1|) \leq V(t) \leq w_2\Phi(|t-1|), \quad \forall t \in (1-\delta_1, 1+\delta_1),$$

( $V_4$ ) there exists  $\omega > 0$  such that

$$V'(t) \leq -\omega\phi(|1-t|)|1-t|, \quad \forall t \in [0, 1],$$

( $V_5$ ) there is  $\delta_0 > 0$  such that  $V'$  is increasing on  $(1-\delta_0, 1)$ ,

( $V_6$ ) there are  $\gamma > 0$  and  $\epsilon > 0$  such that  $\tilde{\Phi}(V'(t)) \leq \gamma\Phi(|1-t|)$  for all  $t \in (1-\epsilon, 1)$

we considered the related quasilinear Allen Cahn model

$$-\Delta_{\Phi}u + V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2. \quad (E_3)$$

where  $\Delta_{\Phi}u = \text{div}(\phi(|\nabla u|)\nabla u)$ . Note that the potential  $V(t) = \Phi(|t^2-1|)$  satisfies ( $V_1$ ) – ( $V_6$ ) and so ( $E_3$ ) reduces to ( $E_1$ ) in the case  $\Phi(t) = |t|^2$  and  $V(t) = (t^2-1)^2$ .

In [14], we refined and adapted the variational procedure introduced in [9] to show that, like in the Laplacian case, ( $E_3$ ) admits transition heteroclinic type solutions and, for each integer number  $k \geq 2$ , a related saddle-type solution with dihedral symmetries of order  $k$ .

In recent years, facing the need of a mathematical description of advanced physical problems there has been a growing number of works involving the  $\Phi$ -laplacian operator  $\Delta_{\Phi}$  and its theory is by now rather developed. As a first example we may consider the case

$$\Phi(t) = |t|^p, \quad t \in \mathbb{R}, \quad p \in (1, +\infty),$$

which is related to the celebrated  $p$ -Laplacian operator that often appears in physical models, for example in Newtonian and non-Newtonian fluids (see [26, 27] and references therein). Motivated by concrete examples of equations arising in fluid mechanics and plasticity theory, Seregin and Fuchs in [34, 35] (see also [33]) were led to the minimization of integrals where appears the logarithmic model

$$\Phi(t) = |t|^p \ln(1+|t|), \quad t \in \mathbb{R}, \quad p \in [1, +\infty),$$

which is an  $N$ -function of the type (1.1). Other model of  $N$ -function of the form (1.1) that often arises in a lot of fields of physics and related sciences such as biophysics and chemical reaction design is

$$\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q, \quad t \in \mathbb{R}, \quad 1 < p < q < +\infty.$$

The differential operator associated with this  $N$ -function is known as the  $(p, q)$ -Laplacian operator and the prototype for these models can be written in the form

$$u_t = -\Delta_{\Phi} u + f(x, u).$$

In this configuration, the function  $u$  generally describes a concentration,  $\Delta_{\Phi}$  corresponds to the diffusion and  $f(x, u)$  is the reaction term that corresponds to source and loss processes. For a quite comprehensive account, the interested reader might start by referring to [16, 32]. Finally, it is worth mentioning that the  $N$ -function of the form (1.1)

$$\Phi(t) = (1 + t^2)^{\gamma} - 1, \quad t \in \mathbb{R}, \quad \gamma > 1,$$

appears in the works [38, 39], where the authors report that studies of quasilinear equations involving the associated operator  $\Delta_{\Phi}$  are motivated by nonlinear elasticity models. For other examples of  $N$ -functions of the type (1.1) and more applications we refer the reader to [33, 36] and the bibliography therein.

In the present paper we continue the study initiated in [14] studying the existence of heteroclinic and related saddle-type weak solutions of the non autonomous version of equation ( $E_3$ )

$$-\Delta_{\Phi} u + A(x, y)V'(u) = 0 \quad \text{in } \mathbb{R}^2, \quad (PDE)$$

where  $A$  is a symmetric positive periodic function satisfying ( $A_1$ ) – ( $A_4$ ).

As a first step in the present study we use variational methods related to the ones introduced in [9] and [14], to establish the existence of (*minimal*) *heteroclinic type solutions* of (PDE), i.e. weak solutions  $v \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$  which are 1 - periodic in the variable  $y$  and such that

$$v(x, y) \rightarrow -1 \text{ as } x \rightarrow -\infty \text{ and } v(x, y) \rightarrow 1 \text{ as } x \rightarrow +\infty, \text{ uniformly in } y \in \mathbb{R}.$$

Here we borrow some ideas developed in [9] and [47] to look for minima of the action functional

$$I(u) = \int_{\mathbb{R}} \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u)) \, dydx,$$

on the class

$$E = \left\{ u \in W_{loc}^{1,\Phi}(\mathbb{R} \times [0, 1]) : 0 \leq u(x, y) \leq 1 \text{ for } x > 0 \text{ and } u \text{ is odd in } x \right\},$$

where  $W_{loc}^{1,\Phi}(\mathbb{R} \times [0, 1])$  denotes the usual Orlicz-Sobolev space. Denoting by  $K$  the set of minima of  $I$  on  $E$ , we show that  $K$  is not empty and constituted by (minimal) heteroclinic type solutions of (PDE).

The minimality properties of these heteroclinic type solutions allows us, as a second step, to build up a variational framework inspired to the one introduced in [9] to detect the existence of saddle type solution of (PDE), characterizing their the asymptotic behavior.

More precisely we have the following results:

**Theorem 1.1.** *Assume  $(\phi_1)$ - $(\phi_2)$ ,  $V \in C^1(\mathbb{R}, \mathbb{R})$ ,  $(V_1)$ - $(V_3)$  and  $(A_1)$ - $(A_3)$ . There exists  $v \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$ , a weak solution of (PDE) that verifies the following:*

- (a)  $v(x, y) = -v(-x, y)$ , for all  $(x, y) \in \mathbb{R}^2$ ;
- (b)  $v(x, y) = v(x, y + 1)$ , for any  $(x, y) \in \mathbb{R}^2$ ;
- (c)  $0 < v(x, y) < 1$  for each  $x > 0$  and  $y \in \mathbb{R}$ .

Moreover,  $v$  is a heteroclinic solution from  $-1$  to  $1$ .

**Theorem 1.2.** *Assume  $(\phi_1)$ - $(\phi_4)$ ,  $V \in C^1(\mathbb{R}, \mathbb{R})$ ,  $(V_1)$ - $(V_4)$  and  $(A_1)$ - $(A_4)$ . There exists  $v \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$ , a weak solution of (PDE) that verifies the following:*

- (a)  $v(x, y) > 0$  on the first quadrant in  $\mathbb{R}^2$ ;
- (b)  $v(x, y) = -v(-x, y) = -v(x, -y)$  for all  $(x, y) \in \mathbb{R}^2$ ;
- (c)  $v(x, y) = v(y, x)$  for any  $(x, y) \in \mathbb{R}^2$ ;
- (d) There is  $u_0 \in K$  such that  $\|v - \tau_j u_0\|_{L^\infty(\mathbb{R} \times [j, j+1])} \rightarrow 0$  as  $j \rightarrow +\infty$ ,

where  $\tau_j u_0(x, y) = u_0(x, y - j)$  for all  $(x, y) \in \mathbb{R}^2$ .

The item (d) of Theorem 1.2 characterizes the asymptotic behavior of  $v$ . It guarantees that along directions parallel to the coordinate axes the saddle solution is asymptotic to (rotated of) the minimal heteroclinic set  $K$ . This implies that along any direction not parallel to the coordinate axes  $v$  is asymptotic at infinity to  $\pm 1$  and so, the saddle solution can be seen as a phase transition solution with cross interface.

We point out that Theorems 1.1, 1.2, improve the results in [14] not only in the fact that the function  $A$  is allowed to be not constant but also because, unlike in [14], the assumptions  $(\phi_5)$  and  $(V_5)$ - $(V_6)$  are not needed. Moreover we note that even though the variational approach is inspired by the one used in [9], many tools used in the classical Laplacian context, such as for example some maximum principles,  $C^2$  regularity, existence and local uniqueness theorems, are no more available in the present framework. The proofs of our results require new estimates based on the Harnack type inequalities found in [55] and on results about  $C^{1,\alpha}$  regularity for quasilinear problems as obtained by Liberman in [45].

This paper is organized as follows. In Section 2, we prove Theorem 1.1, while in Section 3 we show some compactness properties. We build up in Section 5 a renormalized minimization procedure inspired by the one used in [9, 10] (see also [8]) that takes into account refined properties studied in Sections 3 and 4, and then the proof of Theorem 1.2 is given. Finally, we write an Appendix A about some facts involving Orlicz–Sobolev spaces for unfamiliar readers with the topic.

## 2. EXISTENCE OF HETEROCLINIC SOLUTIONS

In this section, we show the existence of a heteroclinic solution from  $-1$  to  $1$  for the quasilinear problem (PDE). To begin with, for  $\Omega_0 = \mathbb{R} \times [0, 1]$  let us consider the set

$$E = \{u \in W_{loc}^{1,\Phi}(\Omega_0) : 0 \leq u(x, y) \leq 1 \text{ for } x > 0 \text{ and } u \text{ is odd in } x\}.$$

In the sequel,  $I : W_{loc}^{1,\Phi}(\Omega_0) \rightarrow \mathbb{R} \cup \{+\infty\}$  designates the functional given by

$$I(u) = \int_{\Omega_0} (\Phi(|\nabla u|) + A(x, y)V(u)) \, dydx.$$

An direct computation shows that

$$(2.1) \quad u_n \rightharpoonup u \text{ in } W_{loc}^{1,\Phi}(\Omega_0) \Rightarrow I(u) \leq \liminf_{n \rightarrow +\infty} I(u_n).$$

Hereinafter, the expression  $u_n \rightharpoonup u$  in  $W_{loc}^{1,\Phi}(\Omega_0)$  means that  $u_n \rightharpoonup u$  in  $W^{1,\Phi}([L, R] \times [0, 1])$  for every  $R, L \in \mathbb{R}$  with  $L < R$ . Setting

$$\mathcal{L}(u) = \Phi(|\nabla u|) + A(x, y)V(u), \quad u \in W_{loc}^{1,\Phi}(\Omega_0),$$

it follows from the definitions of  $\Phi$ ,  $V$  and  $A$  that

$$\mathcal{L}(u) \geq 0, \quad \forall u \in E,$$

and so, the functional  $I$  is bounded from below. Now, it is easy to check that the function  $\varphi_* : \Omega_0 \rightarrow \mathbb{R}$  defined by

$$(2.2) \quad \varphi_*(x, y) = \begin{cases} 1, & \text{if } x > 1 & \text{and } y \in [0, 1], \\ x, & \text{if } -1 \leq x \leq 1 & \text{and } y \in [0, 1], \\ -1, & \text{if } x < -1 & \text{and } y \in [0, 1] \end{cases}$$

belongs to  $E$  with  $I(\varphi_*) < +\infty$ . Therefore, the real number

$$c := \inf_{u \in E} I(u)$$

is well defined.

From now on, for each  $x \in \mathbb{R}$  fixed and  $u \in E$ , we will identify  $u(x, \cdot)$  as being a real function in  $y \in [0, 1]$ . For each  $y \in [0, 1]$  fixed, we will also identify  $u(\cdot, y)$  as being a real function in  $x \in \mathbb{R}$ . Employing Fubini's Theorem, it follows that

$$u(x, \cdot) \in W^{1, \Phi}(0, 1) \text{ a.e. in } x \in \mathbb{R} \text{ and } u(\cdot, y) \in W_{\text{loc}}^{1, \Phi}(\mathbb{R}) \text{ a.e. in } y \in [0, 1].$$

Finally, since the functions in  $E$  have  $L^\infty$ -norm less than or equal to 1, without loss of generality, we can make a modification on function  $V$ , by assuming that it satisfies the following:

$$(2.3) \quad V(t) = V(2), \quad \text{for } |t| \geq 2.$$

Hereafter, we will denote this new modification of  $V$  by itself. Moreover, according to  $(A_1)$ - $(A_4)$ ,

$$0 < \min_{\mathbb{R}^2} A(x, y) \leq A(x, y) \leq \max_{\mathbb{R}^2} A(x, y) < +\infty.$$

In what follows,  $\underline{A} = \min_{\mathbb{R}^2} A(x, y)$  and  $\bar{A} = \max_{\mathbb{R}^2} A(x, y)$ .

Next, we prove an important estimate that will be used often in this paper.

**Lemma 2.1.** *Let  $u \in E$ . If  $x_1, x_2 \in \mathbb{R}$  with  $x_1 < x_2$ , then*

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \leq \frac{\xi_1(|x_1 - x_2|)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(|u_x|) dx dy,$$

where  $\xi_1$  was given in Lemma A.1.

*Proof.* First of all note that from Lemma A.4,  $u \in W_{\text{loc}}^{1, l}(\Omega_0)$ , and hence, by [20, Theorem 8.2],

$$|u(x_2, y) - u(x_1, y)| = \left| \int_{x_1}^{x_2} u_x(x, y) dx \right|.$$

As  $\Phi$  is even,

$$(2.4) \quad \Phi(|u(x_2, y) - u(x_1, y)|) = \Phi\left(\int_{x_1}^{x_2} u_x(x, y) dx\right).$$

Invoking Jensen's Inequality given in [52, Theorem 3.3],

$$(2.5) \quad \Phi\left(\int_{x_1}^{x_2} u_x(x, y) dx\right) \leq \frac{1}{|x_1 - x_2|} \int_{x_1}^{x_2} \Phi((x_2 - x_1)u_x(x, y)) dx,$$

then by (2.4) and (2.5),

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \leq \frac{1}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi((x_2 - x_1)u_x(x, y)) dx dy.$$

According to Lemma A.1,

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \leq \frac{\xi_1(|x_1 - x_2|)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(u_x(x, y)) dx dy,$$



and the lemma follows.  $\square$

As a consequence of the last lemma, we obtain the following result.

**Corollary 2.2.** *If  $u \in E$  and  $I(u) < +\infty$ , then:*

- a) *The function  $x \in \mathbb{R} \mapsto u(x, \cdot) \in L^\Phi(0, 1)$  is uniformly continuous a.e..*
- b) *The function  $x \in \mathbb{R} \mapsto \|u(x, \cdot) - 1\|_{L^\Phi(0,1)}$  is continuous a.e..*

*Proof.* Let be  $x_1, x_2 \in \mathbb{R}$  such that  $x_1 < x_2$ . Since  $\Phi$  is an increasing function in  $(0, +\infty)$  and  $|\partial_x u| \leq |\nabla u|$ , the Lemma 2.1 ensures that

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \leq \frac{\xi_1(|x_1 - x_2|)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(|\nabla u|) dx dy,$$

and so,

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \leq I(u) \max \left\{ |x_1 - x_2|^{l-1}, |x_1 - x_2|^{m-1} \right\}.$$

From this, given  $\epsilon > 0$ , there is  $\delta > 0$  such that

$$\int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy < \epsilon \quad \text{for } |x_1 - x_2| < \delta.$$

The last inequality combined with Lemma A.1 gives

$$\xi_0 \left( \|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^\Phi(0,1)} \right) < \epsilon \quad \text{for } |x_1 - x_2| < \delta.$$

Therefore,

$$|x_1 - x_2| < \delta \Rightarrow \|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^\Phi(0,1)} < \xi_0^{-1}(\epsilon),$$

finishing the proof of a). The item b) follows from a), because we have the inequality below

$$\left| \|u(x_2, \cdot) - 1\|_{L^\Phi(0,1)} - \|u(x_1, \cdot) - 1\|_{L^\Phi(0,1)} \right| \leq \|u(x_2, \cdot) - u(x_1, \cdot)\|_{L^\Phi(0,1)}.$$

This completes the proof.  $\square$

Another important consequence of Lemma 2.1 is the following result.

**Lemma 2.3.** *If  $u \in E$  satisfies*

$$\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq r \quad \text{a.e. in } x \in (x_1, x_2) \subset [0, +\infty),$$

*for some  $r > 0$ , then there exists  $\mu_r > 0$  independent of  $x_1$  and  $x_2$  satisfying*

$$\begin{aligned} \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx &\geq \frac{|x_2 - x_1|}{2\xi_1(|x_2 - x_1|)} \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1| \\ &\geq \mu_r h \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right), \end{aligned}$$

*where  $h(t) = \min \left\{ t^{\frac{1}{l}}, t^{\frac{1}{m}} \right\}$ .*

*Proof.* In what follows, we are going to work with the functional  $F : W^{1,\Phi}(0, 1) \rightarrow \mathbb{R}$  defined by

$$F(v) = \int_0^1 \left( \frac{1}{2} \Phi(|v'|) + \underline{A}V(v) \right) dy.$$

We claim that for any sequence  $(v_n) \subset W^{1,\Phi}(0, 1)$  with  $0 \leq v_n(y) \leq 1$  for all  $y \in (0, 1)$  and  $F(v_n) \rightarrow 0$  as  $n \rightarrow +\infty$ , we must have

$$\|v_n - 1\|_{W^{1,\Phi}(0,1)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$



Indeed, the limit  $F(v_n) \rightarrow 0$  gives

$$\int_0^1 \Phi(|v'_n|) dy \rightarrow 0 \quad \text{and} \quad \int_0^1 V(v_n) dy \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Here we would like point out that by  $(V_1)$  and  $(V_3)$  that there are  $\underline{w}, \bar{w} > 0$  satisfying

$$(2.6) \quad \underline{w}\Phi(|t-1|) \leq V(t) \leq \bar{w}\Phi(|t-1|), \quad \forall t \in [0, 1].$$

In fact, by  $(V_1)$  and the fact that  $\Phi(t) = 0$  if, and only if  $t = 0$ , we have that the function  $\frac{V(t)}{\Phi(|t-1|)}$  is continuous and strictly positive in  $[0, 1 - \delta_1]$ . Hence, there are  $\alpha_1, \alpha_2 > 0$  such that

$$\alpha_1\Phi(|t-1|) \leq V(t) \leq \alpha_2\Phi(|t-1|), \quad \forall t \in [0, 1 - \delta_1].$$

Now (2.6) follows by taking  $\underline{w} = \min\{\alpha_1, w_1\}$  and  $\bar{w} = \max\{\alpha_2, w_2\}$ , where  $w_1$  and  $w_2$  were given in  $(V_3)$ . Thus, since  $0 \leq v_n(y) \leq 1$  for every  $y \in (0, 1)$ , (2.6) ensures that

$$\int_0^1 \Phi(|v_n - 1|) dy \leq \frac{1}{\underline{w}} \int_0^1 V(v_n) dy, \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\int_0^1 \Phi(|v_n - 1|) dy \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

The limits above together with the fact that  $\Phi \in \Delta_2$  yield

$$\|v_n - 1\|_{W^{1,\Phi}(0,1)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which proves the claim. Thereby, if  $v \in W^{1,\Phi}(0, 1)$ ,  $0 \leq v \leq 1$  in  $(0, 1)$  and  $\|v - 1\|_{W^{1,\Phi}(0,1)} \geq r$ , then there exists  $\mu_r \in (0, 1/2)$  such that

$$F(v) \geq (2\mu_r)^{\frac{m}{m-1}}.$$

Now, if  $u \in E$ , we know that  $0 \leq u(x, \cdot) \leq 1$  on  $(0, 1)$  for almost every  $x > 0$ , and so, if  $\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq r$  a.e. in  $(x_1, x_2)$ , we must have

$$F(u(x, \cdot)) \geq (2\mu_r)^{\frac{m}{m-1}} \quad \text{a.e. in } x \in (x_1, x_2),$$

which leads to

$$\begin{aligned} \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx &= \int_{x_1}^{x_2} \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u)) dy dx \\ &\geq \frac{1}{2} \int_{x_1}^{x_2} \int_0^1 \Phi(|\partial_x u|) dy dx + \int_{x_1}^{x_2} \int_0^1 \left( \frac{1}{2} \Phi(|\partial_y u|) + \underline{A}V(u) \right) dy dx \\ &\geq \frac{1}{2} \int_{x_1}^{x_2} \int_0^1 \Phi(|\partial_x u|) dy dx + \int_{x_1}^{x_2} F(u(x, \cdot)) dx \\ &\geq \frac{1}{2} \int_{x_1}^{x_2} \int_0^1 \Phi(|\partial_x u|) dy dx + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1|. \end{aligned}$$

Thanks to Lemma 2.1,

$$\begin{aligned} \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx &\geq \frac{1}{2} \frac{|x_1 - x_2|}{\xi_1(|x_1 - x_2|)} \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1| \\ &\geq \frac{1}{2} \frac{|x_1 - x_2|}{\xi_1(|x_1 - x_2|)} \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy + 2^{\frac{m}{m-1}-1} \mu_r^{\frac{m}{m-1}} |x_2 - x_1|. \end{aligned}$$

Recalling that  $\xi_1(|x_2 - x_1|) = \max\{|x_2 - x_1|^l, |x_2 - x_1|^m\}$ , we will consider the cases  $\xi_1(|x_2 - x_1|) = |x_2 - x_1|^m$  and  $\xi_1(|x_2 - x_1|) = |x_2 - x_1|^l$ . If  $\xi_1(|x_2 - x_1|) = |x_2 - x_1|^m$ ,

$$\begin{aligned} \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx &\geq \frac{1}{2} \frac{1}{|x_1 - x_2|^{m-1}} \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy + 2^{\frac{m}{m-1}-1} \mu_r^{\frac{m}{m-1}} |x_2 - x_1| \\ &\geq \frac{1}{2m} \left[ \frac{1}{|x_1 - x_2|^{\frac{m-1}{m}}} \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{m}} \right]^m + \frac{m-1}{2m} \left( 2\mu_r |x_2 - x_1|^{\frac{m-1}{m}} \right)^{\frac{m}{m-1}}. \end{aligned}$$

Using Young's inequality for the conjugate exponents  $m$  and  $\frac{m}{m-1}$ , we find

$$\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \frac{1}{2} \left[ \frac{1}{|x_1 - x_2|^{\frac{m-1}{m}}} \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{m}} 2\mu_r |x_2 - x_1|^{\frac{m-1}{m}} \right],$$

that is,

$$(2.7) \quad \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \mu_r \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{m}}.$$

If  $\xi_1(|x_1 - x_2|) = |x_1 - x_2|^l$ , a similar argument works to prove that

$$\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \frac{1}{2l} \left[ \frac{1}{|x_1 - x_2|^{\frac{l-1}{l}}} \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{l}} \right]^l + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1|.$$

Now, since  $l \leq m$  and  $0 < 2\mu_r < 1$ , we obtain that  $1 < \frac{m}{m-1} \leq \frac{l}{l-1}$  and  $(2\mu_r)^{\frac{l}{l-1}} \leq (2\mu_r)^{\frac{m}{m-1}}$ . Therefore,

$$\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \frac{1}{2l} \left[ \frac{1}{|x_1 - x_2|^{\frac{l-1}{l}}} \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{l}} \right]^l + (2\mu_r)^{\frac{l}{l-1}} |x_2 - x_1|.$$

Employing again Young's inequality, we derive

$$(2.8) \quad \int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \mu_r \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right)^{\frac{1}{l}}.$$

From (2.7) and (2.8),

$$\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(u) dy dx \geq \mu_r h \left( \int_0^1 \Phi(|u(x_2, y) - u(x_1, y)|) dy \right),$$

where  $h(t) = \min\left\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\right\}$ , which is precisely the assertion of the lemma.  $\square$

The next result characterizes the asymptotic behavior of functions  $u \in E$  with  $I(u) < +\infty$ .

**Lemma 2.4.** *If  $u \in E$  and  $I(u) < +\infty$ , then*

$$\|u(x, \cdot) - 1\|_{L^\Phi(0,1)} \rightarrow 0 \quad \text{as } x \rightarrow +\infty \quad \text{and} \quad \|u(x, \cdot) + 1\|_{L^\Phi(0,1)} \rightarrow 0 \quad \text{as } x \rightarrow -\infty.$$

*Proof.* To begin with, we claim that

$$(2.9) \quad \liminf_{x \rightarrow +\infty} \int_0^1 \Phi(|u(x, y) - 1|) dy = 0.$$

Indeed, if the limit does not hold, then there are  $r > 0$  and  $x_1 > 0$  satisfying

$$\int_0^1 \Phi(|u(x, y) - 1|) dy \geq r, \quad \forall x > x_1.$$

So, the properties of  $\Phi$  together with Lemma A.1 guarantee that

$$\begin{aligned} r &\leq \xi_1 \left( \|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \right) \int_0^1 \Phi \left( \frac{|u(x, y) - 1|}{\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)}} \right) dy \\ &\leq \xi_1 \left( \|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \right) \int_0^1 \Phi \left( \frac{|u(x, y) - 1|}{\|u(x, \cdot) - 1\|_{L^\Phi(0,1)}} \right) dy \\ &\leq \xi_1 \left( \|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \right), \end{aligned}$$

that is,

$$\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq \xi_1^{-1}(r) := r_1 \text{ for all } x > x_1.$$

The last inequality permits to apply Lemma 2.3 to get  $\mu_{r_1} > 0$  satisfying

$$I(u) \geq \int_{x_1}^x \int_0^1 \mathcal{L}(u) dy dx \geq (2\mu_{r_1})^{\frac{m}{m-1}} (x - x_1).$$

Taking the limit of  $x \rightarrow +\infty$  we infer that  $I(u) = +\infty$ , which is absurd, and (2.9) is proved.

As  $\Phi \in \Delta_2$ , the limit in (2.9) is equivalent to

$$(2.10) \quad \liminf_{x \rightarrow +\infty} \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} = 0.$$

Next we are going to show that

$$(2.11) \quad \limsup_{x \rightarrow +\infty} \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} = 0.$$

To see why, assume by contradiction that  $\limsup_{x \rightarrow +\infty} \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} > 0$ . Then, there exists  $r > 0$  such that

$$(2.12) \quad \limsup_{x \rightarrow +\infty} \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} > 2r.$$

By Corollary 2.2, we can assume that the function  $x \in \mathbb{R} \mapsto \|u(x, \cdot) - 1\|_{L^\Phi(0,1)}$  is continuous in  $\mathbb{R}$ . So, according to (2.10) and (2.12), there is a sequence of disjoint intervals  $(\sigma_i, \tau_i)$  with  $0 < \sigma_i < \tau_i < \sigma_{i+1} < \tau_{i+1}$ ,  $i \in \mathbb{N}$ , and  $\sigma_i \rightarrow +\infty$  as  $i \rightarrow +\infty$  such that for each  $i$ ,

$$r \leq \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} \leq 2r \quad \text{for } x \in [\sigma_i, \tau_i]$$

and

$$\|u(\sigma_i, \cdot) - 1\|_{L^\Phi(0,1)} = r \quad \text{and} \quad \|u(\tau_i, \cdot) - 1\|_{L^\Phi(0,1)} = 2r.$$

Due to triangular inequality,

$$(2.13) \quad \|u(\tau_i, \cdot) - u(\sigma_i, \cdot)\|_{L^\Phi(0,1)} \geq r \quad \forall i \in \mathbb{N},$$

from where it follows that there exists  $\epsilon > 0$  such that

$$(2.14) \quad \int_0^1 \Phi(|u(\tau_i, \cdot) - u(\sigma_i, \cdot)|) dy \geq \epsilon, \quad \forall i \in \mathbb{N}.$$

In fact, arguing by contradiction, let us suppose that there is a sequence  $(i_n) \subset \mathbb{N}$  satisfying

$$\int_0^1 \Phi(|u(\tau_{i_n}, \cdot) - u(\sigma_{i_n}, \cdot)|) dy \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since  $\Phi \in \Delta_2$ , the above limit implies that

$$\|u(\tau_{i_n}, \cdot) - u(\sigma_{i_n}, \cdot)\|_{L^\Phi(0,1)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which contradicts (2.13). Consequently, by Lemma 2.3 there exists  $\mu_r > 0$  such that

$$I(u) \geq \sum_{i=1}^{+\infty} \int_{\sigma_i}^{\tau_i} \int_0^1 \mathcal{L}(u) dy dx \geq \sum_{i=1}^{+\infty} \mu_r h \left( \int_0^1 \Phi(|u(\tau_i, \cdot) - u(\sigma_i, \cdot)|) dy \right)$$

that combined with (2.14) provides

$$I(u) \geq \mu_r \sum_{i=1}^{+\infty} h(\epsilon),$$

which is absurd, because  $I(u) < +\infty$ . Now, the lemma follows from (2.10) and (2.11).  $\square$

Our next result is a key point in our approach, because it establishes the existence of heteroclinic solution for a class of problem defined on the strip  $\Omega_0 = \mathbb{R} \times [0, 1]$ , which will be used to prove the existence of heteroclinic solution in whole  $\mathbb{R}^2$ .

**Theorem 2.5.** *There exists  $u \in E$  such that  $I(u) = c$ . Moreover,  $u$  is a weak solution to the quasilinear elliptic problem*

$$\begin{cases} -\Delta_{\Phi} u + A(x, y) V'(u) = 0, & \text{in } \Omega_0 \\ \frac{\partial u}{\partial \eta}(x, y) = 0, & \text{on } \partial\Omega_0. \end{cases} \quad (P)$$

*Proof.* Let  $(u_n) \subset E$  be a minimizing sequence for  $I$ . It is straightforward to check that  $(u_n)$  is bounded in  $W_{\text{loc}}^{1, \Phi}(\Omega_0)$ . Then, by a classical diagonal argument, there are a subsequence of  $(u_n)$ , still denoted by  $(u_n)$ , and  $u \in W_{\text{loc}}^{1, \Phi}(\Omega_0)$  verifying

$$u_n \rightharpoonup u \text{ in } W_{\text{loc}}^{1, \Phi}(\Omega_0) \quad \text{and} \quad u_n(x, y) \rightarrow u(x, y) \text{ a.e. in } \Omega_0.$$

By the pointwise convergence, it is plain that

$$u(x, y) = -u(-x, y) \text{ a.e. in } \Omega_0 \quad \text{and} \quad 0 \leq u(x, y) \leq 1 \text{ for } x \geq 0,$$

from where it follows that  $u \in E$ . Therefore, from (2.1) we may conclude  $I(u) = c$ . To complete the proof, it is sufficient to show that

$$\int_{\Omega_0} (\phi(|\nabla u|) \nabla u \nabla \psi + A(x, y) V'(u) \psi) dy dx \geq 0,$$

for all  $\psi \in X^{1, \Phi}(\Omega_0)$ , where

$$(2.15) \quad X^{1, \Phi}(\Omega_0) = \{w \in W^{1, \Phi}(\Omega_0) \text{ with } w(x, y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\}.$$

Now given  $\psi \in X^{1, \Phi}(\Omega_0)$ , we can write  $\psi(x, y) = \psi_o(x, y) + \psi_e(x, y)$ , where

$$\psi_e(x, y) = \frac{\psi(x, y) + \psi(-x, y)}{2} \quad \text{and} \quad \psi_o(x, y) = \frac{\psi(x, y) - \psi(-x, y)}{2}.$$

Note that  $\psi_o$  is odd in  $x$  and  $\psi_e$  is even in  $x$ . From this, for  $t > 0$  we set

$$\varphi(x, y) = \begin{cases} u(x, y) + t\psi_o(x, y), & \text{if } x \geq 0 \text{ and } u(x, y) + t\psi_o(x, y) \geq 0 \\ -u(x, y) - t\psi_o(x, y), & \text{if } x \geq 0 \text{ and } u(x, y) + t\psi_o(x, y) \leq 0 \\ -\varphi(-x, y) & \text{if } x < 0, \end{cases}$$

from where it follows that  $\varphi$  is odd in the variable  $x$  and  $\varphi(x, y) \geq 0$  if  $x \geq 0$ . Moreover, from (V<sub>2</sub>),  $I(\varphi) = I(u + t\psi_o)$ . Next, putting

$$\tilde{\varphi}(x, y) = \max\{-1, \min\{1, \varphi(x, y)\}\} \quad \text{for } (x, y) \in \Omega_0,$$

a direct computation shows that  $\tilde{\varphi} \in E$  with

$$|\nabla \tilde{\varphi}(x, y)| \leq |\nabla(u + t\psi_o)(x, y)|, \quad \forall (x, y) \in \Omega_0.$$

Furthermore, from  $(V_1)$ - $(V_2)$ ,

$$V(\tilde{\varphi}(x, y)) \leq V((u + t\psi_o)(x, y)), \quad \forall (x, y) \in \Omega_0.$$

Therefore,

$$(2.16) \quad I(u + t\psi_o) = I(\varphi) \geq I(\tilde{\varphi}) \geq c = I(u).$$

On the other hand, according to (A.2),

$$\Phi(|\nabla(u + t\psi)|) - \Phi(|\nabla(u + t\psi_o)|) \geq \phi(|\nabla(u + t\psi_o)|) \nabla(u + t\psi_o) \nabla(t\psi_e),$$

so

$$(2.17) \quad \begin{aligned} & \int_{\Omega_0} (\Phi(|\nabla(u + t\psi)|) - \Phi(|\nabla(u + t\psi_o)|)) dx dy \\ & \geq \int_{\Omega_0} \phi(|\nabla(u + t\psi_o)|) (t \nabla u \nabla \psi_e + t^2 \nabla \psi_o \nabla \psi_e) dx dy. \end{aligned}$$

Since  $I(u) = c$  and  $\psi \in X^{1,\Phi}(\Omega_0)$ , we see that  $I(u + t\psi), I(u + t\psi_o) < +\infty$ , because for  $|x|$  sufficiently large we must have  $u(x, y) + t\psi(x, y) = u(x, y)$  and  $u(x, y) + t\psi_o(x, y) = u(x, y)$ . Thus,

$$\begin{aligned} I(u + t\psi) - I(u + t\psi_o) &= \int_{\Omega_0} (\Phi(|\nabla(u + t\psi)|) - \Phi(|\nabla(u + t\psi_o)|)) dx dy \\ & \quad + \int_{\Omega_0} A(x, y) (V(u + t\psi) - V(u + t\psi_o)) dx dy, \end{aligned}$$

and by (2.17),

$$(2.18) \quad \begin{aligned} I(u + t\psi) - I(u + t\psi_o) &\geq t \int_{\Omega_0} \phi(|\nabla(u + t\psi_o)|) \nabla u \nabla \psi_e dx dy \\ & \quad + t^2 \int_{\Omega_0} \phi(|\nabla(u + t\psi_o)|) \nabla \psi_o \nabla \psi_e dx dy \\ & \quad + \int_{\Omega_0} A(x, y) (V(u + t\psi) - V(u + t\psi_o)) dx dy. \end{aligned}$$

It is easily seen that the functions  $\phi(|\nabla(u + t\psi_o)|) \nabla u \nabla \psi_e$  and  $\phi(|\nabla(u + t\psi_o)|) \nabla \psi_o \nabla \psi_e$  are odd in the variable  $x$ , and so,

$$(2.19) \quad \int_{\Omega_0} \phi(|\nabla(u + t\psi_o)|) \nabla u \nabla \psi_e dx dy = \int_{\Omega_0} \phi(|\nabla(u + t\psi_o)|) \nabla \psi_o \nabla \psi_e dx dy = 0.$$

Substituting (2.19) into (2.18), we infer that

$$I(u + t\psi) - I(u + t\psi_o) \geq \int_{\Omega_0} A(x, y) (V(u + t\psi) - V(u + t\psi_o)) dx dy$$

that combines with (2.16) to give

$$I(u + t\psi) - I(u) \geq \int_{\Omega_0} A(x, y) (V(u + t\psi) - V(u + t\psi_o)) dx dy,$$

and so,

$$\begin{aligned}
 \int_{\Omega_0} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x, y)V'(u)\psi) dx dy &= \lim_{t \rightarrow 0^+} \frac{I(u + t\psi) - I(u)}{t} \\
 &\geq \lim_{t \rightarrow 0^+} \int_{\Omega_0} A(x, y) \frac{V(u + t\psi) - V(u + t\psi_o)}{t} dx dy \\
 (2.20) \quad &\geq \lim_{t \rightarrow 0^+} \int_{\Omega_0} A(x, y) \left( \frac{V(u + t\psi) - V(u)}{t} - \frac{V(u + t\psi_o) - V(u)}{t} \right) dx dy \\
 &\geq \int_{\Omega_0} A(x, y)V'(u)(\psi - \psi_o) dx dy = \int_{\Omega_0} A(x, y)V'(u)\psi_e dx dy.
 \end{aligned}$$

Since the function  $A(x, y)V'(u)\psi_e$  is odd in  $x$ , it follows that

$$(2.21) \quad \int_{\Omega_0} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x, y)V'(u)\psi) dx dy \geq 0,$$

which completes the proof.  $\square$

In what follows, let us consider

$$K = \{u \in E : I(u) = c\}.$$

Invoking Theorem 2.5,  $K \neq \emptyset$  and it consists of critical points of  $I$ . In the sequel, for each  $u \in K$ , we will show that there is a function  $v \in K$  depending on  $u$  such that

$$v(x, 0) = v(x, 1) \text{ for any } x \in \mathbb{R}.$$

To prove this, we define

$$E_p = \{w \in E : w(x, 0) = w(x, 1) \text{ a.e. in } x \in \mathbb{R}\}$$

and

$$c_p = \inf_{w \in E_p} I(w).$$

The next lemma establishes an important relation between  $c$  and  $c_p$ .

**Lemma 2.6.** *It holds that  $c_p = c$ . Moreover, given  $u \in K$  there exists  $v \in K$ , depending on  $u$ , such that  $v(x, 0) = v(x, 1)$  for all  $x \in \mathbb{R}$ .*

*Proof.* Since  $E_p \subset E$ ,  $c \leq c_p$ . Now we are going to prove that  $c_p \leq c$ . To see this, given  $w \in E$ , we write  $I(w) = J_1(w) + J_2(w)$ , where

$$J_1(w) = \int_{\mathbb{R}} \int_0^{1/2} \mathcal{L}(w) dy dx \quad \text{and} \quad J_2(w) = \int_{\mathbb{R}} \int_{1/2}^1 \mathcal{L}(w) dy dx.$$

Let  $u \in K$ . So, if  $J_1(u) \leq J_2(u)$ , we consider the function

$$v(x, y) = \begin{cases} u(x, y), & \text{if } 0 \leq y \leq \frac{1}{2}, \\ u(x, 1 - y), & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$$

that belongs to  $E_p$ . From  $(A_2)$ - $(A_3)$ ,  $J_2(v) = J_1(v) = J_1(u)$ , and hence,

$$I(v) = J_1(v) + J_2(v) = 2J_1(u) \leq J_1(u) + J_2(u) = I(u),$$

showing that  $c_p \leq c$ . For that reason,  $c_p = c$  and  $I(v) = c$  with  $v(x, 0) = v(x, 1)$  for every  $x \in \mathbb{R}$ . On the other hand, if  $J_2(u) \leq J_1(u)$ , we consider

$$\tilde{v}(x, y) = \begin{cases} u(x, 1 - y), & \text{if } 0 \leq y \leq \frac{1}{2} \\ u(x, y), & \text{if } \frac{1}{2} \leq y \leq 1. \end{cases}$$

By a similar argument,  $\tilde{v} \in E_p$  and  $J_1(\tilde{v}) = J_2(\tilde{v}) = J_2(u)$ , from where it follows that  $c_p = c$ , proving the desired result.  $\square$

The Lemma 2.6 shows that the set

$$K_p = \{w \in K : w(x, 0) = w(x, 1) \text{ for all } x \in \mathbb{R}\}$$

is non empty. We would like point out that if  $w \in K_p$ , then it can extend periodicity on  $\mathbb{R}^2$  with period 1. Hereafter, the elements of  $K_p$  will be considered extended in whole  $\mathbb{R}^2$ .

Now, we are ready to prove our main theorem of this section.

**Proof of Theorem 1.1.**

Let  $v \in K_p$ . Then *i)* and *ii)* are immediate. According to the proof of Theorem 2.5,

$$\int_{\Omega_0} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0 \quad \forall \psi \in X^{1, \Phi}(\Omega_0).$$

In the sequel, we fix  $\Omega_1 = \mathbb{R} \times [1, 2]$ ,

$$E_1 = \left\{ w \in W_{\text{loc}}^{1, \Phi}(\Omega_1) : w(x, y) = -w(-x, y), \quad x \in \mathbb{R}, \text{ and } 0 \leq w(x, y) \leq 1 \text{ for } x > 0 \right\},$$

the functional  $I^1 : W_{\text{loc}}^{1, \Phi}(\Omega_1) \rightarrow \mathbb{R} \cup \{+\infty\}$  given by

$$I^1(w) = \int_{\Omega_1} \mathcal{L}(w) dy dx,$$

and the real number  $c^1 = \inf_{w \in E_1} I^1(w)$ . It is easily seen that  $c = c^1$  and

$$\int_{\Omega_1} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0,$$

for each  $\psi \in X^{1, \Phi}(\Omega_1)$ , where

$$(2.22) \quad X^{1, \Phi}(\Omega_1) = \{u \in W^{1, \Phi}(\Omega_1) \text{ with } u(x, y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\}.$$

From this, a straightforward computation ensures that

$$\int_{\mathbb{R} \times [0, 2]} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0,$$

for any  $\psi \in X^{1, \Phi}(\mathbb{R} \times [0, 2])$ , where

$$(2.23) \quad X^{1, \Phi}(\mathbb{R} \times [0, 2]) = \{u \in W^{1, \Phi}(\mathbb{R} \times [0, 2]) \text{ with } u(x, y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\}.$$

A similar argument works to prove that

$$\int_{\mathbb{R} \times [l, k]} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0,$$

for all  $l, k \in \mathbb{Z}$  with  $l < k$  e for any  $\psi \in X^{1, \Phi}(\mathbb{R} \times [l, k])$  where

$$(2.24) \quad X^{1, \Phi}(\mathbb{R} \times [l, k]) = \{u \in W^{1, \Phi}(\mathbb{R} \times [l, k]) \text{ with } u(x, y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\}.$$

So, since  $k$  and  $l$  are arbitrary, we get

$$\int_{\mathbb{R}^2} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0,$$

for any  $\psi \in W^{1, \Phi}(\mathbb{R}^2)$  with compact support in  $\mathbb{R}^2$ . By [45, Theorem 1.7] there exist  $\alpha > 0$  and  $M > 0$  such that  $v \in C_{\text{loc}}^{1, \alpha}(\mathbb{R}^2, \mathbb{R})$  with  $\|v\|_{C_{\text{loc}}^{1, \alpha}(\mathbb{R}^2)} \leq M$ . Next, we will show now that  $v$  is a heteroclinic solution from -1 to 1. To do this, given  $n \in \mathbb{N}$ , we set

$$v_n(x, y) = v(x + n, y), \quad \forall (x, y) \in [0, 1] \times [0, 1].$$



Thereby,  $(v_n)$  is bounded in  $C^{1,\alpha}([0,1] \times [0,1])$ , and so there exists  $v_0 \in C^1([0,1] \times [0,1])$  and a subsequence  $(v_{n_j})$  of  $(v_n)$  such that  $v_{n_j} \rightarrow v_0$  in  $C^1([0,1] \times [0,1])$ . In particular, for  $x \in [0,1]$  fixed,  $v_{n_j}(x, \cdot) \rightarrow v_0(x, \cdot)$  as  $j \rightarrow +\infty$  uniformly in  $y \in [0,1]$ . According to Lemma 2.4,  $v_{n_j}(x, \cdot) \rightarrow 1$  in  $L^\Phi(0,1)$  as  $j \rightarrow +\infty$ . Passing to a subsequence if necessary,  $v_{n_j}(x, y) \rightarrow 1$  for almost every  $y \in [0,1]$ , and hence,  $v_0(x, y) = 1$  in  $[0,1] \times [0,1]$ . Thus,  $v_{n_j}(x, y) \rightarrow 1$  as  $j \rightarrow +\infty$  uniformly in  $y \in [0,1]$ , and consequently,  $v(x, y) \rightarrow 1$  as  $x \rightarrow +\infty$  uniformly in  $y \in [0,1]$ . Since  $v$  is 1-periodic in the variable  $y$  and odd in the variable  $x$ , we conclude

$$v(x, y) \rightarrow -1 \text{ as } x \rightarrow -\infty \text{ and } v(x, y) \rightarrow 1 \text{ as } x \rightarrow +\infty, \text{ uniformly in } y \in \mathbb{R}.$$

Finally, adapting the same arguments explored in reference [14, Lemma 3.9], we conclude that  $0 < v(x, y) < 1$  for all  $x > 0$  and  $y \in \mathbb{R}$ , and the proof is complete.  $\square$

If  $u \in K$ , then we can extend  $u$  by periodicity on  $\mathbb{R}^2$  with period 2 in  $y$  satisfying the equation (PDE). Indeed, defining the function

$$\tilde{u}(x, y) = \begin{cases} u(x, y), & \text{if } (x, y) \in \mathbb{R} \times [0, 1], \\ u(x, 2 - y), & \text{if } (x, y) \in \mathbb{R} \times [1, 2], \end{cases}$$

we have that

$$\tilde{u}(x, 0) = \tilde{u}(x, 2) \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \eta}(x, 0) = 0 = \frac{\partial \tilde{u}}{\partial \eta}(x, 2).$$

Now, we extend  $\tilde{u}$  by periodicity to whole  $\mathbb{R}^2$  by setting  $\bar{u} : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $\bar{u} = \tilde{u}$  in  $\mathbb{R} \times [0, 2]$  and  $\bar{u}(x, y) = \tilde{u}(x, y - 2k)$ , where  $y \in \mathbb{R}$  and  $k \in \mathbb{Z}$  is the only integer such that  $0 \leq y - 2k < 2$ . From now on, without loss of generality, we can assume that  $u \in K$  is a periodic function with period 2 in the variable  $y$ .

Arguing as in the proof of Theorem 1.1, we have the following result.

**Theorem 2.7.** *Assume  $(\phi_1)$ - $(\phi_2)$ ,  $(V_1)$ - $(V_3)$  and  $(A_1)$ - $(A_3)$ . If  $u \in K$ , then  $u$  is a weak solution of (PDE) in  $C_{loc}^{1,\alpha}(\mathbb{R}^2, \mathbb{R})$ , for some positive  $\alpha$ , that verifies the following:*

- i)  $u(x, y) = -u(-x, y)$ , for all  $(x, y) \in \mathbb{R}^2$ ,
- ii)  $u(x, y) = u(x, y + 2)$ , for each  $(x, y) \in \mathbb{R}^2$ ,
- iii)  $0 < u(x, y) < 1$  for any  $x > 0$  and  $y \in \mathbb{R}$ .

Moreover,  $u$  is a heteroclinic solution from -1 to 1, i.e.

$$u(x, y) \rightarrow -1 \text{ as } x \rightarrow -\infty \text{ and } u(x, y) \rightarrow 1 \text{ as } x \rightarrow +\infty, \text{ uniformly in } y \in \mathbb{R}.$$

**Remark 2.1.** *If  $\Phi(t) = \frac{|t|^2}{2}$ , the operator  $\Delta_\Phi$  is the Laplacian operator, and in this case, using a local unique theorem for elliptic equations it is possible to prove that Theorems 1.1 and 2.7 are essentially the same, because every 2-periodic solution of (PDE) is exactly 1-periodic solution, for more details see [9, Lemma 2.4] or [47, Proposition 2.18]. Here, since we are working with a large class of operator we were not able to prove that these theorems are equal.*

**Remark 2.2.** *Here we would like to point out that Theorems 1.1 and 2.7 are valid for the  $p$ -Laplacian operator with  $1 < p < +\infty$ .*

### 3. COMPACTNESS PROPERTIES OF I

In this section, for our purposes, we need to better characterize the compactness properties of  $I$ . For this to happen, given  $L \in (0, +\infty]$  we set  $\Omega_{0,L} = (-L, L) \times [0, 1]$  and

$$I_{0,L}(w) = \int \int_{\Omega_{0,L}} \mathcal{L}(w) dy dx \text{ for } w \in W^{1,\Phi}(\Omega_{0,L}).$$

Note that  $\Omega_{0,+\infty} = \Omega_0$ ,  $I_{0,+\infty} = I$  and  $I_{0,L}$  is also well defined on  $E$  being weakly lower semicontinuous with respect to the  $W^{1,\Phi}(\Omega_{0,L})$  topology. Moreover, given  $u \in E$ , we can identify  $u|_{\Omega_{0,L}}$  with  $u$  itself, and so if  $0 < L_1 < L_2$ , we have

$$I_{0,L_1}(u) \leq I_{0,L_2}(u) \leq I(u), \quad \forall u \in E.$$

From now on, given  $\delta \in (0, 1)$ , we set

$$(3.1) \quad \lambda_\delta = 2^{m+1}\delta^l + \bar{A} \max_{|s-1| \leq \Lambda\delta} V(s) \quad \text{and} \quad l_\delta = \frac{c+1}{(2\mu_\delta)^{\frac{m}{m-1}}},$$

where  $\Lambda > 0$  and  $\mu_\delta > 0$  were given in (A.1) and Lemma 2.3 respectively.

The next lemma is crucial to prove a compactness result involving the functional  $I$ , see Lemma 3.6 for more details.

**Lemma 3.1.** *There exists  $\delta_0 \in (0, \frac{\delta_1}{2})$  such that, for any  $\delta \in (0, \delta_0)$ , if  $u \in E$ ,  $L \in (l_\delta + 1, +\infty]$  and  $I_{0,L}(u) \leq c + \lambda_\delta$ , then the following hold:*

(i) *There exists  $x_+ \in (0, l_\delta)$  verifying*

$$\|u(x_+, \cdot) - 1\|_{W^{1,\Phi}(0,1)} < \delta.$$

(ii) *For  $x_+$  given in (i) we have*

$$\int_{x_+}^L \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u)) dy dx \leq \frac{3}{2}\lambda_\delta.$$

(iii) *For each  $x \in (x_+, L)$ ,*

$$\|u(x, \cdot) - 1\|_{L^\Phi(0,1)} \leq \delta_1.$$

*Proof.* First note that  $\lambda_\delta \rightarrow 0$  as  $\delta \rightarrow 0$ . Then we can fix  $\delta_0 \in (0, \delta_1/2)$  satisfying

$$(3.2) \quad \lambda_\delta < \min \left\{ 1, \frac{2}{3}\mu_{\frac{\delta_1}{2}} \left( \frac{\delta_1}{2} \right)^{\frac{m}{l}} \right\}, \quad \forall \delta \in (0, \delta_0),$$

where  $\delta_1 > 0$  was defined in (V<sub>3</sub>) and  $\mu_{\frac{\delta_1}{2}}$  given in Lemma 2.3 in correspondence to  $r = \frac{\delta_1}{2}$ . Let  $u \in E$ ,  $L \in (l_\delta + 1, +\infty]$  and  $\delta \in (0, \delta_0)$  with  $I_{0,L}(u) \leq c + \lambda_\delta$ . Assuming that (i) is false, we deduce

$$\|u(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq \delta, \quad \forall x \in (0, l_\delta).$$

According to Lemma 2.3, there exists  $\mu_\delta > 0$  such that

$$I_{0,L}(u) \geq \int_0^{l_\delta} \int_0^1 \mathcal{L}(u) dy dx \geq (2\mu_\delta)^{\frac{m}{m-1}} l_\delta = c + 1 > c + \lambda_\delta,$$

which is a contradiction. Therefore, there is  $x_+ \in (0, l_\delta)$  checking item (i).

To prove (ii), let us consider

$$\tilde{u}(x, y) = \begin{cases} u(x, y), & \text{if } 0 \leq x \leq x_+ & \text{and } y \in [0, 1], \\ (x - x_+) + (x_+ + 1 - x)u(x_+, y), & \text{if } x_+ \leq x \leq x_+ + 1 & \text{and } y \in [0, 1], \\ 1, & \text{if } x_+ + 1 \leq x & \text{and } y \in [0, 1], \\ -\tilde{u}(-x, y), & \text{if } x < 0 & \text{and } y \in [0, 1]. \end{cases}$$

Thereby,  $\tilde{u} \in E$  and  $c \leq I(\tilde{u}) = I_{0, x_+ + 1}(\tilde{u})$ . Moreover,

$$\partial_x \tilde{u}(x, y) = 1 - u(x_+, y) \quad \text{and} \quad \partial_y \tilde{u}(x, y) = (x_+ + 1 - x)\partial_y u(x_+, y) \quad \text{in } (x_+, x_+ + 1) \times [0, 1].$$

Using Lemma A.1 and the fact that  $\Phi$  is increasing on  $(0, +\infty)$ , it is possible to show that

$$\Phi(|\nabla \tilde{u}|) \leq 2^m \Phi(|1 - u(x_+, y)|) + 2^m \Phi(|\partial_y u(x_+, y)|) \quad \text{in } (x_+, x_+ + 1) \times [0, 1],$$

from where it follows that

$$(3.3) \quad \int_{x_+}^{x_++1} \int_0^1 \mathcal{L}(\tilde{u}) dy dx \leq 2^m \int_{x_+}^{x_++1} \int_0^1 (\Phi(|1 - u(x_+, y)|) + \Phi(|\partial_y u(x_+, y)|)) dy dx \\ + \int_{x_+}^{x_++1} \int_0^1 A(x, y) V(\tilde{u}) dy dx.$$

Applying again Lemma A.1,

$$(3.4) \quad \int_0^1 \Phi(|u(x_+, y) - 1|) dy = \int_0^1 \Phi \left( \frac{|u(x_+, y) - 1|}{\|u(x_+, \cdot) - 1\|_{L^\Phi(0,1)}} \|u(x_+, \cdot) - 1\|_{L^\Phi(0,1)} \right) dy \\ \leq \xi_1 \left( \|u(x_+, \cdot) - 1\|_{L^\Phi(0,1)} \right) \leq \xi_1(\delta) = \delta^l.$$

A similar argument works to prove that

$$(3.5) \quad \int_0^1 \Phi(|\partial_y u(x_+, y)|) dy \leq \delta^l.$$

Gathering (3.3) with (3.4) and (3.5), we obtain

$$\int_{x_+}^{x_++1} \int_0^1 \mathcal{L}(\tilde{u}) dy dx \leq 2^{m+1} \delta^l + \bar{A} \int_{x_+}^{x_++1} \int_0^1 V(\tilde{u}) dy dx.$$

By item (i) and (A.1),

$$\|\tilde{u}(x, \cdot) - 1\|_{L^\infty(0,1)} \leq \Lambda \delta \quad \forall x \in (x_+, x_+ + 1),$$

and hence

$$(3.6) \quad \int_{x_+}^{x_++1} \int_0^1 \mathcal{L}(\tilde{u}) dy dx \leq 2^{m+1} \delta^l + \bar{A} \max_{|s-1| \leq \Lambda \delta} V(s) = \lambda_\delta.$$

Now, since

$$I_{0,L}(\tilde{u}) = I_{0,x_+}(u) + 2 \int_{x_+}^L \int_0^1 \mathcal{L}(\tilde{u}) dy dx = I_{0,L}(u) + 2 \int_{x_+}^L \int_0^1 \mathcal{L}(\tilde{u}) dy dx - 2 \int_{x_+}^L \int_0^1 \mathcal{L}(u) dy dx,$$

and  $c \leq I_{0,L}(\tilde{u})$  follows from (3.6) that

$$\int_{x_+}^L \int_0^1 \mathcal{L}(u) dy dx \leq \frac{3}{2} \lambda_\delta,$$

which proves (ii).

Finally, if (iii) does not hold, we should find  $\theta \in (x_+, L)$  satisfying

$$\|u(\theta, \cdot) - 1\|_{L^\Phi(0,1)} > \delta_1.$$

Recalling that by (i),

$$\|u(x_+, \cdot) - 1\|_{L^\Phi(0,1)} < \frac{\delta_1}{2},$$

the Corollary 2.2 together with Intermediate Value Theorem guarantees the existence of  $\sigma \in (x_+, \theta)$  such that

$$\|u(\theta, \cdot) - u(\sigma, \cdot)\|_{L^\Phi(0,1)} \geq \frac{\delta_1}{2} \quad \text{and} \quad \|u(x, \cdot) - 1\|_{L^\Phi(0,1)} \geq \frac{\delta_1}{2}, \quad \forall x \in (\sigma, \theta).$$

Invoking Lemma 2.3,

$$\int_{x_+}^L \int_0^1 \mathcal{L}(u) dy dx \geq \mu_{\frac{\delta_1}{2}} h \left( \int_0^1 \Phi(|u(\theta, y) - u(\sigma, y)|) dy \right).$$

On the other hand, from Lemma A.1,

$$\begin{aligned} \int_0^1 \Phi(|u(\theta, y) - u(\sigma, y)|) dy &\geq \xi_0 \left( \|u(\theta, \cdot) - u(\sigma, \cdot)\|_{L^\Phi(0,1)} \right) \int_0^1 \Phi \left( \frac{|u(\theta, y) - u(\sigma, y)|}{\|u(\theta, \cdot) - u(\sigma, \cdot)\|_{L^\Phi(0,1)}} \right) dy \\ &\geq \xi_0 \left( \|u(\theta, \cdot) - u(\sigma, \cdot)\|_{L^\Phi(0,1)} \right) \geq \xi_0 \left( \frac{\delta_1}{2} \right) = \left( \frac{\delta_1}{2} \right)^m. \end{aligned}$$

Hence, by definition of function  $h$  we get the inequality below

$$\int_{x_+}^L \int_0^1 \mathcal{L}(u) dy dx \geq \mu_{\frac{\delta_1}{2}} \left( \frac{\delta_1}{2} \right)^{\frac{m}{l}}$$

that combines with (ii) to give

$$\mu_{\frac{\delta_1}{2}} \left( \frac{\delta_1}{2} \right)^{\frac{m}{l}} \leq \frac{3}{2} \lambda_\delta,$$

which contradicts (3.2), and the lemma follows.  $\square$

From Lemma 3.1, we obtain in particular the following result.

**Lemma 3.2.** *For all  $\epsilon > 0$  there are  $\bar{\lambda}_\epsilon > 0$  and  $\bar{l}_\epsilon > 0$  such that if  $u \in E$  and  $I(u) \leq c + \bar{\lambda}_\epsilon$ , then  $u - 1 \in W^{1,\Phi}(\bar{l}_\epsilon, +\infty) \times (0, 1)$  and*

$$\int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 (\Phi(|u - 1|) + \Phi(|\nabla u|)) dy dx \leq \epsilon.$$

*Proof.* By definition of  $\lambda_\delta$ , see (3.1), we know that  $\lambda_\delta \rightarrow 0$  as  $\delta \rightarrow 0$ . Thereby, given  $\epsilon > 0$  we can choose  $\delta_0 \in (0, \delta_1/2)$  satisfying

$$\frac{3}{2} \lambda_\delta \leq \frac{\epsilon}{\max\left\{1, \frac{1}{A \underline{w}}\right\}}, \quad \forall \delta \in (0, \delta_0),$$

where  $\underline{w}$  was given in (2.6). Denoting  $\bar{\lambda}_\epsilon = \lambda_\delta$ ,  $\bar{l}_\epsilon = l_\delta$  and  $L = +\infty$ , it follows from Lemma 3.1 that

$$(3.7) \quad \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u)) dy dx \leq \frac{3}{2} \lambda_\delta \leq \frac{\epsilon}{\max\left\{1, \frac{1}{A \underline{w}}\right\}}.$$

According to (2.6),

$$\begin{aligned} (3.8) \quad \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 (\Phi(|u - 1|) + \Phi(|\nabla u|)) dy dx &\leq \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 \Phi(|\nabla u|) dy dx + \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 \frac{1}{\underline{w}} V(u) dy dx \\ &\leq \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 \Phi(|\nabla u|) dy dx + \frac{1}{\underline{w} A} \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 A(x, y)V(u) dy dx \\ &\leq \max\left\{1, \frac{1}{A \underline{w}}\right\} \int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 (\Phi(|\nabla u|) + A(x, y)V(u)) dy dx. \end{aligned}$$

From (3.7) and (3.8),  $u - 1 \in W^{1,\Phi}(\bar{l}_\epsilon, +\infty) \times (0, 1)$  with

$$\int_{\bar{l}_\epsilon}^{+\infty} \int_0^1 (\Phi(|u - 1|) + \Phi(|\nabla u|)) dy dx \leq \epsilon,$$

and this is precisely the assertion of the lemma.  $\square$

In order to continue our analysis, we will fix the following set

$$\tilde{E} = \left\{ w \in W_{\text{loc}}^{1,\Phi}(\Omega_0) : w \text{ is odd in } x \text{ and } w - 1 \in W^{1,\Phi}([0, +\infty) \times [0, 1]) \right\}$$

and the real number

$$\tilde{c} = \inf_{w \in \tilde{E}} I(w).$$

It is very important to point out that  $\tilde{E} \neq \emptyset$ , because the function  $\varphi_*$  given in (2.2) belongs to  $\tilde{E}$ . Moreover, it is easy to check that if  $w \in \tilde{E}$ , then  $w + 1 \in W^{1,\Phi}((-\infty, 0] \times [0, 1])$ , and that if  $w_1, w_2 \in \tilde{E}$ , then  $w_1 - w_2 \in W^{1,\Phi}(\Omega_0)$ . Have this in mind, we are able to define on  $\tilde{E}$  the metric  $\rho : \tilde{E} \times \tilde{E} \rightarrow [0, +\infty)$  given by

$$\rho(w_1, w_2) = \|w_1 - w_2\|_{W^{1,\Phi}(\Omega_0)}.$$

A direct computation guarantees that  $(\tilde{E}, \rho)$  is a complete metric space.

The next lemma shows that the numbers  $c$  and  $\tilde{c}$  are equal.

**Lemma 3.3.** *It holds that  $\tilde{c} = c$ . Moreover, if  $(u_n) \subset E$  and  $I(u_n) \rightarrow c$ , then there exists  $n_0 \in \mathbb{N}$  such that  $u_n \in \tilde{E}$  for any  $n \geq n_0$ . Therefore,  $(u_n)$  is a minimizing sequence for  $I$  on  $\tilde{E}$ .*

*Proof.* Let  $(u_n) \subset E$  be a sequence with  $I(u_n) \rightarrow c$ . Thus, given  $\epsilon > 0$  there is  $n_0 \in \mathbb{N}$  verifying  $I(u_n) \leq c + \epsilon$  for any  $n \geq n_0$ . By Lemma 3.2, there exists  $\bar{l}_\epsilon > 0$  such that  $u_n - 1 \in W^{1,\Phi}(\bar{l}_\epsilon, +\infty) \times [0, 1]$  for all  $n \geq n_0$ . Hence,

$$u_n - 1 \in W^{1,\Phi}([0, +\infty) \times [0, 1]), \quad \forall n \geq n_0.$$

From this,  $(u_n) \subset \tilde{E}$  and

$$\tilde{c} \leq I(u_n) = c + o_n(1), \quad \forall n \geq n_0.$$

Taking the limit of  $n \rightarrow +\infty$ , we get  $\tilde{c} \leq c$ . Now, let us consider  $(v_n) \subset \tilde{E}$  with  $I(v_n) \rightarrow \tilde{c}$  and

$$\bar{v}_n(x, y) = \begin{cases} 1, & \text{if } v_n(x, y) \geq 1 \\ v_n(x, y), & \text{if } -1 \leq v_n(x, y) \leq 1 \\ -1, & \text{if } v_n(x, y) \leq -1. \end{cases}$$

From the properties of  $\Phi$ ,  $V$  and  $\bar{v}_n$ ,  $I(\bar{v}_n) \leq I(v_n)$  for every  $n \in \mathbb{N}$ . Setting

$$\tilde{v}_n(x, y) = \begin{cases} \bar{v}_n(x, y), & \text{if } \bar{v}_n \geq 0 \text{ and } x > 0 \\ -\bar{v}_n(x, y), & \text{if } \bar{v}_n \leq 0 \text{ and } x > 0 \\ -\bar{v}_n(-x, y), & \text{if } x \leq 0, \end{cases}$$

it is easy to see that  $(\tilde{v}_n) \subset E$  and  $I(\tilde{v}_n) = I(\bar{v}_n)$  for each  $n \in \mathbb{N}$ . Therefore,

$$c \leq I(\tilde{v}_n) = I(\bar{v}_n) \leq I(v_n) = \tilde{c} + o_n(1).$$

Taking the limit of  $n \rightarrow +\infty$  we obtain  $c \leq \tilde{c}$ , from where it follows that  $c = \tilde{c}$ . Finally, if  $(u_n) \subset E$  and  $I(u_n) \rightarrow c$ , then we already know that there is  $n_0 \in \mathbb{N}$  such that  $u_n \in \tilde{E}$  for  $n \geq n_0$ , and as  $c = \tilde{c}$ , we deduce that  $(u_n)$  is a minimizing sequence for  $I$  on  $\tilde{E}$ .  $\square$

In the sequel, we say that a sequence  $(u_n)$  is a  $(PS)_d$  sequence for  $I$ , with  $d \in \mathbb{R}$ , if  $(u_n) \subset \tilde{E}$  such that

$$I(u_n) \rightarrow d \quad \text{and} \quad \|I'(u_n)\|_* \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

where

$$\|I'(w)\|_* = \sup \left\{ I'(w)\psi : \psi \in X^{1,\Phi}(\Omega_0) \text{ and } \|\psi\|_{W^{1,\Phi}(\Omega_0)} \leq 1 \right\}.$$

**Lemma 3.4.** *If  $(u_n) \subset E$  and  $I(u_n) \rightarrow c$ , then there exists a sequence  $(w_n) \subset \tilde{E}$  such that  $(w_n)$  is a  $(PS)_c$  sequence for  $I$  and*

$$\|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

*Proof.* Let  $(u_n) \subset E$  with  $I(u_n) \rightarrow c$ . As  $(\tilde{E}, \rho)$  is a complete metric space, we can employ the Ekeland's Variational Principle to find a sequence  $(w_n) \subset \tilde{E}$  satisfying:

- (a)  $I(w_n) \leq I(u_n)$  for any  $n \in \mathbb{N}$ ,
- (b)  $\rho(w_n, u_n) \leq \frac{1}{n}$  for all  $n \in \mathbb{N}$ ,
- (c)  $I(w_n) - I(w) < \frac{1}{n} \|w_n - w\|_{W^{1,\Phi}(\Omega_0)}$  for each  $w \in \tilde{E}$  with  $w \neq w_n$ .

Now, given  $\psi \in X^{1,\Phi}(\Omega_0)$  we can write  $\psi = \psi_o + \psi_e$ , where  $\psi_o$  is odd in the variable  $x$  and  $\psi_e$  is even in  $x$ . It is easily seen that  $w_n + t\psi_o \in \tilde{E}$  for all  $n \in \mathbb{N}$  and  $t > 0$ . From (c),

$$\begin{aligned} I(w_n + t\psi) - I(w_n) &= I(w_n + t\psi) - I(w_n + t\psi_o) + I(w_n + t\psi_o) - I(w_n) \\ &\geq I(w_n + t\psi) - I(w_n + t\psi_o) - \frac{1}{n} \|t\psi_o\|_{W^{1,\Phi}(\Omega_0)}, \end{aligned}$$

or equivalently,

$$\frac{I(w_n + t\psi) - I(w_n)}{t} \geq \frac{I(w_n + t\psi) - I(w_n + t\psi_o)}{t} - \frac{1}{n} \|\psi_o\|_{W^{1,\Phi}(\Omega_0)}.$$

Arguing as in the proof of Theorem 2.5, we find

$$(3.9) \quad I'(w_n)\psi \geq -\frac{1}{n} \|\psi_o\|_{W^{1,\Phi}(\Omega_0)}.$$

Here we would like point out that the same arguments found in [14, Lemma 4.6] work to show that

$$(3.10) \quad \|\psi_o\|_{W^{1,\Phi}(\Omega_0)} \leq \|\psi\|_{W^{1,\Phi}(\Omega_0)}.$$

From (3.9)-(3.10) and replacing  $\psi$  by  $-\psi$ , we get

$$|I'(w_n)\psi| \leq \frac{1}{n} \|\psi\|_{W^{1,\Phi}(\Omega_0)}.$$

Thereby,

$$\|I'(w_n)\|_* \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Finally, from Lemma 3.3 and (a),

$$c = \tilde{c} \leq I(w_n) \leq I(u_n) = c + o_n(1),$$

showing that  $I(w_n) \rightarrow c$ . Therefore,  $(w_n)$  is a  $(PS)_c$  sequence for  $I$ , and the lemma is proved.  $\square$

From now on, we consider  $(u_n) \subset E$  and  $(w_n) \subset \tilde{E}$  as in the last lemma. So,  $(w_n)$  is also bounded in  $W_{\text{loc}}^{1,\Phi}(\Omega_0)$ . Indeed, for each  $L > 0$  the Lemma 3.4 ensures that

$$\|w_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \|w_n - u_n\|_{W^{1,\Phi}(\Omega_{0,L})} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \frac{1}{n} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})}.$$

Since  $(u_n)$  is bounded in  $W_{\text{loc}}^{1,\Phi}(\Omega_0)$ , it follows that  $(w_n)$  also is bounded in  $W_{\text{loc}}^{1,\Phi}(\Omega_0)$ . Then, for some subsequence, there is  $u_0 \in W_{\text{loc}}^{1,\Phi}(\Omega_0)$  verifying

$$(3.11) \quad w_n \rightharpoonup u_0 \quad \text{in } W_{\text{loc}}^{1,\Phi}(\Omega_0),$$

$$(3.12) \quad w_n \rightarrow u_0 \quad \text{in } L_{\text{loc}}^\Phi(\Omega_0),$$

$$(3.13) \quad w_n \rightarrow u_0 \quad \text{in} \quad L^1_{\text{loc}}(\Omega_0)$$

and

$$(3.14) \quad w_n(x, y) \rightarrow u_0(x, y) \quad \text{a.e. in} \quad \Omega_0.$$

**Lemma 3.5.** *There exists a subsequence of  $(w_n)$ , still denoted by itself, such that*

$$\nabla w_n(x, y) \rightarrow \nabla u_0(x, y) \quad \text{a.e. in} \quad \Omega_0.$$

*Proof.* Given  $L > 0$ , let us consider  $\psi \in C_0^\infty(\mathbb{R}^2)$  satisfying

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \quad \text{in} \quad \Omega_{0,L} \quad \text{and} \quad \text{supp}(\psi) \subset \Omega_{0,L+1}.$$

From  $(\phi_1)$ - $(\phi_2)$ , it is possible to show that

$$(3.15) \quad \langle \phi(|z_1|)z_1 - \phi(|z_2|)z_2, z_1 - z_2 \rangle > 0, \quad \forall z_1, z_2 \in \mathbb{R}^2, \quad z_1 \neq z_2.$$

Thereby,

$$(3.16) \quad \begin{aligned} 0 &\leq \int_{\Omega_{0,L}} (\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0) dy dx \\ &\leq \int_{\Omega_{0,L+1}} \psi (\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0) dy dx \\ &\leq \int_{\Omega_{0,L+1}} \psi \phi(|\nabla w_n|)\nabla w_n(\nabla w_n - \nabla u_0) dy dx - \int_{\Omega_{0,L+1}} \psi \phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0) dy dx. \end{aligned}$$

Setting the linear functional  $f : W^{1,\Phi}(\Omega_{0,L+1}) \rightarrow \mathbb{R}$  given by

$$f(v) = \int_{\Omega_{0,L+1}} \psi \phi(|\nabla u_0|)\nabla u_0 \nabla v dy dx,$$

we have that it is continuous, because  $\phi(|\nabla u_0|)\nabla u_0 \in L^{\tilde{\Phi}}(\Omega_{0,L+1})$  via Lemma A.3, and so, by Hölder's inequality

$$\left| \int_{\Omega_{0,L+1}} \psi \phi(|\nabla u_0|)\nabla u_0 \nabla v dy dx \right| \leq 2 \|\phi(|\nabla u_0|)\nabla u_0\|_{L^{\tilde{\Phi}}(\Omega_{0,L+1})} \|v\|_{W^{1,\Phi}(\Omega_{0,L+1})},$$

for all  $v \in W^{1,\Phi}(\Omega_{0,L+1})$ . Therefore, (3.11) asserts that  $f(w_n - u_0) \rightarrow 0$ , or equivalently,

$$(3.17) \quad \int_{\Omega_{0,L+1}} \psi \phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0) dy dx \rightarrow 0.$$

Using again the Lemma A.3 and the boundedness of  $(w_n)$  in  $W^{1,\Phi}_{\text{loc}}(\Omega_0)$ , there is  $C > 0$  such that

$$\int_{\Omega_{0,L+1}} \tilde{\Phi}(\phi(|\nabla w_n|)\nabla w_n) dy dx \leq C, \quad \forall n \in \mathbb{N},$$

implying that  $(\phi(|\nabla w_n|)\nabla w_n)$  is bounded in  $L^{\tilde{\Phi}}(\Omega_{0,L+1})$ . So, by (3.12) and Hölder's inequality,

$$(3.18) \quad \int_{\Omega_{0,L+1}} (w_n - u_0)\phi(|\nabla w_n|)\nabla w_n \nabla \psi dy dx \rightarrow 0.$$

Now, considering the sequence  $(\psi w_n)$  we have that  $(\psi w_n) \subset W^{1,\Phi}(\Omega_0)$ , because  $\psi$  has compact support, and by (3.14), passing to a subsequence if necessary, we can assume that

$$\psi w_n \rightharpoonup \psi u_0 \quad \text{in} \quad W^{1,\Phi}(\Omega_{0,L+1}) \quad \text{and} \quad \psi w_n \rightarrow \psi u_0 \quad \text{a.e.} \quad \Omega_0.$$



Consequently,

$$A(x, y)V'(w_n(x, y))(\psi(x, y)w_n(x, y) - \psi(x, y)u_0(x, y)) \rightarrow 0 \text{ a.e. in } \Omega_{0,L+1}.$$

From (2.3) and (3.13), there exist  $h \in L^1(\Omega_{0,L+1})$  and  $\alpha > 0$  such that, along a subsequence,

$$|A(x, y)V'(w_n)(\psi w_n - \psi u_0)| \leq \alpha \bar{A} |\psi| (h + |u_0|) \in L^1(\Omega_{0,L+1}).$$

Applying the Lebesgue's Dominated Convergence Theorem we obtain

$$(3.19) \quad \int_{\Omega_{0,L+1}} A(x, y)V'(w_n)(\psi w_n - \psi u_0) dy dx \rightarrow 0.$$

Finally, we would like point out that

$$(3.20) \quad I'(w_n)(\psi w_n - \psi u_0) \rightarrow 0.$$

In fact, just note that

$$|I'(w_n)(\psi w_n - \psi u_0)| \leq \|I'(w_n)\|_* \|\psi w_n - \psi u_0\|_{W^{1,\Phi}(\Omega_0)},$$

$(\psi w_n) \subset X^{1,\Phi}(\Omega_0)$  is a bounded sequence in  $W^{1,\Phi}(\Omega_0)$  and  $(w_n)$  is a  $(PS)_c$  sequence for  $I$ . Recalling that

$$\begin{aligned} I'(w_n)(\psi w_n - \psi u_0) &= \int_{\Omega_{0,L+1}} \phi(|\nabla w_n|) \nabla w_n \nabla(\psi w_n - \psi u_0) dy dx \\ &\quad + \int_{\Omega_{0,L+1}} A(x, y)V'(w_n)(\psi w_n - \psi u_0) dy dx, \end{aligned}$$

from where it follows by (3.19) and (3.20) that

$$(3.21) \quad \int_{\Omega_{0,L+1}} \phi(|\nabla w_n|) \nabla w_n \nabla(\psi w_n - \psi u_0) dy dx \rightarrow 0.$$

Since  $\nabla(\psi w_n - \psi u_0) = \psi \nabla w_n + w_n \nabla \psi - \psi \nabla u_0 - u_0 \nabla \psi$ , we also have

$$(3.22) \quad \begin{aligned} \int_{\Omega_{0,L+1}} \psi \phi(|\nabla w_n|) \nabla w_n (\nabla w_n - \nabla u_0) dy dx &= \int_{\Omega_{0,L+1}} \phi(|\nabla w_n|) \nabla w_n \nabla(\psi w_n - \psi u_0) dy dx \\ &\quad - \int_{\Omega_{0,L+1}} (w_n - u_0) \phi(|\nabla w_n|) \nabla w_n \nabla \psi dy dx. \end{aligned}$$

From (3.18), (3.21) and (3.22),

$$(3.23) \quad \int_{\Omega_{0,L+1}} \psi \phi(|\nabla w_n|) \nabla w_n (\nabla w_n - \nabla u_0) dy dx \rightarrow 0.$$

Finally, from (3.17), (3.23) and (3.16),

$$\int_{\Omega_{0,L}} (\phi(|\nabla w_n|) \nabla w_n - \phi(|\nabla u_0|) \nabla u_0) (\nabla w_n - \nabla u_0) dy dx \rightarrow 0.$$

This limit combined with (3.15) leads to, along a subsequence,

$$\langle \phi(|\nabla w_n|) \nabla w_n - \phi(|\nabla u_0|) \nabla u_0, \nabla w_n - \nabla u_0 \rangle \rightarrow 0 \text{ a.e. in } \Omega_{0,L}.$$

Applying a result found in Dal Maso and Murat [23], we infer that

$$\nabla w_n(x, y) \rightarrow \nabla u_0(x, y) \text{ a.e. in } \Omega_{0,L}.$$

As  $L > 0$  is arbitrary, there exists a subsequence of  $(w_n)$ , still denoted by itself, such that

$$\nabla w_n(x, y) \rightarrow \nabla u_0(x, y) \text{ almost everywhere in } \Omega_0,$$

finishing the proof of the lemma.  $\square$

The next lemma establishes the strong convergence for minimizing sequences of  $I$  on  $E$ .

**Lemma 3.6.** *Let  $(u_n) \subset E$  with  $I(u_n) \rightarrow c$ . Then, there exists  $u_0 \in K$  such that, along a subsequence,*

$$\|u_n - u_0\|_{W^{1,\Phi}(\Omega_0)} \rightarrow 0.$$

*Proof.* Invoking Lemma 3.4 there is a sequence  $(w_n) \subset \tilde{E}$  with  $I(w_n) \rightarrow c$  and

$$(3.24) \quad \|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Hence there exists  $u_0 \in W_{\text{loc}}^{1,\Phi}(\Omega_0)$  satisfying (3.11)-(3.14). Moreover,

$$(3.25) \quad \|u_n - u_0\|_{L^\Phi(\Omega_{0,L})} \leq \frac{1}{n} + \|w_n - u_0\|_{L^\Phi(\Omega_{0,L})}, \quad \forall L > 0.$$

Thereby, by (3.12),  $u_0$  is the punctual limit of  $(u_n)$ ,  $u_0 \in E$  and  $I(u_0) = c$ , that is,  $u_0 \in K$ . Now, arguing as in [14, Lemma 4.9],

$$\|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \rightarrow 0.$$

From (3.24),

$$\|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \leq \frac{1}{n} + \|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)},$$

implying that

$$(3.26) \quad \|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \rightarrow 0.$$

Finally, according to Lemma 3.2, given  $\epsilon > 0$ , there are  $l_\epsilon > 0$  and  $n_0 \in \mathbb{N}$  such that

$$\int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_0 - 1|) dy dx \leq \frac{\epsilon}{2^m} \quad \text{and} \quad \int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_n - 1|) dy dx \leq \frac{\epsilon}{2^m}, \quad \forall n \geq n_0.$$

So, it is easy to see that

$$(3.27) \quad \int_{l_\epsilon}^{+\infty} \int_0^1 \Phi(|u_n - u_0|) dy dx \leq 2^{m-1} \int_{l_\epsilon}^{+\infty} \int_0^1 (\Phi(|u_n - 1|) + \Phi(|u_0 - 1|)) dy dx \leq \epsilon, \quad \forall n \geq n_0.$$

As  $\Phi \in \Delta_2$ , (3.25) together with (3.27) gives

$$(3.28) \quad \|u_n - u_0\|_{L^\Phi(\Omega_0)} \rightarrow 0.$$

Now, the lemma follows from (3.26) and (3.28).  $\square$

#### 4. THE APPROXIMATING FUNCTIONALS

In the sequel, given  $j \in \mathbb{N} \cup \{0\}$ , let us define the sets

$$\Omega_j = \mathbb{R} \times [j, j+1] \quad \text{and} \quad T_j = \{(x, y) \in \Omega_j : |x| \leq y\}.$$

Associated with sets above, we consider

$$E_j = \{w \in W^{1,\Phi}(T_j) : 0 \leq w(x, y) \leq 1 \text{ for } x > 0 \text{ and } w \text{ is odd in } x\},$$

and the functional  $I_j : W^{1,\Phi}(T_j) \rightarrow \mathbb{R} \cup \{+\infty\}$  given by

$$I_j(w) = \iint_{T_j} \mathcal{L}(w) dy dx.$$

By a direct computation, we see that  $I_j$  is lower semicontinuous with respect to the weak topology of  $W^{1,\Phi}(T_j)$  and bounded from below. Moreover, since  $I_j(0) < +\infty$  the real number

$$c_j := \inf_{w \in E_j} I_j(w)$$

is well defined. For each  $j \in \mathbb{N} \cup \{0\}$  let us also consider

$$K_j = \{w \in E_j : I_j(w) = c_j\}.$$

Arguing as in the proof of Lemma 2.5, it is possible to prove the following result.

**Lemma 4.1.** *For every  $j \in \mathbb{N} \cup \{0\}$ ,  $K_j \neq \emptyset$ . Moreover, if  $u_j \in K_j$ , then  $u_j$  is a weak solution in  $C^{1,\alpha}(T_j)$ , for some  $\alpha > 0$ , of*

$$-\Delta_{\Phi} u_j + A(x, y)V'(u_j) = 0 \quad \text{in } T_j,$$

with  $0 < u_j(x, y) < 1$  for  $x > 0$ ,

$$\partial_y u_j(x, j) = 0 \quad \text{for } |x| < j \quad \text{and} \quad \partial_y u_j(x, j+1) = 0 \quad \text{for } |x| < j+1.$$

As immediate consequence of the last lemma is the corollary below.

**Corollary 4.2.** *For all  $j \in \mathbb{N} \cup \{0\}$  we have  $c_j \leq c_{j+1} < c$ .*

*Proof.* Invoking Lemma 4.1, for each  $j \geq 0$  there exists  $u_{j+1} \in K_{j+1}$ . Now, considering the function

$$\bar{u}_j(x, y) = u_{j+1}(x, y+1) \quad \text{for } (x, y) \in T_j,$$

we see that  $\bar{u}_j \in E_j$  and

$$c_j \leq I_j(\bar{u}_j) \leq I_{j+1}(u_{j+1}) = c_{j+1}.$$

Finally, from Theorem 1.1, there exists  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $v \in E$  with  $I(v) = c$  and  $v$  is 1-periodic in the variable  $y$ . So,  $v \in E_j$  for any  $j \in \mathbb{N} \cup \{0\}$  and

$$c_j \leq I_j(v) < I(v) = c, \quad \forall j \in \mathbb{N} \cup \{0\},$$

showing the desired result.  $\square$

If  $j > 1$  and  $u_j \in K_j$ , then arguing as in the end of Section 3,  $u_j$  have an extension 2-periodic  $v_j$  in  $(-j, j) \times \mathbb{R}$ , i.e., there exists  $v_j : (-j, j) \times \mathbb{R} \rightarrow \mathbb{R}$  that is 2-periodic in the variable  $y$  such that

$$v_j = u_j \quad \text{in } (-j, j) \times (j, j+1).$$

Moreover,  $v_j$  is a weak solution in  $C_{\text{loc}}^{1,\alpha}((-j, j) \times \mathbb{R}, \mathbb{R})$ , for some positive  $\alpha$ , of the equation

$$-\Delta_{\Phi} v_j + A(x, y)V'(v_j) = 0 \quad \text{in } (-j, j) \times \mathbb{R}.$$

An direct computation shows that

$$(4.1) \quad \int_{-j}^j \int_j^{j+1} \mathcal{L}(u_j) dy dx = \int_{-j}^j \int_0^1 \mathcal{L}(v_j) dy dx.$$

From now on, given  $u_j \in K_j$ , with  $j > 1$ , let's fix  $v_j$  as above. Then, we have the following result.

**Lemma 4.3.** *There exists  $L > 0$  such that for  $j > L + \frac{1}{4}$ , if  $u_j \in K_j$  we must have*

$$|u_j(x, y) - 1| \leq \delta_1, \quad \forall (x, y) \in T_j \quad \text{with } x \in \left(L, j - \frac{1}{4}\right),$$

where  $\delta_1$  was given in  $(V_3)$ .

*Proof.* Arguing by contradiction, assume that there is a sequence of indices  $(j_n) \subset (0, +\infty)$  with  $j_n \rightarrow +\infty$  such that for each  $j_n$  there exists  $u_{j_n} \in K_{j_n}$  and points

$$(x_n, y_n) \in \left(0, j_n - \frac{1}{4}\right) \times (j_n, j_n + 1)$$

with  $x_n \rightarrow +\infty$  satisfying

$$(4.2) \quad 1 - \delta_1 > u_{j_n}(x_n, y_n) > 0.$$

Given  $j > 1$ , we fix the rectangles

$$Q_j = \left(-j + \frac{1}{8}, j - \frac{1}{8}\right) \times (j - 1, j + 2) \quad \text{and} \quad \tilde{Q}_j = \left(-j + \frac{1}{4}, j - \frac{1}{4}\right) \times (j, j + 1).$$

Now, taking  $\eta_0 \in (0, \frac{1}{32})$  and  $(x, y) \in \tilde{Q}_j$ , it is clear that

$$B_{\eta_0}(x, y) \subset B_{2\eta_0}(x, y) \subset Q_j.$$

Defining the operator

$$B(x, y) = A(x, y)V'(v_j(x, y)) \quad \text{for } (x, y) \in Q_j,$$

there exists  $A_1 > 0$  such that  $|B(x, y)| \leq A_1$  for every  $(x, y) \in Q_j$ . So, since  $v_j$  is a weak solution of the equation

$$\Delta_{\Phi} w + B(x, y) = 0 \quad \text{in } Q_j$$

with  $\|v_j\|_{L^\infty(Q_j)} \leq 1$ , it follows from [45, Theorem 1.7] that there is  $C > 0$  such that

$$(4.3) \quad \|v_j\|_{C^1(\tilde{Q}_j)} \leq C, \quad \forall j \in \mathbb{N},$$

and so,

$$\|v_j\|_{C^1(B_{\eta_0}(x, y))} \leq C, \quad \forall (x, y) \in \tilde{Q}_j.$$

From this, taking  $\eta < \eta_0$  such that  $C\eta < \delta_1/2$  and invoking the Mean Value Theorem, we arrive at

$$(4.4) \quad |v_{j_n}(x, y) - v_{j_n}(x_n, y_n)| \leq C\eta < \frac{\delta_1}{2}, \quad \forall (x, y) \in B_\eta(x_n, y_n) \quad \text{and} \quad \forall n \in \mathbb{N}.$$

Thereby, from (4.2) and (4.4),

$$|1 - u_{j_n}(x, y)| \geq \frac{\delta_1}{2}, \quad \forall (x, y) \in B_\eta(x_n, y_n) \cap \tilde{Q}_{j_n},$$

leading to

$$\|1 - u_{j_n}(x, \cdot)\|_{L^\infty(j_n, j_n + 1)} > \frac{\delta_1}{2}, \quad \forall x \in (x_n - \eta/2, x_n).$$

As the constant of embedding  $W^{1, \Phi}(j_n, j_n + 1) \hookrightarrow L^\infty(j_n, j_n + 1)$  are independent of  $n \in \mathbb{N}$ , because such constants depend only on the length of the intervals  $(j_n, j_n + 1)$ , then there exists  $r > 0$  such that

$$\|1 - u_{j_n}(x, \cdot)\|_{W^{1, \Phi}(j_n, j_n + 1)} \geq r, \quad \forall x \in (x_n - \eta/2, x_n).$$

Now, setting

$$\tilde{u}_{j_n}(x, y) = u_{j_n}(x, y + j_n), \quad \text{for } (x, y) \in (-j_n, j_n) \times (0, 1),$$

we obtain

$$\|1 - \tilde{u}_{j_n}(x, \cdot)\|_{W^{1, \Phi}(0, 1)} \geq r, \quad \forall x \in (x_n - \eta/2, x_n).$$

From Lemma 2.3, there exists  $\mu_r > 0$  satisfying

$$(4.5) \quad \int_{x_n - \eta/2}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2}, \quad \forall n \in \mathbb{N}.$$

On the other hand, for each  $n \in \mathbb{N}$  it is well known that

$$I_{0,j_n}(\tilde{u}_{j_n}) = \int_{-j_n}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx = \int_{-j_n}^{j_n} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n}) dy dx \leq I(u_{j_n}) = c_{j_n} < c.$$

Using the fact that  $j_n \rightarrow +\infty$ , it follows from the Lemma 3.1 that there are  $x_+ > 0$  and  $n_0 \in \mathbb{N}$  satisfying

$$\int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < \frac{3}{2} \lambda_\delta, \quad \forall n \geq n_0.$$

Next, we take  $\lambda_\delta$  arbitrarily small of such way that

$$\int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2}, \quad \forall n \geq n_0.$$

Therefore, as  $x_n \rightarrow +\infty$ , increasing  $n_0$  if necessary, we find

$$\int_{x_n - \eta/2}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \leq \int_{x_+}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \leq \int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2},$$

for any  $n \geq n_0$ , which contradicts (4.5), and the proof is over.  $\square$

In what follows, our goal is to get an estimate from above of the exponential type for  $c - c_L$ . In order to do that, we fix the real function

$$\zeta(x) = \delta_1 \frac{\cosh\left(a\left(x - \frac{j - \frac{1}{4} + L}{2}\right)\right)}{\cosh\left(a\frac{j - \frac{1}{4} - L}{2}\right)}, \quad x \in \mathbb{R},$$

where  $L > 0$  was given in the Lemma 4.3 for some constant  $a > 0$  that will chose later. A simple computation provides  $\zeta''(x) = a^2 \zeta(x)$  for all  $x \in \mathbb{R}$ , which together with  $(\phi_4)$  permit to use the same idea found in [14] to show that

$$(\phi(|\zeta'(x)|)\zeta'(x))' \leq \kappa a^2 \phi(|\zeta'(x)|)\zeta(x), \quad \forall x \in \mathbb{R}.$$

Since  $|\zeta'(x)| \leq a\zeta(x)$  for each  $x \in \mathbb{R}$ , taking  $a < 1$  and using  $(\phi_3)$ , we get  $\phi(|\zeta'(x)|) \leq \phi(\zeta(x))$  for every  $x \in \mathbb{R}$ , and so,

$$-(\phi(|\zeta'(x)|)\zeta'(x))' + \kappa a^2 \phi(\zeta(x))\zeta(x) \geq 0, \quad \forall x \in \mathbb{R}.$$

Therefore, if we define  $w(x, y) = \zeta(x)$  for each  $(x, y) \in \mathbb{R}^2$ , then

$$(4.6) \quad -\Delta_\Phi w + \kappa a^2 \phi(w)w \geq 0 \quad \text{in } \mathbb{R}^2.$$

Now, fixing  $u_j \in K_j$  satisfying Lemma 4.3 and setting the function

$$\nu(x, y) = 1 - v_j(x, y), \quad (x, y) \in (-j, j) \times \mathbb{R},$$

it follows from Lemma 4.3 that  $0 < v_j(x, y) < 1$  for any  $x \in (0, j)$ , and so, since  $v_j$  is a periodic function in the variable  $y$  and continuous, there exists  $b_j > 0$  verifying

$$0 < b_j \leq v_j(x, y) < 1, \quad \forall (x, y) \in \left[L, j - \frac{1}{4}\right] \times \mathbb{R}.$$

According to (V4),

$$(4.7) \quad V'(v_j) \leq -\omega b_j \phi(\nu)(\nu) \quad \text{in } \left(L, j - \frac{1}{4}\right) \times \mathbb{R}.$$

In what follows, we take  $a > 0$  sufficiently small such that  $\kappa a^2 < \underline{A} b_j \omega$ .

**Claim 4.4.** Let  $j_0 \in \mathbb{N}$  and  $\psi \in X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0))$  with  $\psi \geq 0$ , where

$$X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) = \left\{ u \in W^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) \text{ with } u(x, y) = 0 \text{ for } x \notin \left( L, j - \frac{1}{4} \right) \right\},$$

then

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla \nu|) \nabla \nu \nabla \psi + \kappa a^2 \phi(\nu) \nu \psi) dy dx \leq 0.$$

In fact, from (4.7) it may be concluded that

$$\begin{aligned} \int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla \nu|) \nabla \nu \nabla \psi + \kappa a^2 \phi(\nu) \nu \psi) dy dx &= \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (-\phi(|\nabla v_j|) \nabla v_j \nabla \psi + \kappa a^2 \phi(\nu) \nu \psi) dy dx \\ &= \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x, y) V'(v_j) \psi + \kappa a^2 \phi(\nu) \nu \psi) dy dx \\ &\leq \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x, y) V'(v_j) \psi + \omega A(x, y) b_j \phi(\nu) \nu \psi) dy dx \\ &\leq \int_L^{j-\frac{1}{4}} \int_{-j_0}^{j_0} (A(x, y) V'(v_j) \psi - A(x, y) V'(v_j) \psi) dy dx = 0, \end{aligned}$$

proving the Claim 4.4.

On the other hand, the definitions of  $\nu$  and  $w$  together with Lemma 4.3 ensure that

$$(4.8) \quad \nu(x, y) \leq w(x, y) \quad \text{on} \quad \left\{ L, j - \frac{1}{4} \right\} \times \mathbb{R}.$$

**Lemma 4.5.** It holds that  $\nu(x, y) \leq w(x, y)$  in  $(L, j - 1/4) \times \mathbb{R}$ .

*Proof.* Suppose by contradiction that the lemma is false. Then, we can find  $(x_1, y_1) \in (L, j - 1/4) \times \mathbb{R}$  such that  $\nu(x_1, y_1) > w(x_1, y_1)$ . Let  $j_0 \in \mathbb{N}$  such that  $(x_1, y_1) \in (L, j - 1/4) \times (-j_0, j_0)$ . Now, from (4.8) the function  $\psi_* : \mathbb{R} \times (-j_0, j_0) \rightarrow \mathbb{R}$  given by

$$\psi_*(x, y) = \begin{cases} (\nu - w)^+(x, y), & \text{if } x \in (L, j - 1/4) \\ 0, & \text{if } x \notin (L, j - 1/4) \end{cases}$$

is well defined. Moreover,  $\psi_* \in X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0))$  and  $\psi_*$  is a nonnegative continuous. Therefore, according to Claim 4.4 and (4.6),

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla w|) \nabla w \nabla \psi_* + \kappa a^2 \phi(w) w \psi_*) dy dx \geq 0$$

and

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla \nu|) \nabla \nu \nabla \psi_* + \kappa a^2 \phi(\nu) \nu \psi_*) dy dx \leq 0,$$

which leads to

$$\iint_P ((\phi(|\nabla \nu|) \nabla \nu - \phi(|\nabla w|) \nabla w) \nabla (\nu - w) + \kappa a^2 (\phi(\nu) \nu - \phi(w) w) (\nu - w)) dy dx \leq 0,$$

where  $P = \{(x, y) \in \mathbb{R} \times (-j_0, j_0) : \nu(x, y) \geq w(x, y)\}$ . From (3.15),  $\nu(x, y) \leq w(x, y)$  for all  $(x, y) \in (L, j - 1/4) \times (-j_0, j_0)$ , which is impossible.  $\square$

Now, we are ready to prove an exponential estimate from above to  $c - c_j$ .

**Lemma 4.6.** *There are  $\theta_1, \theta_2 > 0$  such that*

$$0 < c - c_j \leq \theta_1 e^{-\theta_2 j}, \quad \forall j \in \mathbb{N} \cup \{0\}.$$

*In particular,  $c_j \rightarrow c$  as  $j \rightarrow +\infty$ .*

*Proof.* First of all, we note that by Lemma 4.5,

$$|v_j(x, y) - 1| \leq \delta_1 \frac{\cosh\left(a\left(x - \frac{j - \frac{1}{4} + L}{2}\right)\right)}{\cosh\left(a\frac{j - \frac{1}{4} - L}{2}\right)}, \quad \forall (x, y) \in \left(L, j - \frac{1}{4}\right) \times \mathbb{R}.$$

Choosing  $x_+ = \frac{j - \frac{1}{4} + L}{2}$ , we have that

$$|v_j(x_+, y) - 1| \leq \frac{\delta_1}{\cosh\left(a\frac{j - \frac{1}{4} - L}{2}\right)} \quad \forall y \in \mathbb{R},$$

which implies

$$(4.9) \quad |v_j(x_+, y) - 1| \leq 2\delta_1 e^{-\frac{a}{2}(j - \frac{1}{4} - L)} := \rho_j \quad \text{and} \quad \Phi(|v_j(x_+, y) - 1|) \leq \Phi(\rho_j) \quad \forall y \in \mathbb{R}.$$

In the sequel, we fix  $j$  sufficiently large such that  $x_+ + \rho_j \leq j$  and

$$\tilde{v}_j(x, y) = \begin{cases} v_j(x, y), & \text{if } 0 \leq x \leq x_+ & \text{and } y \in \mathbb{R} \\ v_j(x_+, y) + \frac{1}{\rho_j}(x - x_+)(1 - v_j(x_+, y)), & \text{if } x_+ \leq x \leq x_+ + \rho_j & \text{and } y \in \mathbb{R} \\ 1, & \text{if } x_+ + \rho_j \leq x & \text{and } y \in \mathbb{R} \\ -\tilde{v}_j(-x, y), & \text{if } x \leq 0 & \text{and } y \in \mathbb{R}. \end{cases}$$

Hereafter, let us identify  $\tilde{v}_j|_{\Omega_0}$  with the  $\tilde{v}_j$  itself, and consequently  $\tilde{v} \in E$  and  $c \leq I(\tilde{v})$ . Now let us take a look at some important estimates for the end of the proof.

**Claim 4.7.**  $|\partial_x \tilde{v}_j| \leq 1$  in  $(x_+, x_+ + \rho_j) \times \mathbb{R}$ .

Indeed, note that  $\partial_x \tilde{v}_j(x, y) = \frac{1}{\rho_j}(1 - v_j(x_+, y))$  in  $(x_+, x_+ + \rho_j) \times \mathbb{R}$ . From (4.9),

$$|\partial_x \tilde{v}_j(x, y)| \leq \frac{1}{\rho_j} |1 - v_j(x_+, y)| \leq 1, \quad \forall (x, y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

**Claim 4.8.**  $|\partial_y \tilde{v}_j| \leq 2C$  in  $(x_+, x_+ + \rho_j) \times \mathbb{R}$ , where  $C > 0$  was given in (4.3).

By definition of  $\tilde{v}_j$ ,  $|\partial_y \tilde{v}_j(x, y)| \leq 2|\partial_y v_j(x_+, y)|$  in  $(x_+, x_+ + \rho_j) \times \mathbb{R}$ . Now, the definition of  $v_j$  combined with (4.3) leads to

$$|\partial_y \tilde{v}_j(x, y)| \leq 2C \quad \forall (x, y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

**Claim 4.9.**  $A(x, y)V(\tilde{v}_j) \leq \overline{A\bar{w}}\Phi(\rho_j)$  in  $(x_+, x_+ + \rho_j) \times \mathbb{R}$ .

From (2.6),

$$A(x, y)V(\tilde{v}_j(x, y)) \leq \overline{A\bar{w}}\Phi(|\tilde{v}_j(x, y) - 1|) \quad \forall (x, y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Now, the definition of  $\tilde{v}_j$  together with (4.9) yields

$$A(x, y)V(\tilde{v}_j(x, y)) \leq \overline{A\bar{w}}\Phi(|v_j(x_+, y) - 1|) \leq \overline{A\bar{w}}\Phi(\rho_j) \quad \forall (x, y) \in (x_+, x_+ + \rho_j) \times \mathbb{R},$$

proving the Claim 4.9.

According to Claims 4.7, 4.8 and 4.9,

$$\begin{aligned} \int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx &\leq \int_{x_+}^{x_+ + \rho_j} \int_0^1 (2^m \Phi(|\partial_x \tilde{v}_j|) + 2^m \Phi(|\partial_y \tilde{v}_j|) + A(x, y)V(\tilde{v}_j)) dy dx \\ &\leq 2^m \Phi(1) \rho_j + 2^m \Phi(2C) \rho_j + \overline{A\bar{w}}\Phi(\rho_j) \rho_j. \end{aligned}$$



Now, since  $\rho_j \rightarrow 0$  as  $j \rightarrow +\infty$ , there is a constant  $\tilde{M} > 0$ , independent of  $j$  and  $\tilde{v}_j$  such that

$$\int_{x_+}^{x_++\rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \leq \tilde{M} \rho_j,$$

and so, by (4.1),

$$\begin{aligned} c \leq I(\tilde{v}_j) &= \int_{-x_+-\rho_j}^{x_++\rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \leq \int_{-j}^j \int_0^1 \mathcal{L}(v_j) dy dx + 2 \int_{x_+}^{x_++\rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \\ &\leq \int_{-j}^j \int_{j+1}^j \mathcal{L}(u_j) dy dx + 2\tilde{M} \rho_j \leq I_j(u_j) + 2\tilde{M} \rho_j = c_j + 2\tilde{M} \rho_j, \end{aligned}$$

that is,

$$0 < c - c_j \leq 4\tilde{M} \delta_1 e^{-\frac{\alpha}{2}(j-\frac{1}{4}-L)},$$

for  $j$  sufficiently large. Therefore, it is possible to find real numbers  $\theta_1, \theta_2 > 0$  satisfying precisely the assertion of the lemma.  $\square$

Next, we establish further compactness property concerning the functionals  $I_{j_n}$ .

**Lemma 4.10.** *Let  $j_n \rightarrow +\infty$  and  $u_{j_n} \in E_{j_n}$  such that  $I_{j_n}(u_{j_n}) - c_{j_n} \rightarrow 0$  as  $n \rightarrow +\infty$ . Then, there exists  $u_0 \in K$  verifying*

$$\|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}(T_{j_n})} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

where  $\tau_j u_0(x, y) = u_0(x, y - j)$  for all  $j \in \mathbb{N}$ .

*Proof.* Setting

$$w_{j_n}(x, y) = u_{j_n}(x, y + j_n), \text{ for } (x, y) \in (-j_n, j_n) \times [0, 1],$$

it is easily seen that  $I_{0,j_n}(w_{j_n}) \leq I_{j_n}(u_{j_n})$ . Since  $c_{j_n} < c$  for all  $n \in \mathbb{N}$  and  $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$ ,

$$(4.10) \quad I_{0,j_n}(w_{j_n}) < c + o_n(1), \quad \forall n \in \mathbb{N}.$$

We claim that for each  $n \in \mathbb{N}$  there exists  $x_{+,n} \in (\frac{j_n}{2}, j_n)$  satisfying

$$\alpha_n := \|w_{j_n}(x_{+,n}, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Indeed, if the claim is not true, then there is  $r > 0$  such that, for some subsequence,

$$\|w_{j_n}(x, \cdot) - 1\|_{W^{1,\Phi}(0,1)} \geq r, \quad \forall x \in (\frac{j_n}{2}, j_n) \text{ and } \forall n \in \mathbb{N}.$$

Invoking Lemma 2.3, there exists  $\mu_r > 0$  verifying

$$I_{0,j_n}(w_{j_n}) \geq \int_{\frac{j_n}{2}}^{j_n} \int_0^1 \mathcal{L}(w_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{j_n}{2}.$$

Taking  $j_n$  sufficiently large we have  $I_{0,j_n}(w_{j_n}) > c + o_n(1)$ , contrary to (4.10), and the claim is proved. Without loss of generality, we can assume that  $\alpha_n > 0$  for any  $n \in \mathbb{N}$ , and so we define the function  $\tilde{w}_{j_n} : \Omega_0 \rightarrow \mathbb{R}$  by

$$\tilde{w}_{j_n}(x, y) = \begin{cases} w_{j_n}(x, y), & \text{if } 0 \leq x \leq x_{+,n} \\ w_{j_n}(x_{+,n}, y) + \frac{1}{\alpha_n}(x - x_{+,n})(1 - w_{j_n}(x_{+,n}, y)), & \text{if } x_{+,n} \leq x \leq x_{+,n} + \alpha_n \\ 1, & \text{if } x_{+,n} + \alpha_n \leq x \\ -\tilde{w}_{j_n}(-x, y), & \text{if } x \leq 0. \end{cases}$$

Thus,  $\tilde{w}_{j_n} \in E$  and

$$(4.11) \quad c \leq I(\tilde{w}_{j_n}) = I_{0,x_{+,n}}(w_{j_n}) + 2 \int_{x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n}) dy dx.$$

On the other hand, from (A.1),

$$(4.12) \quad |\partial_x \tilde{w}_{j_n}| \leq \Lambda \text{ in } (x_{+,n}, x_{+,n} + \alpha_n) \times (0, 1), \quad \forall n \in \mathbb{N}.$$

Indeed, using (A.1), for each  $(x, y) \in (x_{+,n}, x_{+,n} + \alpha_n) \times (0, 1)$  we have

$$|\partial_x \tilde{w}_{j_n}(x, y)| = \frac{1}{\alpha_n} |1 - w_{j_n}(x_{+,n}, y)| \leq \frac{1}{\alpha_n} \|1 - w_{j_n}(x_{+,n}, \cdot)\|_{L^\infty(0,1)} \leq \Lambda, \quad \forall n \in \mathbb{N}.$$

Moreover, an easy computation shows that

$$(4.13) \quad |\partial_y \tilde{w}_{j_n}(x, y)| \leq 2|\partial_y w_{j_n}(x_{+,n}, y)|, \quad \forall (x, y) \in (x_{+,n}, x_{+,n} + \alpha_n) \times (0, 1).$$

Now, since  $\alpha_n \rightarrow 0$  we can take  $n$  sufficiently large such that  $\alpha_n < 1$ , and for such values of  $n$ , the convexity of  $\Phi$  ensures that

$$\begin{aligned} \int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n}, y)|) dy &= \int_0^1 \Phi \left( \|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)} \frac{|\partial_y w_{j_n}(x_{+,n}, y)|}{\|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)}} \right) dy \\ &\leq \|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)} \int_0^1 \Phi \left( \frac{|\partial_y w_{j_n}(x_{+,n}, y)|}{\|\partial_y w_{j_n}(x_{+,n}, \cdot)\|_{L^\Phi(0,1)}} \right) dy \leq \alpha_n, \end{aligned}$$

that is,

$$(4.14) \quad \int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n}, y)|) dy \leq \alpha_n.$$

A similar argument works to prove that  $A(x, y)V(\tilde{w}_{j_n}) \leq \bar{A}\bar{v}\Phi(|1 - w_{j_n}(x_{+,n}, y)|)$  in  $(x_{+,n}, x_{+,n} + \alpha_n) \times (0, 1)$  and

$$(4.15) \quad \int_0^1 \Phi(|1 - w_{j_n}(x_{+,n}, y)|) dy \leq \alpha_n.$$

Therefore, we conclude from (4.12)-(4.15) that

$$(4.16) \quad \int_{x_{+,n}}^{x_{+,n} + \alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n}) dy dx \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

According to (4.10), (4.11) and (4.16),  $I(\tilde{w}_{j_n}) \rightarrow c$ . From Lemma 3.6, there exists  $u_0 \in K$  such that, along a subsequence,

$$\|\tilde{w}_{j_n} - u_0\|_{W^{1,\Phi}(\Omega_0)} \rightarrow 0.$$

As  $\tilde{w}_{j_n}(x, y) = u_{j_n}(x, y + j_n)$  for  $|x| \leq x_{+,n}$  and  $y \in [0, 1]$ , we deduce

$$(4.17) \quad \|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}([-x_{+,n}, x_{+,n}] \times [j_n, j_n + 1])} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

By definition of  $\tilde{w}_{j_n}$ ,

$$I(\tilde{w}_{j_n}) = \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n + 1} \mathcal{L}(u_{j_n}) dy dx + 2 \int_{-x_{+,n}}^{x_{+,n} + \alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n}) dy dx$$

that combines with (4.16) to provide

$$(4.18) \quad \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n + 1} \mathcal{L}(u_{j_n}) dy dx \rightarrow c.$$

Setting  $R_{+,n} = T_{j_n} \setminus ([-x_{+,n}, x_{+,n}] \times [j_n, j_n + 1])$ , we have

$$\iint_{R_{+,n}} \mathcal{L}(u_{j_n}) dy dx = I_{j_n}(u_{j_n}) - \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n + 1} \mathcal{L}(u_{j_n}) dy dx.$$

Now, the estimate  $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$  together with (4.18) ensures that

$$(4.19) \quad \iint_{R_{+,n}} \mathcal{L}(u_{j_n}) dy dx \rightarrow 0.$$

On the other hand, by (2.6),

$$(4.20) \quad \begin{aligned} \iint_{R_{+,n}} (\Phi(|\nabla u_{j_n}|) + \Phi(|u_{j_n} - 1|)) dy dx &\leq \iint_{R_{+,n}} \left( \Phi(|\nabla u_{j_n}|) + \frac{1}{\underline{w} \underline{A}} A(x, y) V(u_{j_n}) \right) dy dx \\ &\leq \max \left\{ 1, \frac{1}{\underline{w} \underline{A}} \right\} \iint_{R_{+,n}} \mathcal{L}(u_{j_n}) dy dx. \end{aligned}$$

This combined with (4.19) leads to

$$(4.21) \quad \|u_{j_n} - 1\|_{W^{1,\Phi}(R_{+,n})} \rightarrow 0.$$

Finally, by Lemma 3.2, we also have that  $\Phi(|\nabla u_0|), \Phi(|u_0 - 1|) \in L^1(\Omega_0)$ , and so,

$$\iint_{R_{+,n}} \Phi(|\nabla \tau_{j_n} u_0|) dy dx \rightarrow 0 \quad \text{and} \quad \iint_{R_{+,n}} \Phi(|\tau_{j_n} u_0 - 1|) dy dx \rightarrow 0.$$

As  $\Phi \in \Delta_2$ , these limits guarantee that

$$(4.22) \quad \|\tau_{j_n} u_0 - 1\|_{W^{1,\Phi}(R_{+,n})} \rightarrow 0.$$

Now the lemma follows from (4.21), (4.22) and (4.17).  $\square$

## 5. SADDLE-TYPE SOLUTIONS

In this last section we collect the results obtained above to prove Theorem 1.2. To this aim, let us consider

$$\Gamma = \bigcup_{j=0}^{\infty} T_j \quad \text{and} \quad \Gamma_k = \Gamma \cap \{y < k\} \quad \text{for each } k \in \mathbb{N}.$$

Setting

$$E_{\infty} = \left\{ w \in W_{\text{loc}}^{1,\Phi}(\Gamma) : 0 \leq w(x, y) \leq 1 \text{ for } x \geq 0 \text{ and } w \text{ is odd in } x \right\},$$

we infer that if  $w \in E_{\infty}$  then  $w|_{T_j} \in E_j$  for every  $j \in \mathbb{N} \cup \{0\}$ . Hereafter, let us identify  $w|_{T_j}$  with  $w$  itself. With everything, we may define the functional  $J : W_{\text{loc}}^{1,\Phi}(\Gamma) \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$J(w) = \sum_{j=0}^{\infty} (I_j(w) - c_j).$$

Clearly,  $J$  is bounded from below on  $E_{\infty}$ . Here, we would like point out that there exists  $u \in E_{\infty}$  such that  $J(u) < +\infty$ . Indeed, from Theorem 1.1, there exists a function  $u_* : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $u_* \in E_{\infty}$  with  $I(u_*) = c$ . Invoking Lemma 4.6,

$$I_j(u_*) - c_j \leq I(u_*) - c_j = c - c_j \leq \theta_1 e^{-\theta_2 j}, \quad \forall j \in \mathbb{N} \cup \{0\}.$$

Thus,

$$J(u_*) = \sum_{j=0}^{\infty} (I_j(u_*) - c_j) \leq \theta_1 \sum_{j=0}^{\infty} e^{-\theta_2 j} < +\infty,$$

and the real number

$$d_{\infty} := \inf_{w \in E_{\infty}} J(w)$$

is well defined.

In what follows, if  $(u_n) \subset W_{\text{loc}}^{1,\Phi}(\Gamma)$  and  $u \in W_{\text{loc}}^{1,\Phi}(\Gamma)$ , we write  $u_n \rightharpoonup u$  in  $W_{\text{loc}}^{1,\Phi}(\Gamma)$  to denote that  $u_n \rightharpoonup u$  in  $W^{1,\Phi}(\Omega)$  for any  $\Omega$  relatively compact in  $\Gamma$ . Here we would like point out that the same arguments found in [14, Lemma 6.2] work to show that

$$u_n \rightharpoonup u \text{ in } W_{\text{loc}}^{1,\Phi}(\Gamma) \Rightarrow J(u) \leq \liminf_{n \rightarrow +\infty} J(u_n).$$

From this, we are ready to show the following result.

**Lemma 5.1.** *There exists  $\bar{u} \in E_\infty$  such that  $J(\bar{u}) = d_\infty$ .*

*Proof.* Let  $(w_n) \subset E_\infty$  be a minimizing sequence for  $J$ . Then there is  $M > 0$  satisfying  $J(w_n) \leq M$  for every  $n \in \mathbb{N}$ . Thereby, for each  $k \in \mathbb{N}$  fixed,

$$\iint_{\Gamma_k} \Phi(|\nabla w_n|) dy dx \leq \iint_{\Gamma_k} \mathcal{L}(w_n) dy dx \leq \sum_{j=0}^k I_j(w_n) \leq J(w_n) + \sum_{j=0}^k c_j \leq M + (k+1)c$$

that together with  $\|w_n\|_{L^\infty(\Gamma)} \leq 1$  ensures that  $(w_n)$  is bounded in  $W_{\text{loc}}^{1,\Phi}(\Gamma)$ . By a classical diagonal argument, for some subsequence, there exists  $\bar{u} \in W_{\text{loc}}^{1,\Phi}(\Gamma)$  such that

$$w_n \rightharpoonup \bar{u} \text{ in } W_{\text{loc}}^{1,\Phi}(\Gamma) \quad \text{and} \quad w_n(x, y) \rightarrow \bar{u}(x, y) \quad \text{a.e. in } \Gamma.$$

Next, by pointwise convergence,  $\bar{u}(x, y) = -\bar{u}(-x, y)$  for almost every  $(x, y) \in \Gamma$  and  $0 \leq \bar{u}(x, y) \leq 1$  for almost every  $(x, y) \in \Gamma$  with  $x \geq 0$ , that is,  $\bar{u} \in E_\infty$ . Moreover,  $J(\bar{u}) = d_\infty$ , which completes the proof.  $\square$

Setting

$$K_\infty = \{w \in E_\infty : J(w) = d_\infty\},$$

we have by the previous lemma that  $K_\infty \neq \emptyset$ . Repeating the arguments used in the proof of Theorem 2.5, it is possible to prove the following result.

**Lemma 5.2.** *If  $\bar{u} \in K_\infty$ , then for any  $\psi \in W^{1,\Phi}(\mathbb{R}^2)$  with  $\psi$  compact support in  $\mathbb{R}^2$  we have*

$$\iint_{\Gamma} (\phi(|\nabla \bar{u}|) \nabla \bar{u} \nabla \psi + A(x, y) V'(\bar{u}) \psi) dy dx = 0.$$

As a consequence of Lemma 5.2, if  $\bar{u} \in K_\infty$  then  $\bar{u}$  is weak solution of

$$-\Delta_\Phi w + A(x, y) V'(w) = 0 \quad \text{in } \Gamma.$$

Elliptic regularity theory yields that  $\bar{u}$  is a solution in  $C_{\text{loc}}^{1,\alpha}(\Gamma)$ , for some  $\alpha > 0$ . Furthermore, arguing as in the proof of Theorem 1.1 we also have that

$$0 < \bar{u}(x, y) < 1 \quad \text{for } (x, y) \in \Gamma \text{ with } x > 0.$$

Finally, we can now prove our main result.

**Proof of Theorem 1.2.**

The existence of saddle-type solution  $v$  will be done via a recursive reflection of the function  $\bar{u} : \Gamma \rightarrow \mathbb{R}$  given by Lemma 5.1. First of all, let us consider the rotation matrix

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that is,  $T(x, y) = (y, -x)$  for any  $(x, y) \in \mathbb{R}^2$ . Setting  $\Gamma^0 = \Gamma$ , we designate  $\Gamma^i = T^i(\Gamma)$  for  $i = 0, 1, 2, 3$ , i.e.,  $\Gamma^i$  is the  $i\frac{\pi}{2}$ -rotated de  $\Gamma$ . Consequently,

$$\mathbb{R}^2 = \bigcup_{i=0}^3 \Gamma^i, \quad T^{-i}(\Gamma^i) = \Gamma, \quad \text{and} \quad \text{int}(\Gamma^i) \cap \text{int}(\Gamma^j) = \emptyset \quad \text{for } i \neq j.$$

Finally, we define the function  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$v(x, y) = (-1)^i \bar{u}(T^{-i}(x, y)), \quad \forall (x, y) \in \Gamma^i.$$

Note that  $v|_{\Gamma^i}$  is the reflection of  $v|_{\Gamma^{i-1}}$  with respect to the axis separating  $\Gamma^{i-1}$  from  $\Gamma^i$ , for any  $i = 1, 2, 3$ . From the properties of the reflection operator,  $v \in W_{\text{loc}}^{1, \Phi}(\mathbb{R}^2)$ . Now, we note that if  $\psi \in W^{1, \Phi}(\mathbb{R}^2)$  with compact support in  $\mathbb{R}^2$ , then  $\psi \circ T^i \in W^{1, \Phi}(\mathbb{R}^2)$  and has compact support in  $\mathbb{R}^2$ , because  $T^i$  is a linear operator. Moreover, from (A4),

$$A(T^i(x, y)) = A(x, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

Thus, invoking Lemma 5.2,

$$\begin{aligned} & \int_{\Gamma^i} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx \\ &= (-1)^i \int_{\Gamma} (\phi(|\nabla \bar{u}|) \nabla \bar{u} \nabla (\psi \circ T^i) + A(x, y) V'(\bar{u}) (\psi \circ T^i)) dy dx = 0. \end{aligned}$$

Therefore, for any  $\psi \in W^{1, \Phi}(\mathbb{R}^2)$  with compact support in  $\mathbb{R}^2$ ,

$$\begin{aligned} & \int_{\mathbb{R}^2} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx \\ &= \sum_{i=0}^3 \int_{\Gamma^i} (\phi(|\nabla v|) \nabla v \nabla \psi + A(x, y) V'(v) \psi) dy dx = 0. \end{aligned}$$

Furthermore, by regularity arguments,  $v$  is a weak solution of equation (PDE) in  $C_{\text{loc}}^{1, \alpha}(\mathbb{R}^2)$ , for some  $\alpha > 0$ . A direct computation shows that  $v$  checks the conditions (a)-(c) of Theorem 1.2. To complete the proof, we are going to prove that  $v$  satisfies item (d). Since  $J(v) = d_\infty < +\infty$ , we must have  $I_j(v) - c_j \rightarrow 0$  as  $j \rightarrow +\infty$ . By Lemma 4.10, there is  $u_0 \in K$  such that

$$(5.1) \quad \|v - \tau_j u_0\|_{W^{1, \Phi}(T_j)} \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$

Now, we claim that

$$(5.2) \quad \|v - \tau_j u_0\|_{L^\infty(T_j)} \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$

In fact, assume by contradiction that there exists  $\epsilon_0 > 0$  such that for each  $n \in \mathbb{N}$  there are  $j_n > n$  and  $(x_n, y_n) \in T_{j_n}$  satisfying

$$|v(x_n, y_n) - \tau_{j_n} u_0(x_n, y_n)| \geq 3\epsilon_0.$$

From Mean Value Theorem, there is  $\theta > 0$  sufficiently small such that

$$|\tau_{j_n} u_0(x, y) - \tau_{j_n} u_0(x_n, y_n)| \leq \epsilon_0, \quad \forall (x, y) \in B_\theta(x_n, y_n) \cap T_{j_n}$$

and

$$|v(x, y) - v(x_n, y_n)| \leq \epsilon_0, \quad \forall (x, y) \in B_\theta(x_n, y_n) \cap T_{j_n}.$$

Consequently,

$$\iint_{T_{j_n}} \Phi(|v - \tau_{j_n} u_0|) dy dx \geq \Phi(\epsilon_0) |B_\theta(x_n, y_n) \cap T_{j_n}| \geq \beta_0, \quad \forall n \in \mathbb{N},$$

for some  $\beta_0 > 0$ . As  $\Phi \in \Delta_2$ , there is  $r > 0$  such that

$$\|v - \tau_{j_n} u_0\|_{L^\Phi(T_{j_n})} \geq r, \quad \forall n \in \mathbb{N},$$

which contradicts (5.1). Thereby, from (5.2), given  $\epsilon > 0$  there is  $j_0 > 0$  such that

$$|v(x, y) - \tau_j u_0(x, y)| < \frac{\epsilon}{2}, \quad \forall (x, y) \in T_j \text{ and } \forall j > j_0.$$

On the other hand, since  $u_0(x, y) \rightarrow 1$  as  $x \rightarrow +\infty$  uniformly in  $y \in [0, 1]$  we may take  $j_0$  sufficiently large satisfying

$$|\tau_j u_0(x, y) - 1| < \frac{\epsilon}{2}, \quad \forall (x, y) \in T_j \text{ with } x > j_0 \text{ and } j \geq 0.$$

Therefore,

$$|v(x, y) - 1| < \epsilon, \quad \forall x > j_0 \text{ and } y > j_0.$$

A similar argument works to prove that

$$|v(x, y) + 1| < \epsilon, \quad \forall x < -j_0 \text{ and } y > j_0.$$

Gathering these estimates together with (5.2) we conclude the proof the theorem.  $\square$

The above proof suggests the following behavior of the solution  $v$ .

**Corollary 5.3.** *Let  $v$  be given as in Theorem 1.2. Then, the following hold:*

- (a)  $v(x, y) \rightarrow 1$  as  $x \rightarrow +\infty$  and  $y \rightarrow +\infty$ ,
- (b)  $v(x, y) \rightarrow -1$  as  $x \rightarrow -\infty$  and  $y \rightarrow +\infty$ ,
- (c)  $v(x, y) \rightarrow -1$  as  $x \rightarrow +\infty$  and  $y \rightarrow -\infty$ ,
- (d)  $v(x, y) \rightarrow 1$  as  $x \rightarrow -\infty$  and  $y \rightarrow -\infty$ .

#### APPENDIX A. BASIC RESULTS ABOUT ORLICZ-SOBOLEV SPACES

Here we give a brief review of Orlicz-Sobolev spaces. The reader can find more details in [1, 51]. We recall that a continuous function  $\Phi : \mathbb{R} \rightarrow [0, +\infty)$  is a **N-function** if:

- i)  $\Phi$  is convex,
- ii)  $\Phi(t) = 0 \Leftrightarrow t = 0$ ,
- iii)  $\Phi$  is even,
- iv)  $\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$  and  $\lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = +\infty$ .

Moreover, we say that a N-function  $\Phi$  verifies the  $\Delta_2$ -**condition** ( $\Phi \in \Delta_2$  for short) if there are constants  $K > 0$  and  $t_0 \geq 0$  such that

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq t_0. \quad (\Delta_2)$$

Below are some examples of N-functions that satisfy  $(\Delta_2)$  with  $t_0 = 0$ :

- (a)  $\Phi_1(t) = \frac{|t|^p}{p}$  with  $1 < p < +\infty$ ,
- (b)  $\Phi_2(t) = \frac{|t|^p}{p} + \frac{|t|^q}{q}$  for  $1 < p < q < +\infty$ ,
- (c)  $\Phi_3(t) = (1 + |t|) \ln(1 + |t|) - |t|$ ,
- (d)  $\Phi_4(t) = (1 + t^2)^\gamma - 1$  with  $\gamma > 1$ ,
- (e)  $\Phi_5(t) = \int_0^t s^{1-\gamma} (\sinh^{-1} s)^\beta ds$  with  $0 \leq \gamma < 1$  and  $\beta > 0$ .

An N-function that does not satisfy  $(\Delta_2)$  is  $\Phi(t) = (e^{t^2} - 1)/2$ .

If  $\Omega$  is an open set of  $\mathbb{R}^N$  ( $N \geq 1$ ) and  $\Phi$  is a N-function, the Orlicz space associated with  $\Phi$  is defined by

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space  $L^\Phi(\Omega)$  is a Banach space endowed with the Luxemburg norm given by

$$\|u\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

When  $\Phi \in \Delta_2$ ,

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} \Phi(|u|) dx < +\infty \right\} \quad \text{and} \quad \int_{\Omega} \Phi \left( \frac{|u|}{\|u\|_{L^\Phi(\Omega)}} \right) dx = 1.$$

The corresponding Orlicz-Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} = u_{x_i} \in L^\Phi(\Omega), \quad i = 1, \dots, N \right\},$$

endowed with the norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|\nabla u\|_{L^\Phi(\Omega)} + \|u\|_{L^\Phi(\Omega)}.$$

The complementary function  $\tilde{\Phi}$  associated with  $\Phi$  is defined by Legendre's transformation

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

Moreover,  $\tilde{\Phi}$  is an N-function and the functions  $\Phi$  and  $\tilde{\Phi}$  are complementary each other. From inequality,

$$st \leq \Phi(t) + \tilde{\Phi}(s), \quad \forall s, t \geq 0, \quad (\text{Young type inequality})$$

an immediate consequence is the Hölder type inequality

$$\int_{\Omega} |uv| dx \leq 2\|u\|_{L^\Phi(\Omega)} \|v\|_{L^{\tilde{\Phi}}(\Omega)}, \quad \text{for all } u \in L^\Phi(\Omega) \quad \text{and} \quad v \in L^{\tilde{\Phi}}(\Omega).$$

If  $\Phi$  and  $\tilde{\Phi}$  satisfy the  $\Delta_2$ -condition, then the spaces  $L^\Phi(\Omega)$  and  $W^{1,\Phi}(\Omega)$  are reflexive and separable. Under the  $\Delta_2$ -condition,

$$u_n \rightarrow u \quad \text{in } L^\Phi(\Omega) \Leftrightarrow \int_{\Omega} \Phi(|u_n - u|) dx \rightarrow 0$$

and

$$u_n \rightarrow u \quad \text{in } W^{1,\Phi}(\Omega) \Leftrightarrow \int_{\Omega} \Phi(|u_n - u|) dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} \Phi(|\nabla u_n - \nabla u|) dx \rightarrow 0.$$

As is mentioned in [14, 37, 38], we have the next four lemmas.

**Lemma A.1.** *Let  $\Phi$  be a N-function of the form (1.1) satisfying  $(\phi_1)$ - $(\phi_2)$ . Set*

$$\xi_0(t) = \min \{t^l, t^m\} \quad \text{and} \quad \xi_1(t) = \max \{t^l, t^m\}, \quad \forall t \geq 0.$$

*Then  $\Phi$  satisfies*

$$\xi_0(t)\Phi(s) \leq \Phi(st) \leq \xi_1(t)\Phi(s), \quad \forall s, t \geq 0$$

*and*

$$\xi_0 \left( \|u\|_{L^\Phi(\Omega)} \right) \leq \int_{\Omega} \Phi(u) dx \leq \xi_1 \left( \|u\|_{L^\Phi(\Omega)} \right), \quad \forall u \in L^\Phi(\Omega).$$

**Lemma A.2.** *If  $\Phi$  is a N-function of the form (1.1) satisfying  $(\phi_1)$ - $(\phi_2)$ , then  $\Phi, \tilde{\Phi} \in \Delta_2$ .*

**Lemma A.3.** *If  $\Phi$  is a N-function of the form (1.1) satisfying  $(\phi_1)$ - $(\phi_2)$ , then*

$$\tilde{\Phi}(\phi(t)t) \leq \Phi(2t), \quad \forall t \geq 0.$$

**Lemma A.4.** *Let  $\Phi$  be a N-function of the form (1.1) satisfying  $(\phi_1)$ - $(\phi_2)$ . If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , then*

- a)  $L^\Phi(\Omega) \hookrightarrow L^l(\Omega)$ ,
- b)  $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,l}(\Omega)$ .

It is well known that  $W^{1,l}(0,1) \hookrightarrow L^\infty(0,1)$  (see for instance [20, Corollary 9.14]). By Lemma A.4 -b),

$$W^{1,\Phi}(0,1) \hookrightarrow L^\infty(0,1).$$

From now on,  $\Lambda > 0$  is a constant satisfying

$$(A.1) \quad \|u\|_{L^\infty(0,1)} \leq \Lambda \|u\|_{W^{1,\Phi}(0,1)} \quad \forall u \in W^{1,\Phi}(0,1).$$

To end this section, assuming that the N-function  $\Phi$  is  $C^1$  we get

$$(A.2) \quad \Phi(|w|) - \Phi(|z|) \geq \Phi'(|z|) \frac{z}{|z|} \cdot (w - z), \quad \forall w, z \in \mathbb{R}^N, z \neq 0,$$

where  $z \cdot w$  denotes the usual inner product in  $\mathbb{R}^N$ .

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