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A Structural Geometric Approach to the Model Matching Problem for Max-Plus Linear Systems

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Cycle XXXV

Abstract

The max-plus algebra can be used to model discrete event dynamic systems where synchronization is involved, by means of linear dynamic equations. A limitation in the modeling power of conventional max-plus linear systems is the lack of competition. This issue can be partly circumvented by considering non-stationary max-plus linear systems.

The structural geometric approach is an area of systems and control theory in which various properties of the dynamical systems under study are expressed in terms of geometric properties of suitable vector spaces. A wide literature is available on the topic for systems over the conventional algebra, but only a few papers are available for system over the max-plus semiring.

The objective of our work is to extend the available applications of the structural geometric approach to linear systems defined over the max-plus algebra. We provide a new formulation for the model matching problem for stationary linear systems over the max-plus algebra and use the available results on the geometric approach for such class of systems to provide solvability conditions and algorithmic solutions to such problem. Our formulation of the problem closely mimics the classical one used in the framework of the conventional algebra, as the objective is to match exactly a given model with the output of the plant. Moreover, we extend the structural geometric approach to switching and periodic max-plus systems and use the new theory for solving the model matching problem also for systems of these kinds. The solvability conditions and the algorithmic solutions obtained suffer from some issues that are not present in the case of systems over the conventional algebra, so they can be only applied under suitable hypotheses.

Italian title and abstract

Italian title of the thesis: *Approccio geometrico strutturale al problema di inseguimento del modello per sistemi max-plus lineari.*

Abstract (IT)

L'algebra max-plus può essere utilizzata per modellare sistemi dinamici ad eventi discreti con fenomeni di sincronizzazione tra processi, tramite equazioni dinamiche lineari. Una limitazione nella capacità espressiva dei sistemi lineari convenzionali nella max-plus algebra è l'assenza di competizione. Questa problematica può essere in parte risolta ricorrendo a sistemi non stazionari.

L'approccio geometrico strutturale è un'area della teoria dei sistemi e del controllo che permette di esprimere differenti proprietà dei sistemi dinamici oggetto di studio, in termini di proprietà geometriche di particolari spazi vettoriali. Un'ampia letteratura sull'approccio geometrico è disponibile per sistemi definiti nell'algebra convenzionale, ma solo pochi risultati sono stati pubblicati per quanto riguarda i sistemi definiti sul semianello max-plus.

L'obiettivo del nostro lavoro è di estendere le applicazioni dell'approccio geometrico strutturale nell'ambito dei sistemi max-plus lineari. Introduciamo una formulazione del problema di inseguimento del modello per sistemi max-plus lineari stazionari ed utilizziamo i risultati disponibili relativi all'approccio geometrico per fornire condizioni di risolubilità e soluzioni algoritmiche al problema. La nostra formulazione del problema è una generalizzazione diretta di quella classica, utilizzata nell'ambito dell'algebra convenzionale. L'obiettivo è inseguire esattamente un dato modello con l'uscita dell'impianto. Inoltre, estendiamo l'approccio geometrico strutturale a sistemi max-plus switching e periodici e utilizziamo la nuova teoria per risolvere il problema di inseguimento del modello anche per sistemi di queste tipologie. Le condizioni di risolubilità e le soluzioni algoritmiche che otteniamo soffrono di alcune problematiche aggiuntive rispetto al caso di sistemi costruiti nell'algebra convenzionale, che le rendono applicabili solo in casi determinati.

Acknowledgements

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Contents

Chapter 1

Introduction

Linear systems over the max-plus algebra are a suitable formalism to model discrete event dynamic systems where synchronization, without competition, is involved. These are essentially man-made systems in which finite resources (e.g. processors, communication channels or production machinery) are shared by several users (e.g. processes, packets or semi-finished products) to achieve some common objectives (e.g. a parallel computation, the transmission of set of packets or obtaining a finished product from the production line).

The absence of competition implies that the available resources are not equivalent to each other or, otherwise, that the users have been pre-assigned to them, in order to avoid the situation in which a user can be served indifferently by multiple resources.

The expressive potential of the formal structure of linear systems over the max-plus algebra can be extended by considering the possibility that the order of events or the structure of the system can vary. For instance, this is the case when the processing time of a given resource is not constant or when the policy for the allocation of different users to different resources can vary. In this way, we can partly overcome the limitations derived from the absence of competition in stationary max-plus linear systems. If this variation in the structure of the system is arbitrary, we can describe the plant as a switching system. When the variation follows a predetermined periodic schedule the formalization can be carried out more precisely using periodic systems.

Control problems involving systems over the max-plus algebra are important in many applications, especially in industrial engineering, and a number of analysis and control techniques have been developed in the last years. In particular, the development of a structural geometric approach for linear systems over the max-plus algebra has been indicated as a promising research direction [25] and, since then, several authors have obtained interesting results along that line [44], [65], [42], [55].

One of the problems considered in the literature consists in searching for a suitable control law that forces a given plant, modeled as a linear system over the max-plus algebra, to behave accordingly to a given model of the same kind. Different formulations of this problem, usually referred to as the model matching problem, have been given in relation to max-plus systems. In most of them the considered control objective is to obtain an output sequence for the plant that anticipate, or at the most equates, that of the given model.

The problem considered here is slightly different, in fact the objective is to force the plant to generate an output exactly equal to that of a given model. Our formulation of the problem closely mimics the classical one, used for the model matching problem in the framework of linear systems over a field. The main research activity carried out during the PhD, described in this document, concerns the extension of the techniques of the geometric approach to linear systems over the max-plus algebra with the aim to solve the model matching problem.

We find simple structural conditions for checking the solvability of the problem and algorithms for computing a solution, if any exists. These results, that extend the current state of the art, have been developed for stationary [11], switching [12] and periodic [13] max-plus linear systems. Such developments would not have been possible without the construction of a structural geometric approach for max-plus linear system, that was not available for switching and periodic systems [12], [13].

The model matching problem for max-plus linear systems has found many practical applications, in the most varied fields, and may find more in the future. We provide a new methodology to implement an effective control strategy to solve such problem, also in reference to complex systems that cannot be modeled as stationary max-plus linear systems.

The results found highlight new parallels between max-plus linear systems and linear systems over the conventional algebra. The parallelism is built through a generalization process, developed along the same line of research that has led other researchers to the construction of a structural geometric approach for systems over rings. This generalization is non-trivial, and it is sometimes necessary to give up particular properties that cease to apply when generalizing from fields to rings and then to the max-plus semiring. For this reason, the solvability conditions and the computation algorithm are applicable under suitable hypotheses, that are discussed appropriately throughout this thesis.

Chapter 2

Max-plus linear systems

Linear systems over the max-plus algebra, introduced in [23], are a suitable formalism to model and analyze discrete event dynamic systems (DEDS) where synchronization, without competition, is involved. These are essentially man-made systems in which finite resources (e.g. processors, communication channels or production machinery) are shared by several users (e.g. processes, packets or semi-finished products) to achieve some common objectives (e.g. a parallel computation, the transmission of set of packets or obtaining a finished product from the production line).

The absence of competition implies that the available resources are not equivalent to each other or, otherwise, that the users have been pre-assigned to them, in order to avoid the situation in which a user can be served indifferently by multiple resource.

A system of such class can be decomposed into elementary operations that interact with each other in two ways. The first type of interaction is concatenation, in this case a first process (P1) must finish before a second process (P2) can start, and the time at which the compound process complete can be computed as the sum of the completion time of P1 and the time required by P2. An example of this scenario is represented by two subsequent production stages along a production line.

A different interaction is given by synchronization, in this case the processes P1 and P2 can run independently but the overall process cannot be considered completed until both P1 and P2 are concluded. In this case the global completion time is given by the maximum between the conclusion time of P1 and P2. As an example we can consider the manufacturing of two components that need to be assembled together before leaving the production plant.

The presence of the *max* operation, in modeling the synchronization between processes, should highlight the non-linearity of such types of systems,

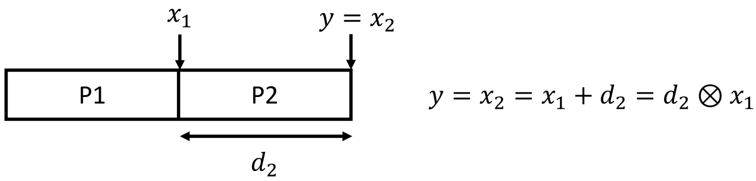

Figure 2.1: Sequential processes: multiplication of a system state and a constant parameter.

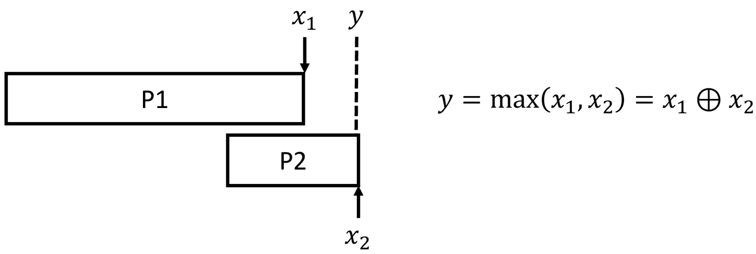

Figure 2.2: Synchronization: sum of system states.

at least in a conventional sense. However, by introducing a specialized algebra, equipped with the two operations $\oplus$ (sum) and $\otimes$ (product) redefined as *max* and $+$ respectively, we can effectively model such DEDS as linear systems. The *max-plus algebra*, whose name appear consistent with its operations, does not enjoy the same properties of the conventional algebra. Moreover, in order to have a neutral element for $\oplus$, the set of numbers must be extended with $-\infty$.

Using the max-plus algebra, we can model the concatenation of processes as multiplication between a system state and a constant parameter as schematized in Figure 2.1, while the synchronization of multiple activities is modeled as a sum of system states as schematized in Figure 2.2.

2.1 Max-plus algebra

Formally, the max-plus algebra $\mathbb{R}_{max}$ consists of the set $\mathbb{R} \cup \{-\infty\}$, equipped with two operations denoted respectively by $\oplus$ and by $\otimes$ and defined by $a \oplus b = \max\{a, b\}$ for $a, b \in \mathbb{R}_{max}$ and by $a \otimes b = a + b$ if a, b belongs to $\mathbb{R}$ or by $(-\infty) \otimes a = a \otimes (-\infty) = -\infty$ for any $a \in \mathbb{R}_{max}$. The neutral element for $\oplus$ and for $\otimes$ are denoted respectively by ϵ and by e, and we have $\epsilon = -\infty$ and $e = 0 \in \mathbb{R}$. As $\otimes$ distributes over $\oplus$ and $\oplus$ is idempotent (i.e. $a \oplus a = a \ \forall a \in \mathbb{R}_{max}$), $\mathbb{R}_{max}$ is an *idempotent semifield* or *diod.* To obtain a more concise notation we will often omit the multiplication symbol

$\otimes$, as is customary in conventional algebra. Moreover, to introduce a total order relation in $\mathbb{R}_{max}$, we say that $a \geq b$ for some $a, b \in \mathbb{R}_{max}$ if and only if $a \oplus b = a$.

Moduloids over semifields are analogous to vector spaces over fields, a more precise definition follows.

Definition 1 (Moduloid [15])**.** *A moduloid $\mathcal{M}$ over an idempotent semifield $\mathcal{K}$ (with operations $\oplus$ and $\otimes$, zero element ϵ and identity element e) is a set endowed with*

- *an internal operation, also denoted $\oplus$, with a zero element, also denoted ϵ;*
- *an external operation defined on $\mathcal{K} \times \mathcal{M}$ with values in $\mathcal{M}$, indicated again as $\otimes$ or by simple juxtaposition;*

which satisfies the following properties:

- *$\oplus$ is associative and commutative;*
- *$\alpha(x \oplus y) = \alpha x \oplus \alpha y$;*
- *$(\alpha \oplus \beta)x = \alpha x \oplus \beta x$;*
- *$\alpha(\beta x) = (\alpha\beta)x$;*
- *$ex = x$;*
- *$\epsilon x = \epsilon$;*

for all $\alpha, \beta \in \mathcal{K}$ and all $x, y \in \mathcal{M}$

In practice, we will be only concerned with $\mathbb{R}^n_{max}$, a moduloid over $\mathbb{R}_{max}$ that consists of the set of all the n-tuples, or vectors, of elements of $\mathbb{R}_{max}$, equipped with the internal component-wise addition denoted $\oplus$ and the scalar product $(\alpha x)_i = \alpha x_i$ for all $\alpha \in \mathbb{R}_{max}$ and $x \in \mathbb{R}^n_{max}$. The zero element of such moduloid is $(\epsilon, ..., \epsilon)^\top$.

Matrices over the max-plus algebra can be introduced as elements of the moduloid $\mathbb{R}^{n\times m}_{max}$, however in this case it is useful to introduce an internal product operation, thus obtaining an *idempotent algebra*:

Definition 2 (Idempotent algebra [15])**.** *A moduloid with an additional internal operation, also denoted as $\otimes$ (or by concatenation), is called an idempotent algebra if:*

- *$\otimes$ is associative;*

- *$\otimes$ has an identity element, denoted as I;*
- *$\otimes$ is distributive with respect to $\oplus$.*

The only idempotent algebra that is relevant to this work is $\mathbb{R}_{max}^{n\times m}$ that consists of the set of matrices with n rows and m columns with values in $\mathbb{R}_{max}$. The $\oplus$ and scalar product operations are those already considered for the moduloid $\mathbb{R}_{max}^{n}$, and the new internal operation, or matrix multiplication, is defined in the standard way in terms of $\oplus$ and $\otimes$ as

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj} \tag{2.1}$$

with $A \in \mathbb{R}_{max}^{n\times m}$ and $B \in \mathbb{R}_{max}^{m\times p}$.

The zero matrix, again denoted ϵ, has all its entries equal to ϵ, while the identity matrix I, is a square matrix that has the diagonal entries equal to e and the other entries equal to ϵ. When confusion can arise, the dimensions of null and identity matrices will be made explicit as subscripts (e.g. $\epsilon_{n\times m}$ or I_n).

In addition to the operations introduced above we can also generalize the order relation to the vector or matrix case, in particular we will say that $A \geq B$ for some $A, B \in \mathbb{R}_{max}^{n\times m}$ if such relation hold element-wise (i.e. $A_{ij} \geq B_{ij}$ for all $i \in [1, n]$ and $j \in [1, m]$). Clearly, this order relation is only partial.

The concepts summarized above can be generalized from the semifield $\mathbb{R}_{max}$ to a semiring. An example of such structure is $\mathbb{N}_{max}$. In such case, there are no inverses with respect to multiplication and the analogous concept of moduloid over idempotent semirings is called *semimodule*. Clearly, every moduloid is also a semimodule and we will generally use the term semimodules to refer to moduloids in $\mathbb{R}_{max}^{n}$.

Moduloids can be dealt with by standard tools of linear algebra, provided the fact that the elements of $\mathbb{R}_{max}$, except ϵ, have no inverse with respect to $\oplus$ is kept into account. Subsemimodules of $\mathbb{R}_{max}^{n}$ are subsets of $\mathbb{R}_{max}^{n}$ that are closed with respect to the component-wise and the scalar operations defined in terms of $\oplus$ and $\otimes$. Subsemimodules will be denoted by script letters, as $\mathcal{V} \subseteq \mathbb{R}_{max}^{n}$. A set of vectors $\{v_1, ..., v_p\} \in \mathcal{V} \subseteq \mathbb{R}_{max}^{n}$ is a set of generators for $\mathcal{V}$ if any element $x \in \mathcal{V}$ can be written as a linear combination of the v_i's, namely $x = \bigoplus_{i=1}^{p} a_i \otimes v_i$ where the coefficients a_i are in $\mathbb{R}_{max}$. Not all the subsemimodules of $\mathbb{R}_{max}^{n}$ have a finite set of generators and those which enjoy of this property are said to be finitely generated.

The condition for the *linear independence* of a set of vectors in the max-plus semimodule $\mathbb{R}_{max}^{n}$ cannot be trivially derived from the definition used in the conventional algebra. Indeed, for any set of vectors $\{v_1, ..., v_p\} \in \mathbb{R}^n_{max} \setminus \{\epsilon\}$ it is never possible to find a set of coefficients $\{a_1, ..., a_p\} \in \mathbb{R}_{max}$ not all equal to ϵ, such that $a_1 v_1 \oplus ... \oplus a_p v_p = \epsilon$. In the rest of the work we do not rely on the concept of linear independence nor on the concept of dimension of a semimodule, which therefore are not explored here. We refer the interested reader to [5], that contains a good summary of the topic and the issues involved. Moreover, an introduction to the theory of max-plus semimodules with application to the study of max-plus dynamic systems can be found in [46], [44] and [65].

The max-plus algebra, after its first thorough and systematic exposition as a modeling tool for dynamic systems [15], has found many applications and connections with a number of other fields, such as operational research [8], [4], game theory [6], reinforcement learning [47], image processing [17], and network calculus [75]. However, here we consider only its usage in the mathematical modeling of DEDS. A recent survey on the state of the art for this subject is available in [70].

2.2 Systems of linear equations

Since inverses with respect to $\oplus$ do not exist, within the max-plus algebra the general system of equations in the vector indeterminate x is of the form

$$Ax \oplus b = Cx \oplus d. \tag{2.2}$$

where A, C are known matrices and b, d are known vectors of compatible dimensions with elements in $\mathbb{R}_{max}$ (see [15]). Equations of such kind are referred as *two-sided affine equations*, as the unknown x appear on both sides of the equation.

General elimination methods can be used to find solutions for linear equations in the max-plus algebra, as described in [19]. A software implementation of the algorithm has been initially developed as a Scilab® toolbox [91], and is now integrated in the Scicoslab software [97]. The general elimination algorithm is directly applicable to two-sided systems of linear equations of the form $Ax = Bx$, but can be easily extended to solve affine systems of the form (2.2) [19].

2.2.1 Systems of the form $x = Ax \oplus b$

As we will see in the following, it is of practical interest to consider a linear system of equations of the form (2.2) which enjoy these properties:

1. C is the identity matrix;
2. d is the null vector;
3. A is a square matrix whose precedence graph contains only circuits of non-positive weights (the definitions of these concepts are reported in Appendix A);

With these premises, we obtain a system of the form

$$x = Ax \oplus b. \tag{2.3}$$

A satisfactory theory for the solution of such systems of linear equations is provided by [104, Theorem 3.17]. The statement, paraphrased, is reported below.

Theorem 1 ([104]). *If we introduce the* kleene star *operator* * *as*

$$A^* = \bigoplus_{n=0}^{+\infty} A^n \tag{2.4}$$

and, if the condition 3 on the associated graph of the matrix A is satified, then A^* *is well defined and the least solution for (2.3) is given by*

$$x = A^* b. \tag{2.5}$$

If A has only circuits of negative weight the solution provided by (2.5) is unique.

Different algorithms for the efficient computation of A^* under different assumptions are available [52], [50].

2.3 Modeling of discrete event systems

We are interested in studying discrete event systems in which events of different types occur with a precise timeline. The most widespread methodology to model a sequence of events as a function over the max-plus algebra is the so called *dater* representation. In modelling a situation in which events of n different types may occur, we can consider an n-dimensional dater function $d(.) : \mathbb{N} \to \mathbb{R}^n_{max}$, whose value at $k \in \mathbb{N}$ is a vector $d(k) = (d_1(k), ..., d_n(k))^\top$ in which the i-th component $d_i(k)$ indicates the time instant at which an event of the i-th type occurs for the k-th time. Dater functions must be

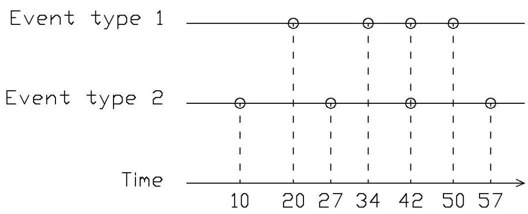

Figure 2.3: A sequence of instances of two types of events.

non-decreasing (i.e. such that $d(k+1) \geq d(k)$ for each $k \in \mathbb{N}$) in order to have physical meaning, as illustrated in the following.

For instance, let us consider the situation in which two types of events can occur and events of the first type happen at instants 20, 34, 42 and 50, for the first four times. The second type of event occurs at 10, 27, 42 and 57. This situation, schematized in Figure 2.3, can be modeled by a dater $d(.) : \mathbb{N} \to \mathbb{R}^2_{max}$ with values:

$$d(1) = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \quad d(2) = \begin{pmatrix} 34 \\ 27 \end{pmatrix} \quad d(3) = \begin{pmatrix} 42 \\ 42 \end{pmatrix} \quad d(4) = \begin{pmatrix} 50 \\ 57 \end{pmatrix} \tag{2.6}$$

If an initial value for d must be specified, we can consider $d(0) = \epsilon$.

In the case in which a specific event can happen for a finite number of times and never again we need to introduce $+\infty$ as a possible value for $d_i(k)$. In this case the dater is a function $d(.) : \mathbb{N} \to \bar{\mathbb{R}}_{max}$ where $\bar{\mathbb{R}}_{max} = \mathbb{R}_{max} \cup \{+\infty\}$. Such case corresponds to a partial or total deadlock for the system and, unless otherwise indicated, we will assume that such behaviour is not possible, so we will refer to daters as functions over $\mathbb{R}_{max}$, or $\mathbb{R}^n_{max}$.

The different types of events that characterize the evolution of a given system can be usually divided into three categories:

- input events: events that are triggered from outside of the system and have direct consequence on the evolution of the system itself (e.g. the arrival of raw parts in the warehouse of a produciton plant);
- internal events: events that are triggered from input events and other internal events and does not have direct consequences outside the system (e.g. the loading of a raw part onto a production machine);
- output events: events that are triggered by internal events of the system whose direct consequences are only relevant outside of the system (e.g. the completion of a product intended to be picked up by the customer).

In case an event does not appear as trivially classifiable could be necessary to consider such event as two coinciding events of different kind. As an example, an event that affects the internal behaviour of the system and is also relevant to the outside, can be modeled as an internal event coinciding with an output event.

Using the above notions we can model a max-plus linear system as a mathematical relation between a dater of input events $u(.) : \mathbb{N} \to \mathcal{U} = \mathbb{R}^m_{max}$ and a dater of output events $y(.) : \mathbb{N} \to \mathcal{Y} = \mathbb{R}^p_{max}$ with an internal dynamic represented by a dater of internal events $x(.) : \mathbb{N} \to \mathcal{X} = \mathbb{R}^n_{max}$. Clearly, the argument of such daters will be the event instance index $k \in \mathbb{N}$. This modeling operation, in general, leads us to linear equations of the form

$$\begin{array}{rcl} x(k) & = & \bigoplus_{i=0}^{M}(A_i x(k-i) \oplus B_i u(k-i)) \\ y(k) & = & \bigoplus_{i=0}^{M} C_i x(k-i) \end{array} \tag{2.7}$$

for some $M \in \mathbb{N}$. By means of appropriate algebraic operations, that are illustrated in the following, it is possible to write equations (2.7) in the so-called *standard form*, which is more compact and formally similar to the well-known structure of dynamic linear systems in the conventional algebra. The first step aim at removing the implicit part $A_0 x(k)$ from (2.7), if any. We can write the first of the two equations (2.7) as

$$x(k) = A_0 x(k) \oplus \bigoplus_{i=1}^{M} A_i x(k-i) \oplus \bigoplus_{i=0}^{M} B_i u(k-i) \tag{2.8}$$

obtaining an equation of the form (2.3) with $A = A_0$, $b = \bigoplus_{i=1}^{M} A_i x(k-i) \oplus \bigoplus_{i=0}^{M} B_i u(k-i)$ and unknown vector $x(k)$. The presence of circuits of positive weight in the graph associated to A_0 would be associated to the pathological situation in which some events of the system cannot be triggered because of a mutual dependence. If this is the case, we can simply ignore such part of the system in the modeling process and obtain a matrix A_0 whose precedence graph contains only circuits of non-positive weights. With these premises we can use (2.5) to obtain

$$x(k) = A_0^* \bigoplus_{i=1}^{M} A_i x(k-i) \oplus A_0^* \bigoplus_{i=0}^{M} B_i u(k-i). \tag{2.9}$$

It is now possible to expand the state $x(k)$ in order to incorporate the original state, the delayed version of the original state, and the delayed version of the input dater u. In this way, we obtain a dynamic object whose evolution is defined by equations of the form

$$\Sigma \equiv \begin{cases} x(k) = A x(k-1) \oplus B u(k) \\ y(k) = C x(k) \\ x(0) = \epsilon \end{cases} \tag{2.10}$$

where $A \in \mathbb{R}_{max}^{n \times n}$, $B \in \mathbb{R}_{max}^{n \times m}$, $C \in \mathbb{R}_{max}^{p \times n}$, and the condition $x(0) = \epsilon$ is taken accordingly to the derivation of the explicit form from the least solution of the implicit equation. Coherently with the usual terminology for dynamical systems, the semimodules $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are called, respectively, the state semimodule, the input semimodule and the output semimodule of the system. Note that in Σ we have n types of internal events, that correspond to the n components of x, m types of input events, that correspond to the m components of u and p types of output events, that correspond to the p components of y. The vector $x(k) = (x_1(k), ..., x_n(k))^\top \in \mathbb{R}_{max}^n$ indicates that the k-th internal event of type i occurs at time $x_i(k)$ and a similar interpretation holds for $u(k)$ and for $y(k)$. A sequence $\{u(k)\}_{k \in \mathbb{N}}$ is viewed as an input to Σ, while a sequence $\{y(k)\}_{k \in \mathbb{N}}$ is viewed as an output of Σ. As discussed, the $(k + 1)$-th occurrence of an event cannot anticipate the k-th one and so, every dater is a non-decreasing sequence. The non-decreasing nature of $\{x(k)\}_{k \in \mathbb{N}}$, that describes the state evolution of the system, implies the following property for the system.

Definition 3 (Non-anticipative)**.** *We say that a max-plus linear system of the form 2.10 is non-anticipative if* $A \geq I_n$.

Decreasing inputs and anticipative systems are not physically realizable and therefore of little interest. For this reason, in the following we only consider non-decreasing daters and non-anticipative systems unless otherwise specified.

In order to model a discrete event system as a linear max-plus system of the form (2.10), some equation manipulation may be required (see also [25] for the details and [51] for an algorithmic procedure that allow to model a given manufacturing system as a max-plus linear system). The following example provides a simple practical application of the concepts expressed in this chapter.

2.4 Example

This section contains an example, originally developed for a paper published during the course of the PhD [11], in which we show how the max-plus formalism can be used to model a manufacturing plant.

Two types of raw components, R1 and R2, are worked in a manufacturing plant. These raw parts are processed by the machines M1 and M2 respectively, in order to obtain the semi-finished parts S1 and S2. These intermediate parts are then assembled together by the machine M3, in order to obtain a unit of final product F1. The machines M1, M2 and M3 require 1, 2

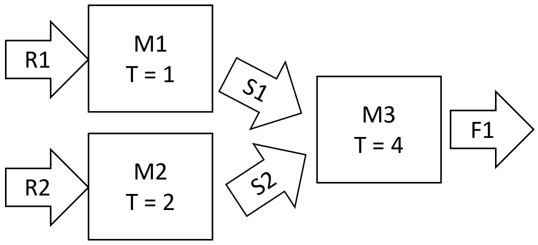

Figure 2.4: Scheme of the plant.

$u_1(k)$	arrival time of the k-th component of type R1
$u_2(k)$	arrival time of the k-th component of type R2
$x_1(k)$	completion time of the k-th cycle on the machine M1
$x_2(k)$	completion time of the k-th cycle on the machine M2
$\left.\begin{array}{c} x_3(k) \\ y(k) \end{array}\right\}$	completion time of the k-th cycle on the machine M3

Table 2.1: Daters and events.

and 4 time units, respectively, to execute their processing cycles. The plant is schematized, from an operational point of view, in Figure 2.4. We assume that all the machines involved can process only one part at a time, and that buffers of infinite capacity are present at each stage of the plant. Clearly, these assumptions can be suitably modified, still obtaining a max-plus linear model.

The only type of output event is “completion of a cycle by the machine M3”. “Arrival of a component of type R1” and “arrival of a component of type R2” are input events. We can consider as internal events the ones of type “completion of a cycle by the machine M1”, “completion of a cycle by the machine M2”, and, again, “completion of a cycle by the machine M3”. We can associate a dater function to each of this events, as reported in Table 5.1.

Clearly, the sequences $\{u_1(k)\}_{k\in\mathbb{N}}$ and $\{u_2(k)\}_{k\in\mathbb{N}}$ need to be provided by some external source, while the values of the other variables can be computed using appropriate rules. For instance, events of type “completion of a cycle on the machine M1” can occur only after one time unit the previous activity on M1 completed and a new input raw part of type R1 became available. Assuming that all the activities start as soon as possible, we can express this statement by the following equation:

$$\begin{aligned} x_1(k) &= \max\{x_1(k-1), u_1(k)\} + 1 = \\ &= \max\{x_1(k-1)+1, u_1(k)+1\} \end{aligned}$$

and, using similar arguments, we have

$$\begin{aligned} x_2(k) &= \max\{x_2(k-1)+2, u_2(k)+2\} \\ x_3(k) &= \max\{x_1(k)+4, x_2(k) \\ &\quad +4, x_3(k)+4\} \\ y(k) &= x_3(k). \end{aligned}$$

These equations are linear in the max-plus algebra and can be written as

$$\begin{cases} x_1(k) = 1 \otimes x_1(k-1) \oplus 1 \otimes u_1(k) \\ x_2(k) = 2 \otimes x_2(k-1) \oplus 2 \otimes u_2(k) \\ x_3(k) = 4 \otimes x_1(k) \oplus \\ \qquad\quad 4 \otimes x_2(k) \oplus 4 \otimes x_3(k-1) \\ y(k) = x_3(k) \end{cases}$$

or, using a matrix notation and omitting the multiplication operator as usual, as:

$$\begin{cases} x(k) = A_0 x(k) \oplus A_1 x(k-1) \oplus \\ \qquad\quad B' u(k) \\ y(k) = C x(k) \end{cases} \tag{2.11}$$

with $A_0 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ 4 & 4 & \epsilon \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & 4 \end{pmatrix}$, $B' = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \\ \epsilon & \epsilon \end{pmatrix}$, $C = \begin{pmatrix} \epsilon & \epsilon & e \end{pmatrix}$.

The first equation of (2.11), due to the presence of the term $A_0 x(k)$, describes an implicit relation that has to be made explicit in order to obtain an expression of the form (2.10). As discussed in Subsection 2.2.1 and Section 2.3, the least solution of the implicit equation $x = Ax \oplus b$ can be expressed as $x = A^* b$, whenever A^* can be given a meaning. In our case, thank to the fact that A_0 is lower triangular and its precedence graph does not have cycles, we have $A_0^i = \epsilon$ for all $i \geq \dim A_0 = 3$ and we can compute A_0^*. We obtain $A_0^* = \begin{pmatrix} e & \epsilon & \epsilon \\ \epsilon & e & \epsilon \\ 4 & 4 & e \end{pmatrix}$ and the following representation of the form (2.10) for the considered plant

$$\begin{cases} x(k) = A_0^* A_1 x(k-1) + A_0^* B' u(k) \\ \qquad\; = A x(k-1) + B u(k) \\ y(k) = C x(k) \\ x(0) = \epsilon \end{cases}$$

with $A = A_0^* A_1 = \begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ 5 & 6 & 4 \end{pmatrix}$, $B = A_0^* B' = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \\ 5 & 6 \end{pmatrix}$, $C = \begin{pmatrix} \epsilon & \epsilon & e \end{pmatrix}$.

Assuming, for sake of illustration, that raw parts of type R1 and R2 arrive together at the time instant 0 and, again, at the instant 1 and simulating the evolution of the system, we get:

$$u(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x(1) = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \quad y(1) = 6$$

$$u(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \quad x(2) = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} \quad y(2) = 10$$

2.5 State of the art and applications

2.5.1 Modeling and analysis

The possibility of modeling a given discrete event system as a max-plus linear system is often used in order to extrapolate useful metrics or to perform specific computations about the expected behaviour of the given plant in different situations [58]. The available literature provides several applications of practical interest.

For instance, the max-plus algebra has been used as the underling technique for developing software that can analyze and simulate the evolution of productive plants. The software developed in [43] has a graphical interface that allows to compose a manufacturing plant by concatenating multiple elementary blocks. The final objective is to determine the required number of workers needed at each stage of the production line and to help the designers in case of reenginering of the production line. In [74] a simulation and analysis software for electronic cards production lines, is developed. The model of the entire production line can be defined by connecting the machine models available in the software library. The obtained program has been used by an industrial partner of the authors to detect bottle necks and predict the requirements of human resources and component supply of a real production plant.

In [61] a suitable max-plus formalism is developed for urban bus networks together with a procedure to compute timetables that maximize the connections between buses from different lines.

In [32] the max-plus algebra is used to model a Kanban control policy as a linear system. The Kanban control system is the most well known pull control policy for manufacturing systems. This technique makes use of authorization cards, called Kanbans, that are used to trigger the production when there is a demand. The max-plus algebra is used by the authors to model the control policy and synthesize a regulator to reduce the work in process without degrading the system performance.

In [2] the max-plus algebra is used to model a networked automation system with a client-server architecture. The response time of the automation system is then computed using the obtained mathematical model and compared with experimental data from a laboratory facility.

Model based on max-plus linear systems are also used in the field of semiconductor manufacturing plants, in order to define their schedules and control them appropriately [112], [67], [68], [69]. Critical time constraints for the permanence of a wafer within a processing chamber must be taken into account, inasmuch the wafer quality is seriously affected by the delay during which it is exposed to high temperature and pressure after completing the processing.

Methodologies to identify and localize faults inside a plant that can be modeled as a max-plus linear system have been proposed in [94], [95], and [76]. Analysis on the performance of max-plus linear systems, when additive inputs are present, has been considered in [54].

2.5.2 Control strategies

Control problems involving systems over the max-plus algebra are important in many applications, especially in industrial engineering, and a number of control techniques have been developed in the last years (a recent review is available in [38]). The most common control problem is to determine appropriate input times (e.g. the instants in which raw components are provided to production machines) for matching or anticipating a specific reference signal on the output of the plant (e.g. time schedule for providing finished products to the consumer). When the request is to anticipate the output reference signal, intended as a dead-line, the just-in-time criterion is often considered. In that case the objective is to delay the input signal as much as possible without exceeding the time schedule for the output of the system [63], [62].

The most widespread approaches are based on, or derived from, the *residuation theory* and the *Model Predictive Control (MPC)*. Residuation basically consists in finding the biggest solution to a system of max-plus inequalities. Approaches that are based on the residuation theory for computing optimal input daters are widely used in the literature [80], [90], [31]. The model predictive control, initially developed for discrete-time linear systems over the conventional algebra, is based on the minimization of a specific cost-function, usually the tracking error, inside a moving window of finite size, also called prediction horizon. Such approach has been extended to max-plus linear systems in [37]. An important benefit is the possibility of easily take into account constraints on the input and output daters. This control technique has found many practical application and several variants have been developed [41], [49], [89].

An efficient implementation of the MPC for the output tracking problem has been developed in [40]. In this work the reference is provided by a P-timed event graph, a more general structure than the timed event graphs considered here. The improved performances of the control algorithm with respect to other similar implementations is due to the fact that the max-plus algebra is used to divide the necessary computations into two parts: the first can be performed offline, while the second, fast and optimized, is performed online.

The case in which the system has significant uncertainties in the time required for the internal tasks has been considered in [105]. In that work an approach based on the analysis and improvement of the critical chain of tasks is proposed.

A relevant distinction between the techniques proposed as solutions for various kind of control objectives on max-plus linear systems is the requirement of an accessible state. When the knowledge of the internal state of the system is necessary we can use an observer to estimate the internal state of the system [42] [56], however, such approach is not always effective as it is usually possible to calculate only an upper and lower bound for the state vector, but not to determine its exact value. An alternative methodology considered by some authors [48] is that of estimating a linear function of the state $Wx(k)$, without knowing the exact value of the state $x(k)$, and use this information in the computation of a feedback for the system. The use of an observer is particularly useful when the system to control is subject to unmeasurable disturbances such as human interventions or failures in the process [39].

In the case of strict time constraints, such as a maximum value for the time distance between two events, a suitable control strategy, that does not require the knowledge of the entire system state has been proposed in [64]. Such control strategy is applicable under suitable hypotheses that are non restrictive for real-world manufacturing systems.

In [93] the authors have implemented a software, that given the model of a max-plus linear plant and a control objective to achieve (disturbance decoupling, model matching or observer-based control), exports a software procedure that can be used to implement a SCADA system.

The approach of this thesis follows a different research line, based on a structural geometric approach. The following chapters will focus on the results obtained along this research direction.

Chapter 3

Geometric approach

3.1 Introduction and state of the art

The area of systems and control theory known as geometric approach finds its roots in the concepts of controlled and conditioned invariance, introduced in [16] and [113] for linear systems over the conventional algebra. The geometric approach is based on the fact that various properties of the systems can be expressed in terms of suitable vector spaces. Analysis and synthesis algorithms can be developed as sums, intersections, and linear transformations of such notable subsets of the state space. In this way deep insights about the behaviour of the system can be obtained together with elegant solutions to several control problems, such as the disturbance decoupling problem and the model matching problem.

The extension of the geometric approach to systems over the max-plus semiring has been indicated as a promising research direction in [25] and, since then, some results have been obtained along that line by different authors. The generalization is not straightforward as max-plus semimodules have different properties than vector spaces. Some of the obstacles to be met have been also addressed by the authors that have extended the geometric approach to linear systems over rings (see [57], [30], [28]), while others are specific to the semiring case. The main difficulty is that a subsemimodule of a finitely generated semimodule does not need to be finitely generated. Moreover, the computational efficiency enjoyed by the geometric approach over the conventional algebra is not preserved and the development of efficient algorithms for computations on max-plus semimodules is largely an open problem, even for finitely generated semimodules.

A first important step towards the extension of geometric concepts to the max-plus semirings has been taken in [44] with the introduction of the concepts of reachable and observable semimodules for max-plus linear systems with coefficients in $\mathbb{Z}_{max}$. These semimodules are not finitely generated in general, however, under the integrity assumption, they belong to the class of rational semimodules, introduced in the same paper.

An important prerequisite for the introduction of a complete theory for the structural geometric approach is to provide a definition of invariant semimodule. The extension of this concept from the framework of the conventional algebra to the semiring case is straightforward. An invariant subsemimodule of a system is a subsemimodule of the state semimodule such that, if the initial state is contained in such a set, then the free evolution of the system takes place entirely within it. By free evolution we mean the evolution of the state in the presence of a zero input (i.e. $u(k) = \epsilon$ for all $k \in \mathbb{N}$). More formally we can write:

Definition 4 (Invariant)**.** *Given a max-plus linear system of the form (2.10) we say that a semimodule $\mathcal{V} \subseteq \mathcal{X}$ is* invariant*, or A-invariant, if, for all $x \in \mathcal{V}$, we have $Ax \in \mathcal{V}$.*

The concept of invariant semimodule can be extended to the case of an arbitrary controllable input for the system by introducing the notion of controlled invariant semimodule.

Definition 5 (Controlled invariant [65])**.** *Given a max-plus linear system of the form (2.10) we say that a semimodule $\mathcal{V} \subseteq \mathcal{X}$ is a* controlled invariant*, or (A, B)-invariant semimodule if, for all $x \in \mathcal{V}$ exists some $u \in \mathcal{U}$ such that $Ax \oplus Bu \in \mathcal{V}$.*

Given a system with an initial state contained inside a controlled invariant subsemimodule of the state semimodule, it is possible to keep the evolution of the system inside the controlled invariant semimodule by means of an appropriate input dater $u(k)$ for all $k \in \mathbb{N}$.

The concept of controlled invariant for max-plus linear systems has been introduced in [65] together with an algorithm for the computation of the maximal controlled invariant contained in a given subsemimodule of the state semimodule. The existence of such maximal element is guaranteed by the following result.

Lemma 1. *Given a max-plus linear system Σ of the form (2.10) and a subsemimodule of its state semimodule $\mathcal{K} \subseteq \mathbb{R}^n_{max}$, the set of all the (A, B)-invariant subsemimodules contained in $\mathcal{K}$ is a sup-semilattice with respect to inclusion and sum of semimodules, so a maximum element, denoted $\mathcal{V}^*(\mathcal{K})$, exists.*

Proof. Given two (A, B)-invariant semimodules $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{K}$ it is always possible to consider their sum semimodule $\mathcal{V}_+ = \{x \in \mathbb{R}^n_{max}$ such that $x = x_1 \oplus x_2$ for some $x_1 \in \mathcal{V}_1$ and $x_2 \in \mathcal{V}_2\}$. Clearly, $\mathcal{V}_1 \subseteq \mathcal{V}_+$, $\mathcal{V}_2 \subseteq \mathcal{V}_+$, and, being $\mathcal{K}$ a semimodule, we also have $\mathcal{V}_+ \subseteq \mathcal{K}$.

The fact that $\mathcal{V}_1$ is a controlled invariant semimodule implies that for each $x_1 \in \mathcal{V}_1$ there exists a vector $u_1 \in \mathbb{R}^m_{max}$ such that $(Ax_1 \oplus Bu_1) \in \mathcal{V}_1$. The same reasoning can be applied to $\mathcal{V}_2$, obtaining $(Ax_2 \oplus Bu_2) \in \mathcal{V}_2$, for each $x_2 \in \mathcal{V}_2$ and some $u_2 \in \mathbb{R}^m_{max}$. So we have that for each $x = (x_1 \oplus x_2) \in \mathcal{V}_+$ there exists $u = (u_1 \oplus u_2) \in \mathbb{R}^m_{max}$ such that $(Ax \oplus Bu) = ((Ax_1 \oplus Bu_1) \oplus (Ax_2 \oplus Bu_2))$ belongs to $\mathcal{V}_+$. □

The possibility of determining the maximal invariant contained in a given subsemimodule of the state semimodule, as we will see, represent a keystone in the solution of various problems using the geometric approach. The algorithm presented by [65] represents a generalization of the one proposed in [113] for conventional linear systems.

Theorem 2 ([65])**.** *The sequence of semimodules $\mathcal{V}_k$ defined by*

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{K} \\ \mathcal{V}_k &= \mathcal{V}_{k-1} \cap A^{-1}(\mathcal{V}_{k-1} \ominus \mathrm{Im} B) \end{aligned} \tag{3.1}$$

where $A^{-1}(\mathcal{Y}) = \{v \in \mathbb{R}^n_{max}, \text{such that } Av \in \mathcal{Y}\}$ *and* $\mathcal{V}_{k-1} \ominus \mathrm{Im}\, B = \{x \in \mathbb{R}^n_{max}, \text{for which there exists } u \in \mathbb{R}^m_{max} \text{ such that } x \oplus Bu \in \mathcal{V}_{k-1}\}$*, is such that $\mathcal{V}_{k+1} \subseteq \mathcal{V}_k$ for all $k \in \mathbb{N}$. If we define $\mathcal{V}_\infty = \lim_{k\to\infty}\mathcal{V}_k = \bigcap_{k\in\mathbb{N}} \mathcal{V}_k$, then every (A, B)-invariant semimodule contained in $\mathcal{K}$ is also contained in $\mathcal{V}_\infty$. Moreover, $\mathcal{V}_{k+1} = \mathcal{V}_k$ if and only if $\mathcal{V}_k$ is (A, B)-invariant and in such case $\mathcal{V}_\infty = \mathcal{V}_k = \mathcal{V}^*(\mathcal{K})$.*

As it happens for systems with coefficients in a ring, the sequence (3.1) does not necessarily converge in a finite number of steps and therefore it does not provide, in general, an algorithm for the computation of $\mathcal{V}^*(\mathcal{K})$, as its counterpart does for systems with coefficients in a field [113].

Remark 1. *If $\mathcal{K}$ is finitely generated, then the semimodules $\mathcal{V}_k$ are finitely generated for all $k \in \mathbb{N}$. In fact, given some finitely generated semimodules $\mathcal{Z}$ and $\mathcal{Y}$, the semimodules $\mathcal{Y} \ominus \mathcal{Z}$, $A^{-1}(\mathcal{Y})$, and $\mathcal{Y} \cap \mathcal{Z}$ are all finitely generated [46, Corollary 86]. As explained in [65, Remark 1], their generators can be obtained as the set of solutions of appropriate systems of linear equations over the max-plus algebra.*

A more restrictive property is the *controlled invariance of feedback type*, that requires not only the existence of an input that makes the semimodule invariant, but also such input to be calculable as a linear transformation of the state of the system (i.e. as a linear feedback). The definition of controlled invariance of feedback type for max-plus linear system has been introduced in [65] (therein such semimodules are referred to as algebraically controlled invariants) and is reported below.

Definition 6 (Controlled invariant of feedback type [65])**.** *Given a max-plus linear system of the form (2.10) we say that a semimodule $\mathcal{V} \subseteq \mathcal{X}$ is a controlled invariant of feedback type, or (A, B)-invariant of feedback type, if some matrix $F \in \mathbb{R}_{max}^{m \times n}$ exists, such that, for all $x \in \mathcal{V}$, $(A \oplus BF)x \in \mathcal{V}$.*

Clearly, (A, B)-invariance of feedback type implies (A, B)-invariance. In the framework of systems with coefficients in a field, the two properties are equivalent [16], [113], but this is not true in the case of systems with coefficients in a ring [57], [30] or in a semiring [65]. This is a significant drawback. The power of the geometric approach in the conventional algebra is tied to the ability to find a controlled invariant space with suitable properties and assume that the construction of a linear feedback that makes it invariant is possible. In the max-plus case the calculation is more complex and the convergence is not guaranteed, moreover, even in the case of success, the sub-semimodule found may not be of feedback type. A remedy for this problem is provided in [20] and it essentially consists in extending the state space of the closed loop system by means of a controller that is implemented as a dynamic feedback, instead of a static feedback. The authors prove that this technique can be applied to every (A, B)-invariant semimodule.

Another relevant notion for the geometric approach is that of conditioned invariance, such property in the max-plus context was studied in [42]. The duality between conditioned and controlled invariance holds, as in the conventional algebra. However, the practical computations are possible only under restrictive assumptions. We do not rely on the concept of conditioned invariant subsemimodule in the following of this work, so we refer the interested reader to the sources for any further information on the topic.

One of the main problems that find an effective and elegant solution in the geometric approach over the conventional algebra is the disturbance decoupling problem (DDP). Different formulations of such problem has been provided for systems over the max-plus algebra and tackled using different techniques [78] [101] [100] [102] [92] [79], among them the geometric approach [55]. However, the transposition of the disturbance decoupling problem to max-plus linear systems seems to have a different practical meaning with respect of the same problem for systems over the conventional algebra. In fact, the controller can only slow down the operation of the plant, that is the same effect that the disturbance has. Consequently, the disturbance cannot really be compensated for. At most, it is possible to slow down the system more than what the disturbance would have. In this way the effect of the disturbance is not directly visible in the output. As an example, it would be possible to implement a controller that, in the event of a failure, could limit the flow of new inputs to a production machinery. In this way the exit delay due to the fault would be hidden by the delay due to the absence of parts to be machined. Although this problem is of practical interest in the minimization of work in process, it differs significantly from the interpretation of the DDP in the conventional algebra, in which the goal is to compensate for the effects of the fault.

3.2 Model matching problem

The model matching problem (MMP) for linear systems over the conventional algebra is another problem that finds a simple and elegant solution in the geometric approach. Such problem consists in finding a suitable control law that forces a given plant, modeled as a linear system, to behave accordingly to a given model of the same kind.

The adaptation of the problem to max-plus linear systems has been interpreted in different ways by different authors. In its most widespread formulation, the model matching problem is considered as the problem of finding an appropriate control for a given plant, in order to maintain its output smaller or equal to the output of a given model. In practice, the output of the model is interpreted as a deadline for the corresponding output event of the plant. As an additional requirement, the just-in-time criterion can be considered [7], in this case the objective is to delay the input of the plan as much as possible, without violating the model matching condition. This definition of the model matching problem is coherent with the concept of output target tracking provided in [26], or the concept of output reference control [15] [90] [108], except for the fact that the target is not given directly, instead it can be obtained as the output sequence of a given model. Several authors studied and solved this formulation of the model matching problem, using the concept of transfer matrix of a max-plus linear system and specific properties of formal power series over a dioid. In [96] an adaptive output feedback control for SISO systems is proposed. In [81] a different representation, based on the $(min, +)$ diod, is employed to model the plant.

In [31], the matching condition consists in minimizing the distance, in a suitable sense, between the output of the plant and that of a reference model while delaying as much as possible the control action. The tools used for solving the problem are based on residuation theory and some properties of the Kleene star operator. A similar formulation is considered in [85], where the problem is tackled by developing a three-block control structure that implements a precompensation and a feedback action.

Somewhat similar to the model matching problem is the control design for temporal constraint meeting tackled in [10]. In this case the aim is to assure that the time between two instances of internal events does not exceed a given threshold. Conditions for the solvability of the problem and an appropriate causal feedback controller can be found by solving appropriate systems of linear equations in the max-plus algebra.

In our work [11], we formulated the model matching problem as that of forcing a max-plus linear plant to generate an output that equals, exactly, that of a given model of the same kind. This formulation represents a straightforward generalization of the classical model matching problem [71] to systems whose coefficients are not in a field, but in an idempotent semiring. Throughout this thesis we will use the same approach and we will adopt the definition reported below.

Problem 1 (Model Matching Problem [11]). *Given a linear max-plus system*

$$\Sigma_P \equiv \begin{cases} x_P(k) = A_P x_P(k-1) \oplus B_P u_P(k) \\ y_P(k) = C_P x_P(k) \\ x_P(0) = \epsilon \end{cases} \tag{3.2}$$

of the form (2.10), called the plant, and a linear max-plus system

$$\Sigma_M \equiv \begin{cases} x_M(k) = A_M x_M(k-1) \oplus B_M u_M(k) \\ y_M(k) = C_M x_M(k) \\ x_M(0) = \epsilon \end{cases} \tag{3.3}$$

of the form (2.10), called the model, with $x_P : \mathbb{N} \to \mathbb{R}^{n_P}_{max}$, $x_M : \mathbb{N} \to \mathbb{R}^{n_M}_{max}$, $u_P : \mathbb{N} \to \mathbb{R}^{m_P}_{max}$, $u_M : \mathbb{N} \to \mathbb{R}^{m_M}_{max}$ and $y_P, y_M : \mathbb{N} \to \mathbb{R}^{p}_{max}$, the Model Matching Problem (MMP) consists in finding, for all possible non-decreasing input sequences $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model, an appropriate non-decreasing control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for the plant, such that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of this latter equals the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model, i.e. $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

Our formulation of the problem has several points of contact with the one considered recently in [88] for the forced synchronization problem. However, important differences are present. In particular, in the formulation provided in [88] the model is an autonomous system that has not inputs. Moreover, in that work the initial state of both the plant and the model is arbitrary while we limit our analysis to the case of null initial state for both the systems involved. As a possible future research direction, the theory presented here could be combined with the one provided in that work in order to obtain more general results that account for both a model subject to an arbitrary input and an arbitrary initial state for both the plant and the model.

In addition to the one formalized above, we introduced a more restrictive formulation of the MMP that requires the control signal to be a linear function of the state of the plant, the state of the model, and the input of the model. Such control signal can be viewed as a feedback map.

Problem 2 (Feedback Model Matching Problem [11]). *Given a plant of the form (3.2) and a model of the form (3.3), the Feedback Model Matching Problem (FMMP) consists in finding, for all possible non-decreasing input sequences* $\{u_M(k)\}_{k\in\mathbb{N}}$ *of the model, two appropriate matrices* $F \in \mathbb{R}_{max}^{m_P\times(n_P+n_M)}$ *and* $G \in \mathbb{R}_{max}^{m_P\times m_M}$ *such that the control input sequence* $\{u_P(k)\}_{k\in\mathbb{N}}$ *defined by*

$$u_P(k) = F\begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus Gu_M(k) \text{ for } k \geq 1 \tag{3.4}$$

is a solution for the corresponding MMP.

Remark 2. *If* $x_P(k)$, $x_M(k)$ *and* $u_M(k)$ *are all non-decreasing daters, then the control input sequence* $u_P(k)$ *defined by (3.4) is also non-decreasing. In fact, in the max-plus algebra every linear function is monotone, and so are both the terms in the right-hand term of equation (3.4).*

Remark 3. *In the given formulation of the FMMP it is not required that the matrices* F *and* G *contain only non-negative or* ϵ *values, so the solution can be a feedback controller whose outputs anticipate its inputs. In this case, the practical implementation is possible only if the values of the model input dater are known with some advance. However, not necessarily the entire sequence has to be known from the beginning.*

3.3 Solution

Given the plant Σ_P described by (3.2) and the model Σ_M described by (3.3), let us consider the joint internal event dater $x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathbb{R}_{max}^{(n_P+n_M)}$ and the related joint dynamics, which is described by the equation

$$x_E(k) = A_E x_E(k-1) \oplus B_1 u_P(k) \oplus B_2 u_M(k) \tag{3.5}$$

with $A_E = \begin{pmatrix} A_P & \epsilon \\ \epsilon & A_M \end{pmatrix}$, $B_1 = \begin{pmatrix} B_P \\ \epsilon \end{pmatrix}$, $B_2 = \begin{pmatrix} \epsilon \\ B_M \end{pmatrix}$ and $x_E(0) = \epsilon$.

It is possible to reformulate the control problem expressed in Problem 5 as that of finding, for any input sequence of the model $\{u_M(k)\}_{k\in\mathbb{N}}$, a control input sequence for the plant $\{u_P(k)\}_{k\in\mathbb{N}}$ that keeps $\{x_E(k)\}_{k\in\mathbb{N}}$ inside the *output equalizer* subsemimodule $\mathcal{K} \subseteq \mathbb{R}_{max}^{(n_p+n_M)}$ defined by

$$\mathcal{K} = \{x_E = \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathbb{R}_{max}^{(n_p+n_M)}, \text{ such that } C_P x_P = C_M x_M\}. \tag{3.6}$$

In the framework of max-plus systems, control problems in which the control objective is that of constraining the system state $x(k)$ inside a given subsemimodule, have been considered and dealt with by employing a structural geometric approach in [65], [86] and [84]. Viewing $\{u_M(k)\}_{k\in\mathbb{N}}$ as a disturbance input and $\{u_P(k)\}_{k\in\mathbb{N}}$ as a control input we can see similarities with the disturbance decoupling problem over the conventional algebra. In that case the objective is to constrain the state of the joint dynamics inside the null space of the output matrix. The parallelism is evident although there are some differences which we discuss in the following Remark.

Remark 4. *In the semiring case, contrary to what happens for systems over rings [29] and fields, it is not possible to consider the difference between the output of the plant and that of the model and to describe the resulting control problem as that of keeping the state inside the kernel of such function. Differences cannot be computed in the max-plus algebra and, in general, it is not possible to find a matrix C_E whose kernel is equal to the output equalizer subsemimodule $\mathcal{K}$ defined in (3.6). This is true both using the classical definition of kernel (i.e.* $\operatorname{Ker} A = \{x \in \mathbb{R}_{max}^{n_r}, \textit{such that } Ax = \epsilon\}$, *where n_r is the number of rows of A), and the alternative definition, more convenient in the max-plus case, used by some authors (for instance [24], [42]) in terms of congruence (i.e.* $\operatorname{Ker} = \{(x, y) \in \mathbb{R}_{max}^{2n_r}, \textit{such that } Ax = Ay\}$*). This means that it is not possible to associate to the dynamics (3.5) a linear output map that represents the difference between the output of the plant and that of the model. Note, however, that the output equalizer subsemimodule $\mathcal{K}$ defined in (3.6) is the pull-back, in the category of modules over $\mathbb{R}_{max}$, of the pair of maps (C_P, C_M) and, as such, it generalizes the notion of kernel of the difference between these two maps [83]. It would be possible to proceed by introducing the operator $\oslash$, defined over $\mathbb{R}_{max}$ by $a \oslash b = a - b$ for $a, b \in \mathbb{R}$, by $\epsilon \oslash a = a \oslash \epsilon = a$ for any $a \in \mathbb{R}_{max}$ and by $\epsilon \oslash \epsilon = e$. This, actually makes it possible to write*

$$y_E(k) = y_P(k) \oslash y_M(k) = C_P x(k) \oslash C_M x_M(k) \tag{3.7}$$

and the MMP reduces to find, for any disturbance input $\{u_M(k)\}_{k\in\mathbb{N}}$, a control input $\{u_P(k)\}_{k\in\mathbb{N}}$ such that $y_E(k) = e$ for all $k \in \mathbb{N}$. However, the output equation (3.7) is not linear in the max-plus algebra and it cannot be conveniently handled. For such reason, we do not follow this approach.

The formulation of the MMP that consists of keeping $x_E(k)$ inside the output equalizer subsemimodule $\mathcal{K}$ can be tackled by means of a structural geometric approach. In this way, we have obtained a necessary and sufficient condition for the existence of a solution for the MMP.

Theorem 3 ([11])**.** *Given a non-anticipative plant Σ_P of the form (3.2) and a non-anticipative model Σ_M of the form (3.3), the related MMP is solvable if and only if for all $x \in \operatorname{Im} B_2 = \operatorname{Im}\begin{pmatrix}\epsilon\\ B_M\end{pmatrix}$ there exists $z \in \operatorname{Im} B_1 = \operatorname{Im}\begin{pmatrix}B_P\\ \epsilon\end{pmatrix}$ such that $x \oplus z$ belongs to $\mathcal{V}^*(\mathcal{K})$, where $\mathcal{V}^*(\mathcal{K})$ is the maximum (A_E, B_1)-invariant semimodule contained in the output equalizer semimodule $\mathcal{K}$ defined by (3.6).*

Proof. If. By the (A_E, B_1)-controlled invariance of $\mathcal{V}^*(\mathcal{K})$, it follows that given $x_E(k-1) \in \mathcal{V}^*(\mathcal{K})$, there exists $u_1(k) \in \mathbb{R}^{m_P}_{max}$ such that $A_E x_E(k-1) \oplus B_1 u_1(k)$ belongs to $\mathcal{V}^*(\mathcal{K})$. Moreover, by hypothesis, given $u_M(k) \in \mathbb{R}^{m_M}_{max}$, there exists $u_2(k) \in \mathbb{R}^{m_P}_{max}$ such that $B_1 u_2(k) \oplus B_2 u_M(k)$ belongs to $\mathcal{V}^*(\mathcal{K})$. We can then construct recursively a control input $\{u_P(k)\}_{k\in\mathbb{N}}$ for the dynamics (3.5) as

$$u_P(k) = \begin{cases} u_2(1) \text{ for } k = 1 \\ u_1(k) \oplus u_2(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases} \tag{3.8}$$

More precisely, we start by taking $u_2(1)$ such that $B_1 u_2(1) \oplus B_2 u_M(1) \in \mathcal{V}^*(\mathcal{K})$ and we set $u_P(1) = u_2(1)$. Then, we compute $x_E(1)$ by means of (3.5), $x_E(0)$, $u_P(1)$ and $u_M(1)$, and we take $u_1(2)$ and $u_2(2)$ such that $A_E x_E(1) \oplus B_1 u_1(2) \in \mathcal{V}^*(\mathcal{K})$ and $B_1 u_2(2) \oplus B_M u_M(2) \in \mathcal{V}^*(\mathcal{K})$. We set $u_P(2) = u_1(2) \oplus u_2(2) \oplus u_P(1)$ and we iterate the same procedure increasing by 1 the index k at each step. Note that the sequence $\{u_P(k)\}_{k\in\mathbb{N}}$, thanks to the presence of the term $u_P(k)$ in the second equation of (3.8), is non decreasing and it gives rise to the following state evolution

$$x_E(k) = \begin{cases} A_E x_E(0) \oplus B_1 u_1(1) \oplus B_2 u_M(1) \quad \text{for } k = 1 \\ A_E x_E(k-1) \oplus B_1 u_1(k) \oplus (B_1 u_2(k) \\ \oplus B_2 u_M(k)) \oplus B_1 u_P(k-1) \quad \text{for } k > 1 \end{cases}. \tag{3.9}$$

In equation (3.9), the term $B_1 u_P(k-1)$ is irrelevant, since, we have $x_E(k-1) \geq B_1 u_P(k-1)$ and hence, thanks to the assumption of non-anticipativity that implies $A_E \geq I$, also $A_E x_E(k-1) \geq B_1 u_P(k-1)$. Then, disregarding this last term, we can show by induction that the state evolution $\{x_E(k)\}_{k\in\mathbb{N}}$ given in (3.9) is contained in $\mathcal{V}^*(\mathcal{K})$. In fact, $x_E(0) = \epsilon$ belongs to $\mathcal{V}^*(\mathcal{K})$. Moreover, by the definition of $u_1(.)$ it follows that the summand $(A_E x_E(k-1) \oplus B_1 u_1(k))$ in the right-hand term of (3.9) is contained in $\mathcal{V}^*(\mathcal{K})$ if $x_E(k-1)$ is contained in $\mathcal{V}^*(\mathcal{K})$. Finally, the second summand $(B_1 u_2(k) \oplus B_2 u_M(k))$ in the right-hand term of (3.9) is contained in $\mathcal{V}^*(\mathcal{K})$ by the definition of $u_2(k)$. Since $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$, by the definition of $\mathcal{K}$ given in (3.6), it follows that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of the plant generated by the input $\{u_P(k)\}_{k\in\mathbb{N}}$ defined by (3.8) is equal to the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model generated by the input $\{u_M(k)\}_{k\in\mathbb{N}}$ and the MMP is solved.

Only if. If the condition of the theorem does not hold, there exists an input vector $\bar{u}_M$ such that $B_2\bar{u}_M \oplus B_1 u_P \notin \mathcal{V}^*(\mathcal{K})$ for any $u_P \in \mathbb{R}_{max}^{m_P}$. Then, for the constant input $u_M(k) = \bar{u}_M$ for $k \in \mathbb{N}$, we have, from (3.5), that $x_E(1) = A_E x_E(0) \oplus B_1 u_P(1) \oplus B_2 u_M(1) = A_E\epsilon \oplus B_1 u_P(1) \oplus B_2\bar{u}_M = B_1 u_P(1) \oplus B_2\bar{u}_M$ does not belong to $\mathcal{V}^*(\mathcal{K})$ for any value $u_P(1) \in \mathbb{R}_{max}^{m_P}$ and also $x_E(1) \geq B_2\bar{u}_M$. The latter inequality, thank to the fact that $A_E \geq I$, implies recursively $x_E(k) = A_E x_E(k-1) \oplus B_1 u_P(k) \oplus B_2 u_M(k) = A_E x_E(k-1) \oplus B_1 u_P(k) \oplus B_2\bar{u}_M = A_E x_E(k-1) \oplus B_1 u_P(k)$, while the fact that $x_E(1)$ does not belong to $\mathcal{V}^*(\mathcal{K})$ implies that for any input $\{u_P(k)\}_{k\in\mathbb{N}}$ there exists $\bar{k} \in \mathbb{Z}$ such that $x_E(\bar{k}) = A_E x_E(\bar{k}-1) \oplus B_1 u_P(\bar{k}) \notin \mathcal{K}$. In other words, $x_E(k)$ cannot be kept indefinitely inside the subsemimodule $\mathcal{K}$ and the MMP cannot be solved. □

Remark 5. *The condition expressed by Theorem 3 can be equivalently written, using the* $\ominus$ *operator introduced in (3.1), as* $\mathrm{Im}B_2 \subseteq \mathcal{V}^*(\mathcal{K}) \ominus \mathrm{Im}B_1$, *and it can be practically checked using the techniques described in Remark 1. Solutions, if any exists, can be constructed by solving the linear equations considered in the proofs of the theorems by means of general elimination methods (see [65], [19]). A Scilab® toolbox that implements such methods is described in [91] and is now integrated in the Scicoslab software [97].*

For the feedback version stated in Problem 6, we derived a specialized, more restrictive condition in [11]. The result is the following.

Theorem 4 ([11])**.** *Given a plant* Σ_P *of the form (3.2) and a model* Σ_M *of the form (3.3) as in Theorem 3, the related FMMP is solvable if and only if there exists an* (A_E, B_1)*-invariant subsemimodule* $\mathcal{V}$ *of feedback type contained in the output equalizer subsemimodule* $\mathcal{K}$ *such that for all* $x \in \mathrm{Im}\,B_2 = \mathrm{Im}\begin{pmatrix}\epsilon\\ B_M\end{pmatrix}$ *there exists* $z \in \mathrm{Im}\,B_1 = \mathrm{Im}\begin{pmatrix}B_P\\ \epsilon\end{pmatrix}$ *with* $x \oplus z \in \mathcal{V}$.

Proof. If. Let $\mathcal{V} \subseteq \mathcal{K}$ be an (A_E, B_1)-invariant subsemimodule of feedback type for which the condition of the theorem holds. Then, by hypothesis, there exists a matrix F such that for each $v \in \mathcal{V}$, $(A_E \oplus B_1 F)v$ belongs to $\mathcal{V}$ and a matrix G such that the columns of the matrix $\begin{pmatrix} \epsilon \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P \\ \epsilon \end{pmatrix} G = \begin{pmatrix} B_P G \\ B_M \end{pmatrix}$ belong to $\mathcal{V}$. We can then construct recursively a control input $\{u_P(k)\}_{k\in\mathbb{N}}$, of the form (3.4), for the dynamics (3.5) as

$$u_P(k) = F x_E(k-1) \oplus G u_M(k). \tag{3.10}$$

More precisely, we start by taking $u_P(1) = F x_E(0) \oplus G u_M(1) = G u_M(1)$. Then, we compute $x_E(1)$ by means of (3.5), $x_E(0)$, $u_P(1)$ and $u_M(1)$, and we take $u_P(2) = F x_E(1) \oplus G u_M(2)$. We iterate the same procedure increasing by 1 the index k at each step. The sequence $\{x_E(k)\}_{k\in\mathbb{N}}$ is non-decreasing, since $A_E \geq I$. Since also $\{u_M(k)\}_{k\in\mathbb{N}}$ is assumed to be non-decreasing, the control sequence $u_P(.)$ that is a linear function of $x_E(.)$ and $u_M(.)$ turns out to be non-decreasing, as linear functions are monotone in the max-plus algebra. The resulting state evolution

$$x_E(k) = (A_E \oplus B_1 F) x_E(k-1) \oplus \begin{pmatrix} B_P G \\ B_M \end{pmatrix} u_M(k) \tag{3.11}$$

clearly evolves in $\mathcal{V} \subseteq \mathcal{K}$ and, hence, the FMMP is solved.

Only if. Assume that the FMMP is solved by a control law of the form (3.4). Then, the set of reachable states for the dynamics (3.11) is an (A, B)-invariant subsemimodule of feedback type contained in $\mathcal{K}$ that contains all the columns of the matrix $\begin{pmatrix} B_P G \\ B_M \end{pmatrix} = \begin{pmatrix} \epsilon \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P \\ \epsilon \end{pmatrix} G$. This clearly implies the condition of the theorem. □

Remark 6. *The solvability condition for the FMMP given in Theorem 4 is strictly stronger than the solvability condition for the MMP given in Theorem 3, since any (A, B)-invariant of feedback type $\mathcal{V}$ contained in $\mathcal{K}$ is also contained in $\mathcal{V}^*(\mathcal{K})$ due to the maximality of the latter, and $\mathcal{V}^*(\mathcal{K})$ is not necessarily of feedback type. Therefore, the solvability of the FMMP implies the solvability of the MMP and any solution for the first is a solution also for the second. In particular, if the solvability condition given in Theorem 4 is satisfied, the solution $\{u_P(k)\}_{k\in\mathbb{N}}$ given by (3.10), being non-decreasing, can be expressed as*

$$u_P(k) = \begin{cases} G u_M(1) \text{ for } k = 1 \\ F x_E(k-1) \oplus G u_M(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases} \tag{3.12}$$

Letting $u_1(k) = Fx_E(k)$ and $u_2(k) = Gu_M(k)$, (3.12) shows that the procedure indicated in the proof of Therorem 4 provides the same control input we can have from (3.8) in the proof of Theorem 3.

Remark 7. *In practice, the condition of Theorem 3 can be checked and the elements $u_1(k)$, $u_2(k)$ that are needed in (3.8) to construct the control input $\{u_P(k)\}_{k\in\mathbb{N}}$ can be found by solving systems of linear equations of the form (2.2) that involve the matrices A_E, B_1, B_2 and the generators of $\mathcal{V}^*(\mathcal{K})$. The same holds for the condition of Theorem 3 and for the matrices F, G that are needed in (3.10). An example of how to deal in practice with the computations involved in solving the FMMP is given in the following section.*

3.4 Example

We have shown, in [11], a practical example where the results of Theorem 4 can be used to solve the feedback model matching problem for a max-plus linear system. Such example is reported in this section. Let us consider a plant Σ_P of the form (3.2) described by

$$\Sigma_P \equiv \begin{cases} x_P(k) &= \begin{pmatrix} e & e \\ 2 & e \end{pmatrix} x_P(k-1) \oplus \begin{pmatrix} 2 \\ e \end{pmatrix} u_P(k) \\ y_P(k) &= \begin{pmatrix} e & e \end{pmatrix} x_P(k) \\ x_P(0) &= \epsilon \end{cases}$$

and a model Σ_M of the form (3.3) described by

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 3x_M(k-1) \oplus 4u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} .$$

The related joint dynamics is given by

$$x_E(k) = A_E x_E(k-1) \oplus B_1 u_P(k) \oplus B_2 u_M(k)$$

with

$$A_E = \begin{pmatrix} e & e & \epsilon \\ 2 & e & \epsilon \\ \epsilon & \epsilon & 3 \end{pmatrix},\ B_1 = \begin{pmatrix} 2 \\ e \\ \epsilon \end{pmatrix},\ B_2 = \begin{pmatrix} \epsilon \\ \epsilon \\ 4 \end{pmatrix}$$

The output equalizer subsemimodule $\mathcal{K}$ is easily seen to be given by the set of vectors in $\mathbb{R}^3_{max}$ whose last component is equal to the maximum of the first two components, namely

$$\mathcal{K} = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3_{max} \text{ with } x_1 \oplus x_2 = x_3 \} \tag{3.13}$$

and we also have

$$\mathcal{K} = \operatorname{Im} K = \operatorname{Im} \begin{pmatrix} e & \epsilon \\ \epsilon & e \\ e & e \end{pmatrix}.$$

Note that $\mathcal{K}$ is not A_E-invariant, since e.g.

$$\begin{pmatrix} e & e & \epsilon \\ 2 & e & \epsilon \\ \epsilon & \epsilon & 3 \end{pmatrix} \begin{pmatrix} e \\ \epsilon \\ e \end{pmatrix} = \begin{pmatrix} e \\ 2 \\ 3 \end{pmatrix} \notin \mathcal{K}$$

but it can be proved to be (A_E, B_1)-invariant of feedback type and, hence, $\mathcal{V}^*(\mathcal{K}) = \mathcal{K}$. To show that $\mathcal{K}$ is (A_E, B_1)-invariant of feedback type, we need to solve the two sided linear max-plus system of equations

$$(A_E \oplus B_1 F)K = KQ \tag{3.14}$$

whose set of solutions (F, Q) is a finitely generated max-plus set [65]. This can be done, first, by rewriting (3.14) as a system of max-plus linear equations of the form (2.2) and, then, by solving it by implementing the technique presented in [19]. To accomplish the first step, let us consider separately each column K_i, with $i = 1, 2$, of K in the left side of (3.14), so obtaining a set of equations of the form

$$(A_E \oplus B_1 F)K_i = A_E K_i \oplus B_1 F K_i = KQ_i \text{ for } i = 1, 2. \tag{3.15}$$

Now, for the max-plus product FK_i, we can write $FK_i = \Delta(K_1^\top, K_2^\top)f$ where f is a vector in $\mathbb{R}_{max}^{(n_M+n_P)u_P}$ that consists of all the columns of F stacked on top of each other in lexicographic order and $\Delta(K_1^\top, K_2^\top)$ is a block-diagonal matrix in $\mathbb{R}_{max}^{m_P\times((n_P+n_M)m_P)}$ whose diagonal blocks are all equal to $K_i^\top$. Substituting in (3.15), we get

$$B_1\Delta(K_1^\top, K_2^\top)f \oplus A_E K_i = KQ_i \text{ for } i = 1, 2. \tag{3.16}$$

Such equation can be rewritten as

$$M\xi \oplus \alpha = N\xi \oplus \beta \tag{3.17}$$

where the matrices $M, N \in \mathbb{R}_{max}^{(n_P+n_M)\times((n_P+n_M)m_P+2)}$ have the structure

$$M = \begin{pmatrix} B_1\Delta(K_1^\top, K_2^\top) & \epsilon \end{pmatrix}, \qquad N = \begin{pmatrix} \epsilon & K \end{pmatrix},$$

the constant vectors $\alpha, \beta \in \mathbb{R}_{max}^{n_P+n_M}$ are $\alpha = A_E K_i,\ \beta = \epsilon$, and

$$\xi = \begin{pmatrix} f \\ Q_i \end{pmatrix} \in \mathbb{R}_{max}^{(n_P+n_M)m_P+2}.$$

Equation (3.17) clearly represents a two-sided affine linear equation of the form (2.2) in the vector indeterminate ξ. Using elimination methods, as mentioned in Remark 7, it is possible to find a solution of (3.17) and then a solution

$$(F,Q) = (\begin{pmatrix}1 & 1 & \epsilon\end{pmatrix}, \begin{pmatrix}3 & 3\\ 2 & 1\end{pmatrix})$$

of (3.14) (we do not report here the single elementary steps of the computation, but the reader can easily verify that the pair (F,Q) given above solves the equation). In particular, the control law $u_P(k+1) = Fx_E(k)$ keeps inside $\mathcal{K}$ any state evolution which starts in $\mathcal{K}$.

The condition of Theorem 4 is verified for $\mathcal{V} = \mathcal{V}^*(\mathcal{K}) = \mathcal{K}$. In fact, any $x \in \operatorname{Im} B_2$ is of the form

$$x = B_2\, a = \begin{pmatrix}\epsilon\\ \epsilon\\ 4\end{pmatrix} a = \begin{pmatrix}\epsilon\\ \epsilon\\ 4a\end{pmatrix} \text{ with } a \in \mathbb{R}_{max}$$

and, taking

$$z = B_1\, b = \begin{pmatrix}2\\ e\\ \epsilon\end{pmatrix} b = \begin{pmatrix}2b\\ b\\ \epsilon\end{pmatrix} \text{ with } b = 2a \in \mathbb{R}_{max},$$

we have that $x \oplus z = \begin{pmatrix}4a\\ 2a\\ 4a\end{pmatrix}$ belongs to $\mathcal{K}$. In particular, the columns of the matrix $\begin{pmatrix}\epsilon\\ B_M\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_P\\ \epsilon\end{pmatrix} G = \begin{pmatrix}B_P G\\ B_M\end{pmatrix} = \begin{pmatrix}2G\\ G\\ 4\end{pmatrix}$ belongs to $\mathcal{K}$ for $G = (2)$. Hence, according to the proof of Theorem 4, a solution to the FMMP turns out to be given by the control law

$$\begin{aligned} u_P(k) &= Fx_E(k-1) \oplus Gu_M(k)\\ &= (1 \quad 1 \quad \epsilon)x_E(k-1) \oplus 2u_M(k)\\ &= 1x_{P1}(k-1) \oplus 1x_{P2}(k-1) \oplus 2u_M(k) \end{aligned} \tag{3.18}$$

that is of the form (3.4). In fact, by substituting $u_P(k)$ with the above expression in the joint dynamics (3.5) we get

$$\begin{pmatrix}x_{P1}(k)\\ x_{P2}(k)\\ x_M(k)\end{pmatrix} = \begin{pmatrix}3 & 3 & \epsilon\\ 2 & 1 & \epsilon\\ \epsilon & \epsilon & 3\end{pmatrix} \begin{pmatrix}x_{P1}(k-1)\\ x_{P2}(k-1)\\ x_M(k-1)\end{pmatrix} \oplus \begin{pmatrix}4\\ 2\\ 4\end{pmatrix} u_M(k)$$

and

$$\begin{aligned} y_P(k) &= x_{P1}(k) \oplus x_{P2}(k) \\ &= 3x_{P1}(k-1) \oplus 3x_{P2}(k-1) \oplus 4u_M(k) \\ &= 3y_P(k-1) \oplus 4u_M(k); \\ y_M(k) &= x_M(k) \\ &= 3x_M(k-1) \oplus 4u_M(k) \\ &= 3y_M(k-1) \oplus 4u_M(k). \end{aligned}$$

From the last expression, since $x_P(0) = x_M(0) = \epsilon$, it is easy to see by induction that $\{y_P(k)\}_{k\in\mathbb{N}}$ is equal to $\{y_M(k)\}_{k\in\mathbb{N}}$. Obviously, the found solution of the FMMP is a solution also of the MMP and, as explained in Remark 6, the control law (3.18) satisfy the requirement of Problem 5.

Chapter 4

Max-plus switching linear systems

The max-plus linear systems considered in the previous chapters are stationary, so they are time-invariant and event-invariant. The first property refer to the fact that the system behaviour is the same in every time instant, while the second property takes into account the variations of the structure in the events domain. An event-variant system is such that the effects of the events depend on the instance index of the considered events. For instance, a model for a crossroad controlled by a traffic light has a time-varying behaviour because the color of the traffic light depends on time and not on the sequential number of the car that is passing through the considered intersection. A different situation can arise in production plants where the allocation of the available resources rely on the sequential number of the part that is currently in production, as a part that has undergone some work with the plant in a given configuration will have to coherently complete all the subsequent production phases. In the rest of the work our objective is to model in an appropriate way, using the max-plus algebra, discrete event systems that are event-varying.

4.1 State of the art and applications

As discussed in the previous chapter, max-plus linear systems can be straightforward used to model synchronization phenomena, but only in absence of shared resources. The extension of the max-plus formalism to the event-varying case allow to consider more complex situations, in which conflicts arise and are solved using an approach that depends on the index of the current event. Moreover, we can consider more general scenarios in which subsequent inputs of the system need to follow a different path inside the plant. In order to model such situations, several approaches have been proposed by different authors. Each of these approaches has some advantages and some limitations and refers to a different set of dynamic systems. In [45] a formalism, based on the height of heaps of pieces, is introduced, in order to compare in an efficient way the evolution of safe jobshops for different working sequences. In [3] conflicting timed event graphs are modeled as max-plus linear systems by adding a new term to the dynamics equation whose aim is to model the availability of a shared resource and its allocation to a specific process.

We will follow a different approach, based on the introduction of switching max-plus linear systems. The first use of such formalism can be traced back to [110] in order to model max-plus linear systems that can switch between different modes of operation. The switching mechanism can be exploited to model changes in the structure of the system, to break the synchronization or to modify the order of events.

Since their introduction, switching max-plus linear systems have found various applications. For instance, in [82] such formalism is used to model gait reference generators for legged autonomous robots. The synchronization constraints that define a specific gait in the motion of a robot with several legs are expressed as max-plus linear equations, obtaining a model that is a max-plus linear system. The transition between different gaits, that is needed in a number of situations, is modeled as the switching between different system matrices, obtaining a switching max-plus linear system. The obtained gait reference generator is then used to design a dynamic tracking controller for the legged robot.

In [66] the Dutch railway network is modeled as a switching max-plus linear system and an appropriate strategy to determine optimal dispatching actions, based on the model predictive control, is proposed. The obtained model is pretty general and shows the great modeling power of such kind of systems. The order of the trains can be changed, connections can be broken, joined trains can be split up, and trains can change track if there are multiple tracks available.

Recently, switching max-plus linear systems have been used to model cubes packing systems where different manipulators can be selected for the handling of a given resource, taking into account possible faults of the system [87]. In the same work a fault-tolerant control is proposed and compared with a simpler, but less effective, model predictive control strategy.

In [22] a control technique, called event-driven Model Predictive Control (e-MPC) is demonstrated to be effective in minimizing the work in progress in serial production system while overcoming some inconveniences that affect alternative control strategies. In that work the system under study is modeled as a time-varying max-plus linear system in order to take into account unexpected events that can affect the evolution of the plant.

The problem of controlling a switching max-plus linear system in order to fulfill strict time constraints relative to the temporal distance between some events of the system has been solved by means of a causal state feedback in [1]. The authors provide also an example of application of the proposed theory to a crossing railways system.

In the case of shared resources or conflicts the system evolution depends on the choice relative to resource allocation. Different problems can be considered, for instance the resource allocation can be chosen in order to maximize some performance metrics. In this case the aim is to compare the possible evolution of the system relative to different choices for conflicts resolution.

A different situation arise when the switching sequence is given and the aim is to control the event-varying system by generating an appropriate input by means of a control mechanism. This situation has been considered in [111], where a solution based on model predictive control is proposed.

The approach of this thesis, instead, is relative to the study of the structural properties of the systems. These are independent from the specific resource allocation policy, which is seen as an exogenous decision. Using this approach, we obtain control techniques that are applicable for each possible resource allocation decision. This is made possible by the development of a structural geometric approach for switching max-plus linear systems. As far as we know, the extension of the geometric approach to systems of the class considered here has never been tackled before our work on the topic [12].

4.2 System model

An event-varying, switching max-plus linear system is a max-plus system in which the matrices that define the linear maps between the input, internal and output daters of the system can assume different values for different event instance indexes $k \in \mathbb{N}$. The possible values for such matrices are taken by a finite set, whose cardinality is denoted as I.

The formal representation of that idea leads to a system Σ described by equations of the form

$$\Sigma \equiv \begin{cases} x(k) = A_{\sigma(k)}x(k-1) \oplus B_{\sigma(k)}u(k) \\ y(k) = C_{\sigma(k)}x(k) \\ x(0) = \epsilon \end{cases} \tag{4.1}$$

whose variables have the meanings described in the following. As in the case of stationary max-plus linear systems, $k \in \mathbb{N}$ is the event instance index, $x(.) : \mathbb{N} \to \mathcal{X} = \mathbb{R}^n_{max}$, $u(.) : \mathbb{N} \to \mathcal{U} = \mathbb{R}^m_{max}$ and $y(.) : \mathbb{N} \to \mathcal{Y} = \mathbb{R}^p_{max}$ are the daters of internal, input and output events, respectively. Every possible configuration for the switching system Σ is a max-plus linear system and is called a *mode* of the switching system. In particular, at each $k \in \mathbb{N}$ the relation between the input, internal and output daters is that of a max-plus linear system, called the *active mode* for the event index k. The function $\sigma(k) : \mathbb{N} \to \mathcal{I} = \{1, ..., I\}$ is the relation that associates to each $k \in \mathbb{N}$ the index of the active mode. $\{A_i\}_{i \in \mathcal{I}}$ is a family of matrices whose entries $A_i \in \mathbb{R}^{n \times n}_{max}$ are the dynamic matrices of the different modes of the switching system. Similar considerations hold for $B_{\sigma(k)} \in \mathbb{R}^{n \times m}_{max}$, and $C_{\sigma(k)} \in \mathbb{R}^{p \times n}_{max}$ that are the input and output matrices for the mode $\sigma(k)$. The semi-modules $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are the state, input and output semimodules, respectively. The internal dater of the system $x(k)$, like all daters, must be non-decreasing in order to be physically realizable. For this reason, it is of limited interest to consider systems that admit as a possible state evolution sequences that are not monotone or decreasing. The following definition provides a basic requirement for the systems we consider in this work.

Definition 7 (Non-anticipative)**.** *We say that a switching max-plus linear system of the form 4.1 is non-anticipative, if $A_i \geq I_n$ for all $i \in \mathcal{I}$.*

Non-anticipative systems are such that, for each possible input sequence $\{u(k)\}_{k \in \mathbb{N}}$ and switching sequence $\{\sigma(k)\}_{k \in \mathbb{N}}$, the dater of internal events $x(k)$ is non-decreasing.

Remark 8. *Given the structure of max-plus linear systems, together with the fact that all the daters must be non-decreasing, we obtain a first-in first-out (FIFO) behaviour and, contrary to what happens for stationary systems, this represents an effective limitation in the modeling power of switching max-plus linear systems.*

As an illustrative example, let us consider the case of a plant that can process input raw parts of a single type in two different modes. In the first mode the parts need to be processed by a machine M1 and then by a machine M2, while in the second mode the operation on M1 is skipped and the parts get processed immediately by M2, before leaving the plant. Both M1 and M2 can process only one part at a time. If the first raw input part P1 is worked in the first mode, it could happen that, when the second raw part P2 become available in the input of the plant, P1 is still in the machine M1. In practice, would be reasonable to imagine that, if P2 needs to skip the process on M1, then its working starts immediately in the machine M2, overtaking P1 that *is still in the machine M1. However, the overtaking cannot be modeled using the techniques described in this work, at least not directly. Going back to the example, if we want to represent the output of such a system by means of a non-decreasing dater $y(.)$, then the instant in which the first finished part exits from the system should be computed as $y(1)$. However, if we want to take possible overtaking into account, this value depends on the mode of the plant that will be active at $k = 2$, and this dependence is not present in the structure of switching max-plus linear systems defined in (4.1). If the plant is modeled using the techniques presented here, then P2 must wait that P1 gets processed by the machines M1 and M2, because M2 can process P2 only after having completed P1.*

4.3 Example

In this section we show how, in practice, a manufacturing plant can be modeled as a switching linear system. In doing this we use an extended version of an example that we have developed for a paper on the topic [12].

Let us consider a plant that receives raw components in pairs. The pieces of each pair are separated and fed into machine M1 and M2 respectively. The machine M1 takes 2 time units to perform its operation. The machine M2 usually takes 1 time unit, but sometimes additional activities are performed in order to obtain an acceptable quality for the semi-finished product. In this case, the machine M2 requires 3 time units to perform its activity. After the completion of the activities by M1 or M2, the semi-finished parts are processed by the machines M3 or M4. These machines perform the same activity in 2 and 4 time units respectively (so M3 is faster than M4). A switch allows to choose whether the part exiting from M1 should enter the machine M3 or M4. The other piece of the original pair, after being processed in M2, will go in the other machine (M4 or M3 respectively). We assume that the switching mechanism is controlled from outside of the system and it is operated in order to optimize some production metrics. The pair of semi-finished product is then assembled into a single finished product by the machine M5, in 1 time unit. The situation is graphically represented in Figure 4.1. All the machines involved can process only one part at a time, and buffers of very big (assumed as infinite) capacity are present at each stage of the plant.

The only type of input event is “arrival of a pair of raw components”. We can consider as internal events the ones of type “completion of a cycle by the machine M1”, “completion of a cycle by the machine M2”, “completion of a cycle by the machine M3”, and “completion of a cycle by the machine M4”.

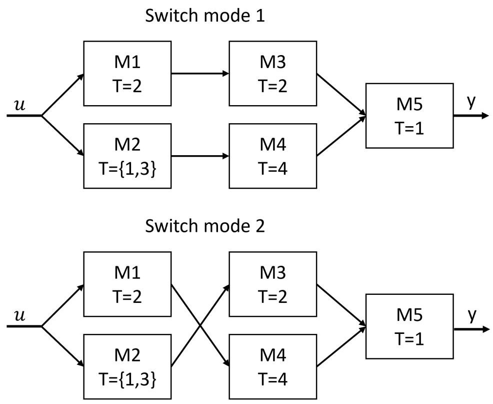

Figure 4.1: Representation of the plant.

$u(k)$	arrival time of a pair of raw components
$x_1(k)$	completion time of the k-th cycle on the machine M1
$x_2(k)$	completion time of the k-th cycle on the machine M2
$x_3(k)$	completion time of the k-th cycle on the machine M3
$x_4(k)$	completion time of the k-th cycle on the machine M4
$x_5(k)$, $y(k)$	completion time of the k-th cycle on the machine M5

Table 4.1: Daters and events.

The event "completion of a cycle by the machine M5" is both an internal event and an output event. We can associate a dater function to each of this events, as reported in Table 4.1.

We can model the plant as a switching max-plus system with 4 modes that describe the possible configurations:

1. M2 requires 1 time unit and its output goes to M4;
2. M2 requires 3 time units and its output goes to M4;
3. M2 requires 1 time unit and its output goes to M3;
4. M2 requires 3 time units and its output goes to M3.

We start by modeling the mode 1 of the switching system. Events of type "completion of a cycle on the machine M1" can occur, in this configuration of the system, only after two time units the previous activity on M1 completed and a new input raw part became available. Assuming that all the activities start as soon as possible, we can express this statement by the following equation:

$$\begin{aligned} x_1(k) &= \max\{x_1(k-1), u(k)\} + 2 = \\ &= \max\{x_1(k-1)+2, u(k)+2\} \end{aligned}$$

and, using similar arguments, we have

$$\begin{aligned} x_2(k) &= \max\{x_2(k-1)+1, u(k)+1\} \\ x_3(k) &= \max\{x_1(k)+2, x_3(k-1)+2\} \\ x_4(k) &= \max\{x_2(k)+4, x_4(k-1)+4\} \\ x_5(k) &= \max\{x_3(k)+1, x_4(k)+1, x_5(k-1)+1\} \\ y(k) &= x_5(k), \end{aligned}$$

and, using the max-plus algebra, we can write

$$\begin{cases} x_1(k) = 2 \otimes x_1(k-1) \oplus 2 \otimes u(k) \\ x_2(k) = 1 \otimes x_2(k-1) \oplus 1 \otimes u(k) \\ x_3(k) = 2 \otimes x_1(k) \oplus 2 \otimes x_3(k-1) \\ x_4(k) = 4 \otimes x_2(k) \oplus 4 \otimes x_4(k-1) \\ x_5(k) = 1 \otimes x_3(k) \oplus 1 \otimes x_4(k) \oplus 1 \otimes x_5(k-1) \\ y(k) = x_5(k) \end{cases}.$$

Such equations can be written in the standard form (4.1) by removing all the references to $x_i(k)$ on the right-hand terms of all of them. This can be done using the approach based on the Kleene star operator, as in the example provided in Section 2.4, or by substitution. We use here the latter approach. For instance, we rewrite the equation for $x_3(k)$ as

$$\begin{aligned} x_3(k) &= 2 \otimes (2 \otimes x_1(k-1) \oplus 2 \otimes u(k)) \oplus 2 \otimes x_3(k-1) = \\ &= 4 \otimes x_1(k-1) \oplus 2 \otimes x_3(k-1) \oplus 4 \otimes u(k) \end{aligned}$$

and the same can be done for $x_4(k)$, and, finally $x_5(k)$. The resulting system can be written using a matrix notation in the form (4.1) with

$$A_1 = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon & \epsilon & \epsilon \\ 4 & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & 5 & \epsilon & 4 & \epsilon \\ 5 & 6 & 3 & 5 & 1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \; C_1 = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix}.$$

And the same reasoning can be applied to the other configurations of the system, obtaining

$$A_2 = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 3 & \epsilon & \epsilon & \epsilon \\ 4 & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & 7 & \epsilon & 4 & \epsilon \\ 5 & 8 & 3 & 5 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon & \epsilon & \epsilon \\ \epsilon & 3 & 2 & \epsilon & \epsilon \\ 6 & \epsilon & \epsilon & 4 & \epsilon \\ 7 & 4 & 3 & 5 & 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 3 & \epsilon & \epsilon & \epsilon \\ \epsilon & 5 & 2 & \epsilon & \epsilon \\ 6 & \epsilon & \epsilon & 4 & \epsilon \\ 7 & 6 & 3 & 5 & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 2 & 3 & 4 & 7 & 8 \end{pmatrix}^\top, B_3 = \begin{pmatrix} 2 & 1 & 3 & 6 & 7 \end{pmatrix}^\top, B_4 = \begin{pmatrix} 2 & 1 & 3 & 6 & 7 \end{pmatrix}^\top$$

$$C_i = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix} \text{ for all } i \in \mathcal{I} = \{1, 2, 3, 4\}.$$

4.4 Geometric approach

The geometric approach, together with the concepts of invariance and controlled invariance, can be extended to the framework of switching max-plus linear systems, as shown by us in a recent paper [12]. In this section we analyze how such generalization can be carried out. We take for granted the general concepts underlying the structural geometric approach for dynamic systems, as those illustrated in Section 3.1 apply also to the switching case.

We begin our generalization process with the definition of invariant semimodule for switching max-plus linear systems.

Definition 8 (Invariant semimodule)**.** *Given a switching max-plus linear system Σ of the form (4.1), a semimodule $\mathcal{V} \subseteq \mathbb{R}^n_{max}$ is said to be invariant, or A_σ-invariant, if for all $v \in \mathcal{V}$ and for all $i \in \mathcal{I}$, $A_i v$ belongs to $\mathcal{V}$.*

Also the basic notion of (A_σ, B_σ)-invariant semimodule can be easily generalized.

Definition 9 (Controlled invariant semimodule [12])**.** *Given a switching max-plus linear system Σ of the form (4.1), a semimodule $\mathcal{V} \subseteq \mathbb{R}^n_{max}$ is said to be controlled invariant, or (A_σ, B_σ)-invariant, if for all $v \in \mathcal{V}$ and for all $i \in \mathcal{I}$ there exists $u \in \mathbb{R}^m_{max}$ such that $(A_i v \oplus B_i u)$ belongs to $\mathcal{V}$.*

As in the case of stationary systems, given a switching max-plus system Σ of the form (4.1) and a subsemimodule of its state semimodule $\mathcal{K} \subseteq \mathbb{R}^n_{max}$, the set of all the (A_σ, B_σ)-invariant semimodules contained in $\mathcal{K}$ is a semilattice with respect to inclusion and sum of semimodules. Hence, a maximum element of that set, denoted by $\mathcal{V}^*(\mathcal{K})$, exists.

Theorem 5 ([12])**.** *The sequence of semimodules $\mathcal{V}_r$ defined by*

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{K}, \\ \mathcal{V}_r &= \mathcal{V}_{r-1} \cap \left(\bigcap_{i \in \mathcal{I}} A_i^{-1}(\mathcal{V}_{r-1} \ominus \mathrm{Im} B_i)\right), \end{aligned} \tag{4.2}$$

where $A_i^{-1}(\mathcal{Y}) = \{v \in \mathbb{R}^n_{max}, \text{ such that } A_i v \in \mathcal{Y}\}$ and $\mathcal{V}_{r-1} \ominus \mathrm{Im}\, B_i = \{x \in \mathbb{R}^n_{max}, \text{ for which there exists } u \in \mathbb{R}^m_{max} \text{ such that } x \oplus B_i u \in \mathcal{V}_{r-1}\}$, has the following properties:

1. *$\mathcal{V}_r \subseteq \mathcal{V}_{r-1}$ for all $r \in \mathbb{N}$;*

2. *letting* $\mathcal{V}_\infty = \lim_{r\to\infty}\mathcal{V}_r = \bigcap_{r\in\mathbb{N}} \mathcal{V}_r$*, then every* (A_σ, B_σ)*-invariant semimodule contained in* $\mathcal{K}$ *is also contained in* $\mathcal{V}_\infty$*;*

3. $\mathcal{V}_r = \mathcal{V}_{r-1}$ *if and only if* $\mathcal{V}_{r-1}$ *is an* (A_σ, B_σ)*-invariant semimodule and, in such case,* $\mathcal{V}_\infty = \mathcal{V}_{r-1} = \mathcal{V}^*(\mathcal{K})$*.*

Proof. (1) The statement is a consequence of the definition of $\mathcal{V}_r$.
(2) Let $\mathcal{X} \subseteq \mathcal{K} = \mathcal{V}_0$ be an (A_σ, B_σ)-invariant semimodule and assume that, for some $r \in \mathbb{N}$, we have $\mathcal{X} \subseteq \mathcal{V}_{r-1}$. Then, since $\mathcal{X} \subseteq A_i^{-1}(\mathcal{X} \ominus \mathrm{Im}B_i) \subseteq A_i^{-1}(\mathcal{V}_{r-1} \ominus \mathrm{Im}B_i)$, we also have $\mathcal{X} \subseteq \mathcal{V}_r$ and the conclusion follows by induction.
(3) The relation $\mathcal{V}_r = \mathcal{V}_{r-1}$ holds if and only if $\mathcal{V}_{r-1} \subseteq \bigcap_{i\in\mathcal{I}} A_i^{-1}(\mathcal{V}_{r-1} \ominus \mathrm{Im}B_i)$, which, in turn, holds if and only if $\mathcal{V}_{r-1} \subseteq A_i^{-1}(\mathcal{V}_{r-1} \ominus \mathrm{Im}B_i)$ for all $i \in I$. This is equivalent to the fact that $\mathcal{V}_{r-1}$ is an (A_σ, B_σ)-invariant semimodule. In the considered case, the equality $\mathcal{V}_\infty = \mathcal{V}_{r-1}$ is obvious and $\mathcal{V}_\infty = \mathcal{V}^*(\mathcal{K})$ follows from (2). □

The algorithm provided by Theorem 5 suffers from the same issue analyzed for stationary systems, in fact the sequence (4.2) does not necessarily converge in a finite number of steps and it cannot be considered a general algorithm for the computation of $\mathcal{V}^*(\mathcal{K})$. Also, Remark 1 is still valid for the switching case and we can assert that if $\mathcal{K}$ is finitely generated, so are all the semimodules $\mathcal{V}_r$ in the sequence (4.2).

Definition 10 (Controlled invariant of feedback type [12])**.** *Given a switching max-plus linear system* Σ *of the form (4.1), a semimodule* $\mathcal{V} \subseteq \mathbb{R}^n_{max}$ *is said to be controlled invariant of feedback type, or* (A_σ, B_σ)*-invariant of feedback type, if there exists a family of matrices* $F_i \in \mathbb{R}^{m\times n}_{max}$ *such that* $(A_i \oplus B_i F_i)v$ *belongs to* $\mathcal{V}$ *for all* $v \in \mathcal{V}$ *and for all* $i \in \mathcal{I}$*.*

As happens in the stationary case, the property of controlled invariance of feedback type implies the property of controlled invariance, but the converse is not true.

4.5 Invariant families of semimodules

As illustrated in [14], in the framework of switching max-plus linear systems, it is useful to consider an alternative definition for the concept of invariant. A family of semimodules, and not a single semimodule, can be considered as an invariant for the system throughout its evolution. As we will see in Section 4.7, such approach leads to necessary and sufficient results, while considering a single invariant semimodule leads to more conservative conditions.

For switching max-plus systems of the form (4.1), we consider families of subsemimodules of the state semimodule, indexed by the set $\mathcal{I} = \{1, ..., I\}$ and denote a family of semimodules of such kind, coherently with the notation used for family of matrices, by using the switching function as a subscript, for instance $\mathcal{V}_\sigma$. We denote as $\mathcal{V}_i$ the semimodule of the family $\mathcal{V}_\sigma$ associated to a given $i \in \mathcal{I}$. Because families of semimodules are widely used in the following, it is convenient to introduce some notations to improve the readability of the provided results. When no confusion can arise we refer to family of semimimodules simply as semimodules. For instance, we refer to both $\mathcal{V}_\sigma$ and $\mathcal{X}$ as semimodules, but the first is a family of semimodules, while the latter is a proper semimodule.

We say that a family of semimodules $\mathcal{E}_\sigma$ is a subsemimodule of a semimodule $\mathcal{X}$, and denote such relation as $\mathcal{E}_\sigma \subseteq \mathcal{X}$, if $\mathcal{E}_i \subseteq \mathcal{X}$ for each $i \in \mathcal{I}$. Moreover, given two families of subsemimodules $\mathcal{E}_\sigma, \mathcal{I}_\sigma \subseteq \mathbb{R}^n_{max}$ we say that:

- $\mathcal{E}_\sigma$ is finitely generated if, for all $i \in \mathcal{I}$, $\mathcal{E}_i$ is finitely generated;
- $\mathcal{E}_\sigma$ is equal to $\mathcal{I}_\sigma$, or $\mathcal{E}_\sigma = \mathcal{I}_\sigma$, if $\mathcal{E}_i = \mathcal{I}_i$ for all $i \in \mathcal{I}$;
- $\mathcal{E}_\sigma$ is contained in $\mathcal{I}_\sigma$, or $\mathcal{E}_\sigma \subseteq \mathcal{I}_\sigma$, if $\mathcal{E}_i \subseteq \mathcal{I}_i$ for all $i \in \mathcal{I}$;
- a family of semimodules $\mathcal{S}_\sigma$ is the sum of $\mathcal{E}_\sigma$ and $\mathcal{I}_\sigma$, if $\mathcal{S}_i$ is the sum of $\mathcal{E}_i$ and $\mathcal{I}_i$ for all $i \in \mathcal{I}$;
- a family of semimodules $\mathcal{R}_\sigma$ is the intersection of $\mathcal{E}_\sigma$ and $\mathcal{I}_\sigma$, or $\mathcal{R}_\sigma = \mathcal{E}_\sigma \cap \mathcal{I}_\sigma$, if $\mathcal{R}_i = \mathcal{E}_i \cap \mathcal{I}_i$ for all $i \in \mathcal{I}$.

We can now introduce the notion of (A_σ, B_σ)-invariant family of semimodules for switching max-plus linear systems.

Definition 11 ([14]). *Given a switching max-plus linear system Σ of the form (4.1), a family of semimodules $\mathcal{V}_\sigma \subseteq \mathbb{R}^n_{max}$ is said to be (A_σ, B_σ)-invariant if for all $i, j \in \mathcal{I}$ and for all $v \in \mathcal{V}_j$ there exists $u \in \mathbb{R}^m_{max}$ such that $(A_i v \oplus B_i u)$ belongs to $\mathcal{V}_i$.*

Given a switching max-plus system Σ of the form (4.1) and a subsemimodule of its state semimodule $\mathcal{K}_\sigma \subseteq \mathbb{R}^n_{max}$, the set of all the (A_σ, B_σ)-invariant families of semimodules contained in $\mathcal{K}_\sigma$ is a semi-lattice with respect to inclusion and sum of families of semimodules. Hence, a maximum element of that set, denoted by $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$, exists. The following theorem provides, under suitable hypoteses, a procedure to compute $\mathcal{V}^*_\sigma(\mathcal{K}_\sigma)$.

Theorem 6 ([14]). *The sequence of families of semimodules $\mathcal{V}_\sigma^r$ defined by*

$$\begin{array}{rcl} \mathcal{V}_i^0 &=& \mathcal{K}_i, \\ \mathcal{V}_i^r &=& \mathcal{V}_i^{r-1} \cap (\bigcap_{j\in\mathcal{I}} A_j^{-1}(\mathcal{V}_j^{r-1} \ominus \mathrm{Im} B_j)), \end{array} \tag{4.3}$$

where $i \in \mathcal{I} = \{1, ..., I\}$, $A_i^{-1}(\mathcal{Y}) = \{v \in \mathbb{R}^n_{max}, \text{ such that } A_i v \in \mathcal{Y}\}$ and $\mathcal{V}_i^{r-1} \ominus \mathrm{Im}\, B_i = \{x \in \mathbb{R}^n_{max}, \text{ for which there exists } u \in \mathbb{R}^m_{max} \text{ such that } x \oplus B_i u \in \mathcal{V}_i^{r-1}\}$, has the following properties:

1. *$\mathcal{V}_\sigma^r \subseteq \mathcal{V}_\sigma^{r-1}$ for all $r \in \mathbb{N}$;*
2. *letting $\mathcal{V}_\sigma^\infty = \lim_{r\to\infty} \mathcal{V}_\sigma^r = \bigcap_{r\in\mathbb{N}} \mathcal{V}_\sigma^r$, then every (A_σ, B_σ)-invariant family of semimodules contained in $\mathcal{K}_\sigma$ is also contained in $\mathcal{V}_\sigma^\infty$;*
3. *$\mathcal{V}_\sigma^r = \mathcal{V}_\sigma^{r-1}$ if and only if $\mathcal{V}_\sigma^r$ is an (A_σ, B_σ)-invariant family of semimodules and, in such case, $\mathcal{V}_\sigma^\infty = \mathcal{V}_\sigma^r = \mathcal{V}_\sigma^*(\mathcal{K}_\sigma)$.*

Proof. (1) The statement is a consequence of the definition of $\mathcal{V}_\sigma^r$.
(2) Let $\mathcal{P}_\sigma \subseteq \mathcal{K}_\sigma = \mathcal{V}_\sigma^0$ be an (A_σ, B_σ)-invariant family of semimodules and assume that, for some $r \in \mathbb{N}$, we have $\mathcal{P}_\sigma \subseteq \mathcal{V}_\sigma^{r-1}$. Then, since $\mathcal{P}_\sigma \subseteq A_i^{-1}(\mathcal{P}_i \ominus \mathrm{Im} B_i) \subseteq A_i^{-1}(\mathcal{V}_i^{r-1} \ominus \mathrm{Im} B_i)$ for all $i \in \mathcal{I}$, we also have $\mathcal{P}_\sigma \subseteq \mathcal{V}_\sigma^r$ and the conclusion follows by induction.
(3) The relation $\mathcal{V}_\sigma^r = \mathcal{V}_\sigma^{r-1}$ holds if and only if $\mathcal{V}_\sigma^{r-1} \subseteq \bigcap_{i\in\mathcal{I}} A_i^{-1}(\mathcal{V}_i^{r-1} \ominus \mathrm{Im} B_i)$, which, in turn, holds if and only if $\mathcal{V}_j^{r-1} \subseteq A_i^{-1}(\mathcal{V}_i^{r-1} \ominus \mathrm{Im} B_i)$ for all $i, j \in I$. This is equivalent to the fact that $\mathcal{V}_\sigma^{r-1}$ is an (A_σ, B_σ)-invariant family of semimodules. In the considered case, the equality $\mathcal{V}_\sigma^\infty = \mathcal{V}_\sigma^{r-1} = \mathcal{V}_\sigma^r$ is obvious and $\mathcal{V}_\sigma^\infty = \mathcal{V}_\sigma^*(\mathcal{K}_\sigma)$ follows from (2). □

As usual, the sequence (4.3) does not necessarily converge in a finite number of steps and it cannot be considered a general algorithm for the computation of $\mathcal{V}^*(\mathcal{K})$. Also, Remark 1 is still valid and we can assert that if $\mathcal{K}$ is finitely generated, so are all the semimodules $\mathcal{V}_r$ in the sequence (4.3).

The maximal invariant family contained in a given family of semimodules enjoy a property that will be useful in proving some results in Section 4.7 and is stated in the following technical lemma.

Lemma 2 ([14]). *The maximal (A_σ, B_σ)-invariant family of semimodules $\mathcal{V}_\sigma^*$ contained in a family of semimodules $\mathcal{K}_\sigma$ fulfill the property*

$$(\mathcal{V}_j^* \setminus \mathcal{V}_i^*) \cap \mathcal{K}_i = \emptyset \textit{ for all } i, j \in \mathcal{I} \tag{4.4}$$

Proof. Given some $i, j \in \mathcal{I}$, we have that a vector v_i belongs to $\mathcal{V}_i^*$ if and only if $v_i \in \mathcal{K}_i$ and for each $q \in \mathcal{I}$ there exists $u_q \in \mathbb{R}^m_{max}$ such that $A_q v_i \oplus B_q u_q$ belongs to $\mathcal{V}_q^*$. Analogously, v_j belongs to $\mathcal{V}_j^*$ if and only if it belongs to $\mathcal{K}_j$

and for each $q \in \mathcal{I}$ there exists $u_q \in \mathbb{R}^m_{max}$ such that $A_q v_j \oplus B_q u_q$ belongs to $\mathcal{V}_q^*$. The only difference between v_i and v_j is that the former belongs to $\mathcal{K}_i$, while the latter belongs to $\mathcal{K}_j$, but the other condition required on them is identical. We can state that if a vector $z \in \mathbb{R}^n_{max}$ belongs to $\mathcal{K}_i$ and $\mathcal{V}_j$, it must also belong to $\mathcal{V}_i$, and the conclusion is straightforward. □

4.6 Model matching problem

Similarly to the situation described in Section 3.2 for stationary systems, the existing literature has considered the problem of tracking a specific output in terms of due-date signal. In practice, the reference is not interpreted as a sequence that need to be matched exactly, but as a dead-line not to be exceeded, possibly also delaying the control input as much as possible in a just-in-time fashion [111], [9]. However, the literature available for switching systems is much more limited.

In [9] the just-in-time control of switching max-plus linear system is solved by means of the residuation theory [18]. The same problem is considered, with a similar approach, in [36] for max-plus linear systems with set-based constraints. Systems of such class are non-stationary max-plus systems that can be practically modeled as max-plus switching linear systems, however such representation is not directly employed, and a different modeling technique, based on transfer matrices, is considered by the authors.

Coherently with the approach used for stationary systems in Section 3.2 our formulation of the MMP differs from the one mainly adopted in the literature as we require an exact correspondence between the output of the plant and that of the model. Such approach leads to a definition of the model matching problem that is a straightforward generalization of the classical one employed in the framework of dynamic systems over a field.

Problem 3 (Model Matching Problem (MMP) [12]). *Given a non-anticipative switching max-plus linear system*

$$\Sigma_P \equiv \begin{cases} x_P(k) &= A_{P\sigma(k)} x_P(k-1) \oplus B_{P\sigma(k)} u_P(k) \\ y_P(k) &= C_{P\sigma(k)} x_P(k) \\ x_P(0) &= \epsilon \end{cases} \tag{4.5}$$

of the form (4.1), called the plant, and a non-anticipative switching max-plus linear system

$$\Sigma_M \equiv \begin{cases} x_M(k) &= A_{M\sigma(k)} x_M(k-1) \oplus B_{M\sigma(k)} u_M(k) \\ y_M(k) &= C_{M\sigma(k)} x_M(k) \\ x_M(0) &= \epsilon \end{cases} \tag{4.6}$$

of the form (4.1), called the model, with $x_P : \mathbb{N} \to \mathbb{R}^{n_P}_{max}$*,* $x_M : \mathbb{N} \to \mathbb{R}^{n_M}_{max}$*,* $u_P : \mathbb{N} \to \mathbb{R}^{m_P}_{max}$*,* $u_M : \mathbb{N} \to \mathbb{R}^{m_M}_{max}$ *and* $y_P, y_M : \mathbb{N} \to \mathbb{R}^{p}_{max}$*, the model matching problem consists in finding, for all possible non-decreasing input sequences* $\{u_M(k)\}_{k\in\mathbb{N}}$ *of the model and all possible switching sequences* $\{\sigma(k)\}_{k\in\mathbb{N}}$*, a non-decreasing control input sequence* $\{u_P(k)\}_{k\in\mathbb{N}}$ *for the plant, such that the output* $\{y_P(k)\}_{k\in\mathbb{N}}$ *of this latter equals the output* $\{y_M(k)\}_{k\in\mathbb{N}}$ *of the model, i.e.* $y_P(k) = y_M(k)$ *for all* $k \in \mathbb{N}$*.*

As in Section 3.2, a more restrictive formulation of the MMP is obtained by requiring the control input $u_P(k)$ to be, for each value of the switching signal $\sigma(k)$, a linear function of the state of both the plant and the model (i.e. $x_P(k-1)$ and $x_M(k-1)$) as well as of the input of the model $u_M(k)$. We refer to this formulation of the problem as the feedback model matching problem.

Problem 4 (Feedback Model Matching Problem (FMMP) [12])**.** *Given a plant of the form (5.3) and a model of the form (5.4), the feedback model matching problem consists in finding, for all possible non-decreasing input sequences* $\{u_M(k)\}_{k\in\mathbb{N}}$ *of the model and for all possible switching sequences* $\{\sigma(k)\}_{k\in\mathbb{N}}$*, two families of matrices* $F_i \in \mathbb{R}^{m_P\times(n_P+n_M)}_{max}$ *and* $G_i \in \mathbb{R}^{m_P\times m_M}_{max}$*, with* $i \in \mathcal{I}$*, such that the control input sequence* $\{u_P(k)\}_{k\in\mathbb{N}}$ *defined by*

$$u_P(k) = \begin{cases} F_{\sigma(1)} \begin{pmatrix} x_P(0) \\ x_M(0) \end{pmatrix} \oplus G_{\sigma(1)} u_M(1) \text{ for } k = 1 \\ F_{\sigma(k)} \begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus G_{\sigma(k)} u_M(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases} \tag{4.7}$$

is a solution for the corresponding MMP.

Remark 9. *In the definitions of the MMP and the FMMP we assume the switching sequences of the plant and the model to coincide in* $\sigma(k)$*. This is not a restrictive hypotesis, in fact given a plant with switching sequence* $\sigma_P(k) : \mathbb{N} \to \{1, ..., I_P\}$ *and a model with switching sequence* $\sigma_M(k) : \mathbb{N} \to \{1, ..., I_M\}$*, also in the worst case in which the two switching sequences are arbitrary and not related to each other, it is trivial to redefine the families of system matrices of the plant and the model in order to have a common switching sequence* $\sigma(k) : \mathbb{N} \to \{1, ..., I_P \times I_M\}$*.*

Remark 10. *Contrary to what happens in the framework of stationary systems (see Remark 2, Section 3.2), it is necessary to introduce the term* $u_P(k-1)$ *at the second member of equation (5.5) for* $k > 1$ *to assure that the resulting input dater* $u_P(k)$ *is non-decreasing. The new term introduces a dynamic component in the feedback controller.*

As in the stationary case, we do not require the causality of the feedback controller, the consequences are the same that are mentioned in Remark 3, in Section 3.2.

4.7 Solution

In this section we use the geometric approach to provide solvability conditions and procedures to compute a solution for the MMP and the FMMP in the framework of switching max-plus linear systems. The approach is the same followed for stationary systems in Section 3.3. However, in the switching case we obtain weaker results as some new assumptions are necessary.

Given a plant Σ_P described by (4.5) and a model Σ_M described by (4.6), we can consider the extended system Σ_E, whose dynamics is described by

$$\Sigma_E \equiv \begin{cases} x_E(k) &= A_{E\sigma(k)} x_E(k-1) \oplus B_{1\sigma(k)} u_P(k) \oplus B_{2\sigma(k)} u_M(k) \\ x_E(0) &= \epsilon \end{cases} \tag{4.8}$$

where $x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}_{max}^{(n_P+n_M)}$ is the internal event dater, $A_{E\sigma(k)} = \begin{pmatrix} A_{P\sigma(k)} & \epsilon \\ \epsilon & A_{M\sigma(k)} \end{pmatrix}$, $B_{1\sigma(k)} = \begin{pmatrix} B_{P\sigma(k)} \\ \epsilon \end{pmatrix}$, and $B_{2\sigma(k)} = \begin{pmatrix} \epsilon \\ B_{M\sigma(k)} \end{pmatrix}$.

A control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ that, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, forces $x_E(k)$ to evolve, for each possible switching signal $\{\sigma(k)\}_{k\in\mathbb{N}}$, inside the *output equalizer* semimodule

$$\mathcal{K} \subseteq \mathcal{X}_E \tag{4.9}$$

defined by $\mathcal{K} = \bigcap_{i\in\mathcal{I}} \mathcal{K}_i$ where

$$\mathcal{K}_i = \left\{ x_E = \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E \,,\ \text{s. t. } C_{Pi} x_P = C_{Mi} x_M \right\} \tag{4.10}$$

is clearly a solution for the MMP. In particular, if it is of the form (4.7), then it is a solution also for the FMMP. However, if a control sequence with such property does not exist we cannot say that the MMP is not solvable. In fact, a possible solution for the problem can be represented by a control sequence that forces the state of the system to "jump" appropriately between the semimodules $\mathcal{K}_i$ without keeping it inside $\mathcal{K}$. This fact marks an important difference with respect to the solution of the MMP for switching systems over the conventional algebra. This inconsistency is due to the different structure of switching systems defined over the max-plus algebra, as described in the following remark.

Remark 11. *In the framework of dynamic systems over the conventional algebra the geometric approach based on the concept of invariant sequence of semimodules is well known and widely used for periodic systems [27], [53], but cannot be used for switching systems. This is due to the fact that the formal structure of switching linear systems over the conventional algebra is slightly different from the one described by Equation (4.1). In switching max-plus linear systems, for the event instance index* $k \in \mathbb{N}$*, the active mode* $\sigma(k)$ *influences the value of* $x(k)$ *(through* $A_{\sigma(k)}$ *and* $B_{\sigma(k)}$*) and, in turn* $y(k)$ *(through* $C_{\sigma(k)}$*). In the corresponding structure usually employed over the conventional algebra, the active mode* $\sigma(k)$ *has a role in the calculation of* $x(k+1)$ *(through* $A_{\sigma(k)}$ *and* $B_{\sigma(k)}$*), which, in turn is used to compute* $y(k+1)$*, by means of the mode* $\sigma(k+1)$*, which is unknown at instant* $k \in \mathbb{N}$*. In practice, at instant* $k \in \mathbb{N}$ *we must provide an input* $u(k)$ *that keeps* $x(k+1)$ *inside a subspace that leads to an acceptable output* $y(k+1)$*, for each possible* $\sigma(k+1)$*, as its value is not yet known and so is* $C_{\sigma(k+1)}$*.*

In the framework of switching max-plus systems, in order to obtain a formulation of the model matching problem that is equivalent to the one provided in Problem 3, we need to consider the problem of keeping the extended state $x_E(k)$ inside the family of semimodules $\mathcal{K}_\sigma$.

Before stating new solvability conditions for the MMP and FMMP in the framework of switching max-plus linear systems, we need to formally introduce a property that we call *strong non-anticipativeness*.

Definition 12 (Strong non-anticipativeness [12])**.** *We say that a switching max-plus linear system* Σ *of the form (4.1) is strongly non-anticipative if it is non-anticipative (i.e.* $A_i \geq I_n$ *for all* $i \in \mathcal{I}$*) and* $A_i B_j \geq B_i$ for all $i, j \in \mathcal{I}$.

The influence of such property in the system dynamics is explained in the following technical lemma.

Lemma 3. *If a switching max-plus linear system* Σ *of the form (4.1) is strongly non-anticipative and* $u(k+1) = u(k)$ *for some* $k \in \mathbb{N}$*, then the term* $B_{\sigma(k+1)}u(k+1)$ *does not influence the state evolution of the system.*

Proof. Given a switching linear max-plus system Σ of the form (4.1), let $u(k+1) = u(k)$. Then, we have

$$\begin{aligned} x(k) &= A_{\sigma(k)}x(k-1) \oplus B_{\sigma(k)}u(k) \\ x(k+1) &= A_{\sigma(k+1)}x(k) \oplus B_{\sigma(k+1)}u(k+1) \\ &= A_{\sigma(k+1)}x(k) \oplus B_{\sigma(k+1)}u(k) \end{aligned}$$

and, by substitution, we get

$$\begin{aligned} x(k+1) = {} & A_{\sigma(k+1)}A_{\sigma(k)}x(k-1) \oplus A_{\sigma(k+1)}B_{\sigma(k)}u(k) \\ & \oplus B_{\sigma(k+1)}u(k) \end{aligned}$$

By strong non-anticipativeness, it follows $A_{\sigma(k+1)}B_{\sigma(k)} \geq B_{\sigma(k+1)}$, and hence $x(k+1) = A_{\sigma(k+1)}A_{\sigma(k)}x(k-1) \oplus A_{\sigma(k+1)}B_{\sigma(k)}u(k)$ □

Remark 12. *A non-anticipative system with constant input matrix (i.e. $B_i = \bar{B} \in \mathbb{R}_{max}^{n\times m}$ for all $i \in \mathcal{I}$) is strongly non-anticipative.*

Intuitively, a system is strongly non-anticipative if its dynamics is slow enough to filter the effect of the switching in the input matrix, when the input is constant. If the plant is not strongly non-anticipative we can find ourselves in the situation in which the switching to a new active mode forces the state, and so the output, of the system to increment even if the control input sequence provided keeps constant. A decreasing input is not physically realizable and this makes in general the problem unsolvable if the model is capable of generating output sequences that are incompatible with the minimum increment of the output of the plant in correspondence with a switching event.

More formally, we obtain the following sufficient condition for the solvability of the MMP.

Theorem 7 ([12])**.** *Given a strongly non-anticipative plant Σ_P of the form (4.5) and a non-anticipative model Σ_M of the form (4.6), consider the extended system Σ_E given by (4.8). Then, the related MMP is solvable if for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon\\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi}\\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}^* \subseteq \mathcal{X}_E$, where $\mathcal{V}^*$ is the maximum $(A_{E\sigma}, B_{1\sigma})$-invariant semimodule for Σ_E contained in the output equalizer semimodule $\mathcal{K} \subseteq \mathcal{X}_E$ defined by (4.10).*

Proof. By the controlled invariance of $\mathcal{V}^*$, it follows that for each $x_E \in \mathcal{V}^*$ and for each $i \in \mathcal{I}$ there exists a vector $u_{1i} \in \mathbb{R}_{max}^{m_P}$ such that $A_{Ei}x_E \oplus B_{1i}u_{1i}$ belongs to $\mathcal{V}^*$. Moreover, by hypothesis, for each $i \in \mathcal{I}$ and each $u_M \in \mathbb{R}_{max}^{m_M}$ there exists $u_{2i} \in \mathbb{R}_{max}^{m_P}$ such that

$$\begin{pmatrix}\epsilon\\ B_{Mi}\end{pmatrix} u_M \oplus \begin{pmatrix}B_{Pi}\\ \epsilon\end{pmatrix} u_{2i} \in \mathcal{V}^*. \tag{4.11}$$

Then, for each input $\{u_M(k)\}_{k\in\mathbb{N}}$, using the dynamics of Σ_E, we can recursively construct a control input $\{u_P(k)\}_{k\in\mathbb{N}}$ for Σ_E as

$$u_P(k) = \begin{cases} u_{2\sigma(k)}(1) \text{ for } k = 1\\ u_{1\sigma(k)}(k) \oplus u_{2\sigma(k)}(k) \oplus u_P(k-1) \text{ for } k > 1\end{cases}$$

and the corresponding state evolution $\{x_E(k)\}_{k\in\mathbb{N}}$ as

$$x_E(k) = \begin{cases} B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k) \quad \text{for } k=1 \\ (A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))\oplus \\ (B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))\oplus \\ B_{1\sigma(k)}u_P(k-1) \quad \text{for } k>1 \end{cases}$$

and we can show by induction that $x_E(k)$ belongs to $\mathcal{V}^* \subseteq \mathcal{K}$ for all $k \in \mathbb{N}$ and therefore $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$. In fact, $x_E(1)$ belongs to $\mathcal{V}^*$ by the definition of $u_{2\sigma(k)}(.)$. For $k > 1$, we have, by the definition of $u_{1\sigma(k)}(.)$, that $(A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))$ belongs to $\mathcal{V}^*$ if $x_E(k-1)$ does and, by the definition of $u_{2\sigma(k)}(.)$, that $(B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))$ also belongs to $\mathcal{V}^*$. Moreover, since $A_{E\sigma(k)}x_E(k-1) \geq A_{E\sigma(k)}B_{1\sigma(k-1)}u_P(k-1) \geq B_{1\sigma(k)}u_P(k-1)$ due to the fact that the plant is strongly non-anticipative, the term $B_{1\sigma(k)}u_P(k-1)$ in $x_E(k)$ can be disregarded. □

An analogous result about the FMMP can be stated as follows.

Theorem 8 ([12])**.** *Given a strongly non-anticipative plant Σ_P of the form (4.5) and a non-anticipative model Σ_M of the form (4.6), consider the extended system Σ_E given by (4.8). Then, the related FMMP is solvable if there exists an $(A_{E\sigma}, B_{1\sigma})$-invariant semimodule $\mathcal{V}$ of feedback type contained in the output equalizer semimodule $\mathcal{K}$ defined by (4.10) such that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ with $x \oplus z \in \mathcal{V}$.*

Proof. Let $\mathcal{V} \subseteq \mathcal{K}$ be an $(A_{E\sigma}, B_{1\sigma})$-invariant semimodule of feedback type for which the condition of the theorem holds. Then there exists a family of matrices F_i such that, for each $x_E(k-1) \in \mathcal{V}$ and $i \in \mathcal{I}$, $(A_{Ei} \oplus B_{1i}F_i)x_E(k-1)$ belongs to $\mathcal{V}$ and a family of matrices G_i such that, for each $i \in \mathcal{I}$, the columns of the matrix $\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} G_i = \begin{pmatrix}B_{Pi}G_i \\ B_{Mi}\end{pmatrix}$ belong to $\mathcal{V}$. Then, applying a control law recursively defined as in equation (4.7), with the above defined matrices families F_i and G_i, we get the compensated dynamics

$$\begin{aligned} x_E(k) = {} & (A_{E\sigma(k)} \oplus B_{1\sigma(k)}F_{\sigma(k)})x_E(k-1)\oplus \\ & \oplus \begin{pmatrix}B_{P\sigma(k)}G_{\sigma(k)} \\ B_{M\sigma(k)}\end{pmatrix} u_M(k) \oplus B_{1\sigma(k)}u_P(k-1) \end{aligned} \tag{4.12}$$

where $u_P(0) = \epsilon$. Since the plant is strongly non-anticipative, $A_{E\sigma(k)}x_E(k-1) \geq A_{E\sigma(k)}B_{1\sigma(k-1)}u_P(k-1) \geq B_{1\sigma(k)}u_P(k-1)$ holds and the last summand of the right-hand term of equation (4.12) does not interfere with the state of the system, that evolves in $\mathcal{V} \subseteq \mathcal{K}$. Therefore, $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$. □

Relying on the concept of invariant family of semimodules, we can state stronger results, that are necessary and sufficient.

Theorem 9 ([14]). *Given a strongly non-anticipative plant Σ_P of the form (4.5) and a strongly non-anticipative model Σ_M of the form (4.6), consider the extended system Σ_E given by (4.8). Then, the related MMP is solvable if and only if for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}_i^* \subseteq \mathcal{X}_E$, where $\mathcal{V}_\sigma^*$ is the maximum $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules for Σ_E contained in the output equalizer family of semimodules $\mathcal{K}_\sigma \subseteq \mathcal{X}_E$ defined by (4.10).*

Proof. If. By the controlled invariance of $\mathcal{V}_\sigma^*$, it follows that for each $i \in \mathcal{I}$ and each $x_E \in \mathcal{V}_\sigma^*$ there exists a vector $u_{1i} \in \mathbb{R}_{max}^{m_P}$ such that $A_{Ei}x_E \oplus B_{1i}u_{1i}$ belongs to $\mathcal{V}_i^*$. Moreover, by hypothesis, for each $i \in \mathcal{I}$ and each $u_M \in \mathbb{R}_{max}^{m_M}$ there exists $u_{2i} \in \mathbb{R}_{max}^{m_P}$ such that

$$\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} u_M \oplus \begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} u_{2i} \in \mathcal{V}_i^*. \tag{4.13}$$

Then, for each input $\{u_M(k)\}_{k\in\mathbb{N}}$, using the dynamics of Σ_E, we can recursively construct a control input $\{u_P(k)\}_{k\in\mathbb{N}}$ for Σ_E as

$$u_P(k) = \begin{cases} u_{2\sigma(k)}(1) \text{ for } k = 1 \\ u_{1\sigma(k)}(k) \oplus u_{2\sigma(k)}(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases}$$

and the corresponding state evolution $\{x_E(k)\}_{k\in\mathbb{N}}$ as

$$x_E(k) = \begin{cases} B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k) \quad \text{for } k = 1 \\ (A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))\oplus \\ (B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))\oplus \\ B_{1\sigma(k)}u_P(k-1) \quad \text{for } k > 1 \end{cases}$$

and we can show by induction that $x_E(k)$ belongs to $\mathcal{V}_{\sigma(k)}^* \subseteq \mathcal{K}$ for all $k \in \mathbb{N}$ and therefore $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$. In fact, $x_E(1)$ belongs to $\mathcal{V}_{\sigma(1)}^*$ by the definition of $u_{2\sigma(k)}(.)$. For $k > 1$, we have, by the definition of

$u_{1\sigma(k)}(.)$, that $(A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_{1\sigma(k)}(k))$ belongs to $\mathcal{V}^*_{\sigma(k)}$ if $x_E(k-1)$ belongs to $\mathcal{V}^*_\sigma$ and, by the definition of $u_{2\sigma(k)}(.)$, that $(B_{1\sigma(k)}u_{2\sigma(k)}(k) \oplus B_{2\sigma(k)}u_M(k))$ also belongs to $\mathcal{V}^*_{\sigma(k)}$. Moreover, since $A_{E\sigma(k)}x_E(k-1) \geq A_{E\sigma(k)}B_{1\sigma(k-1)}u_P(k-1) \geq B_{1\sigma(k)}u_P(k-1)$ due to the fact that the plant is strongly non-anticipative, the term $B_{1\sigma(k)}u_P(k-1)$ in $x_E(k)$ can be disregarded.

Only if. If the condition of the theorem does not hold, there exists $\bar{u}_M$ and $i \in \mathcal{I}$ such that $B_{2i}\bar{u}_M \oplus B_{1i}u_P \notin \mathcal{V}^*_i$ for any $u_P \in \mathbb{R}^{m_P}_{max}$. Then, taking the constant input $\{u_M(k)\}_{k\in\mathbb{N}}$ with $u_M(k) = \bar{u}_M$ and a switching signal $\sigma(.)$ with $\sigma(1) = i$, we have that $x_E(1) = B_{1i}u_P(1) \oplus B_{2i}\bar{u}_M$ does not belong to $\mathcal{V}^*_i$ for any value $u_P(1) \in \mathbb{R}^{m_P}_{max}$. We can write, recursively, for $k \geq 2$, $x_E(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k) \oplus B_{2\sigma(k)}u_M(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k) \oplus B_{2\sigma(k)}\bar{u}_M$ and, thanks to the strong non-anticipativeness of the model, $x_E(k) = A_{E\sigma(k)}x_E(k-1) \oplus B_{1\sigma(k)}u_P(k)$. By Lemma 2, the fact that $x_E(1)$ does not belong to $\mathcal{V}^*_{\sigma(1)}$ implies that either $x_E(1)$ does not belong to $\mathcal{K}_{\sigma(1)}$ or that it does not belong to $\mathcal{V}^*_\sigma$. If $x_E(1) \notin \mathcal{K}_{\sigma(1)}$ then $y_M(1) \neq y_P(1)$ and the problem cannot be solved. If $x_E(1) \notin \mathcal{V}^*_\sigma$, then for any input $\{u_P(k)\}_{k\in\mathbb{N}}$ there exist a switching signal $\sigma(.)$ and some $\bar{k} \in \mathbb{N}$ such that $x_E(\bar{k}) \notin \mathcal{K}_{\sigma(\bar{k})}$. In other words, $x_E(k)$ cannot be forced to evolve inside the family of semimodules $\mathcal{K}_\sigma$ and therefore the MMP cannot be solved. □

Also this stronger condition can be easily adapted to solve the FMMP, as stated in the following.

Theorem 10 ([14]). *Given a strongly non-anticipative plant Σ_P of the form (4.5) and a strongly non-anticipative model Σ_M of the form (4.6), consider the extended system Σ_E given by (4.8). Then, the related FMMP is solvable if and only if there exists an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules $\mathcal{V}_\sigma$ of feedback type contained in the output equalizer family of semimodules $\mathcal{K}_\sigma$ defined by (4.10) such that, for each $i \in \mathcal{I}$ and for each $x \in \operatorname{Im} B_{2i} = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_{1i} = \operatorname{Im}\begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ with $x \oplus z \in \mathcal{V}_i$.*

Proof. Let $\mathcal{V}_\sigma \subseteq \mathcal{K}_\sigma$ be an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules of feedback type for which the condition of the theorem holds. Then there exists a family of matrices F_i such that, for each $x_E(k-1) \in \mathcal{V}_\sigma$ and $i \in \mathcal{I}$, $(A_{Ei} \oplus B_{1i}F_i)x_E(k-1)$ belongs to $\mathcal{V}_i$ and a family of matrices G_i such that, for each $i \in \mathcal{I}$, the columns of the matrix $\begin{pmatrix}\epsilon \\ B_{Mi}\end{pmatrix} I_{m_M} \oplus \begin{pmatrix}B_{Pi} \\ \epsilon\end{pmatrix} G_i = \begin{pmatrix}B_{Pi}G_i \\ B_{Mi}\end{pmatrix}$ belong to $\mathcal{V}_i$. Then, applying a control law recursively defined as in equation (4.7), with the above defined matrices families F_i and G_i, we get the compensated dynamics

$$\begin{aligned} x_E(k) = (A_{E\sigma(k)} \oplus B_{1\sigma(k)} F_{\sigma(k)}) x_E(k-1) \oplus \\ \oplus \begin{pmatrix} B_{P\sigma(k)} G_{\sigma(k)} \\ B_{M\sigma(k)} \end{pmatrix} u_M(k) \oplus B_{1\sigma(k)} u_P(k-1) \end{aligned} \tag{4.14}$$

where $u_P(0) = \epsilon$. Since the plant is strongly non-anticipative, $A_{E\sigma(k)} x_E(k-1) \geq A_{E\sigma(k)} B_{1\sigma(k-1)} u_P(k-1) \geq B_{1\sigma(k)} u_P(k-1)$ holds and the last summand of the right-hand term of equation (4.14) does not interfere with the state of the system, that evolves, for all $k \in \mathbb{N}$, inside $\mathcal{V}_{\sigma(k)} \subseteq \mathcal{K}_{\sigma(k)}$. Therefore, $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

Only if. Assume that the FMMP is solved by a control law of the form (4.7). Then, the family of sets of reachable states for the dynamics (4.8) indexed by the last active mode $\sigma(k)$ is an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules of feedback type contained in $\mathcal{K}_\sigma$ whose semimodules contain all the columns of the matrix $\begin{pmatrix} B_{P\sigma(k)} G_{\sigma(k)} \\ B_{M\sigma(k)} \end{pmatrix} = \begin{pmatrix} \epsilon \\ B_{M\sigma(k)} \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_{P\sigma(k)} \\ \epsilon \end{pmatrix} G_{\sigma(k)}$. This clearly implies the condition of the theorem. □

Remark 13. *If the plant is not strongly non-anticipative, the condition expressed in Theorem 9 and Theorem 10 are necessary but not sufficient. Dually, if the model is not strongly non-anticipative the conditions are sufficient, but not necessary. In fact, these assumptions are used in the proofs of the theorems only in the corresponding sections. The fact that these assumptions are necessary for the Theorem to hold is proved by counterexamples in Section 4.10.*

Considerations about the computational aspects of the provided conditions and how to determine appropriate solutions can be easily derived from those relative to stationary systems, mentioned in Remark 5.

4.8 Example

In this section we provide a practical example of model matching using the results exposed in this chapter. The objective is to control the plant modeled in Section 4.3 in order to match the output of a model defined by equations of the form (4.6)

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 5x_M(k-1) \oplus 5u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} \tag{4.15}$$

where all the daters have image in $\mathbb{R}_{max}$. The plant is strongly non-anticipative and the model is non-anticipative, so the assumptions of Theorem 3 and 4 are fulfilled. Using suitable Scicoslab [97] procedures, we can compute the output equalizer semimodule $\mathcal{K}$ and the maximal $(A_{E\sigma}, B_{1\sigma})$-invariant subsemimodule $\mathcal{V}^*$ contained in it. The sequence of semimodules considered in Theorem 5 converges after two iterations (i.e. $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}^*$), and we get:

$$\mathcal{K} = \mathrm{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & e \\ \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix}, \quad \mathcal{V}^* = \mathrm{Im}\begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon & e \\ 1 & \epsilon & \epsilon & e & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon & \epsilon & \epsilon \\ 4 & e & e & 3 & e & 2 \\ 4 & e & e & 3 & e & 2 \end{pmatrix}$$

The condition of Theorem 3 is satisfied, so the MMP is solvable. Moreover, $\mathcal{V}^*$ is of feedback type, so the FMMP is also solvable by Theorem 4. A control signal $u_P(k)$ of the form (5.5) that solves the problem is obtained by taking

$$\begin{array}{c} F_1 = \begin{pmatrix} 1 & \epsilon & \epsilon & \epsilon & -1 & \epsilon \end{pmatrix}, \quad F_2 = \begin{pmatrix} \epsilon & \epsilon & -5 & \epsilon & -3 & \epsilon \end{pmatrix} \\ F_3 = \begin{pmatrix} \epsilon & \epsilon & -4 & \epsilon & -2 & \epsilon \end{pmatrix}, \quad F_4 = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & -2 & \epsilon \end{pmatrix} \\ G_1 = -1, \quad G_2 = -3, \quad G_3 = -2, \quad G_4 = -2 \end{array} \tag{4.16}$$

In order to check the correctness of the proposed solution, let us simulate the evolution of the plant and of the model with, e.g., the input sequence for the model and the switching signal given in Table 4.2.

k	1	2	3	4	5
$u_M(k)$	1	3	9	13	19
$\sigma(k)$	1	3	1	2	3

Table 4.2: Considered sequence

The simulation gives the result shown in Figure 5.2, with $y_P(k) = y_M(k)$ for all $k \in \{1, ..., 5\}$, as expected. Each one of the first five columns in the figure is associated to a different machine of the plant, and the last one is associated to the model, represented as a single machine that requires 5 time units to process the input. Time is represented on the vertical axis. Each rectangle is associated to a time interval in which the corresponding machine is busy. The different textures used to fill the rectangles characterize the various parts that are processed. The same texture is used for different parts i and j if they are processed by the system in the same configuration (i.e. if $\sigma(i) = \sigma(j)$).

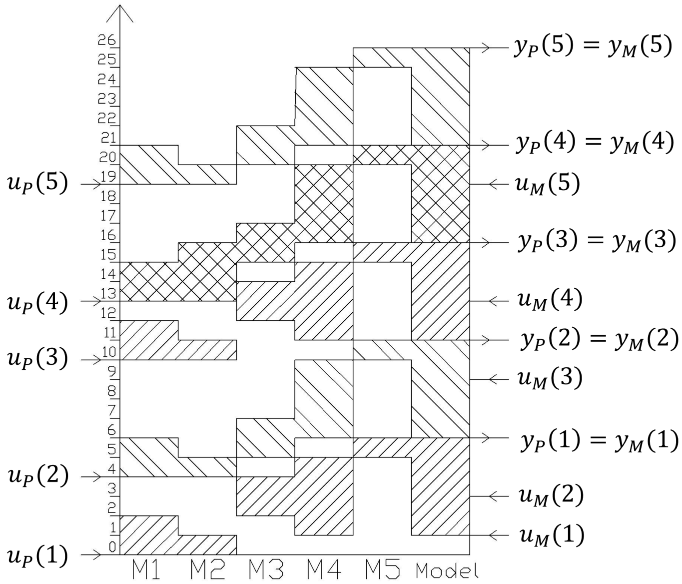

Figure 4.2: Timing of the example

4.9 Counterexample to the necessity of the conditions of Theorem 7 and 8

The conditions provided by Theorem 7 and Theorem 8 are stated as sufficient conditions. In this section we prove that they are not necessary, by means of a counterexample that was published in a recent paper of ours [12] and is a modified version of the one produced by Prof. Davide Zorzenon, Technische Universität Berlin in a personal communication, in the process of reviewing such a paper.

Consider the max-plus linear plant defined by

$$\Sigma_P \equiv \begin{cases} x_P(k) = x_P(k-1) \oplus u_P(k) \\ y_P(k) = x_P(k) \\ x_P(0) = \epsilon \end{cases}$$

and the switching max-plus linear model defined by

$$\Sigma_M \equiv \begin{cases} x_M(k) = 2x_M(k-1) \oplus u_M(k) \\ y_M(k) = C_{M\sigma(k)}x_M(k) \\ x_M(0) = \epsilon \end{cases}$$

where

$$C_{M\sigma(k)} = \begin{cases} 1 \text{ for } \sigma(k) = 1 \\ 2 \text{ for } \sigma(k) = 2 \end{cases} .$$

Note that the first equation of Σ_P simplifies to $x_P(k) = u_P(k)$ for any non-decreasing input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$. Since

$$\mathcal{K}_1 = \{\begin{pmatrix} x_P & x_M \end{pmatrix}^\top \text{such that } x_P = 1x_M\}$$

and

$$\mathcal{K}_2 = \{\begin{pmatrix} x_P & x_M \end{pmatrix}^\top \text{such that } x_P = 2x_M\},$$

we have $\mathcal{K} = \{\epsilon\}$ and the conditions of Theorem 3 and Theorem 4 are not satisfied.

Nevertheless, the fact that the FMMP is solvable can be proved by using the necessary and sufficient condition provided by Theorem 10. It is not difficult to show that the algorithm (4.3) converges at the first step (i.e. $\mathcal{V}^1_\sigma = \mathcal{V}^0_\sigma = \mathcal{K}_\sigma$). So, $\mathcal{K}_\sigma$ is an $(A_{E\sigma}, B_{1\sigma})$-invariant family of semimodules. Moreover, $\mathcal{K}_\sigma$ is of feedback type with $F_1 = \begin{pmatrix} \epsilon & 3 \end{pmatrix}$ and $F_2 = \begin{pmatrix} \epsilon & 4 \end{pmatrix}$. The set of linear equations $B_{2i} \oplus B_{1i} g_i \in \mathcal{K}_i$ admits as an unique solution $g_1 = 1$ and $g_2 = 2$. The FMMP (and hence the MMP) has the solution

$$u_P(k) = F_{\sigma(k)} x_E(k-1) \oplus g_{\sigma(k)} u_M(k) \oplus u_P(k-1)$$

with $u_P(0) = \epsilon$. In fact, the input $u_P(k)$ makes the state $x_E(k)$ stay in $\mathcal{K}_1$ or in $\mathcal{K}_2$ if, respectively, $\sigma(1)$ equals 1 or 2 and $\sigma(k)$ keeps constant, while the same input makes the state jump from $\mathcal{K}_1$ to $\mathcal{K}_2$ and viceversa each time $\sigma(k)$ changes value. Then, $y_P(k)$ equals $y_M(k)$ for all $k \in \mathbb{N}$.

4.10 Necessity of strong non-anticipativeness

In Section 4.7 we have provided some conditions for the solvability of the MMP and the FMMP for the class of switching max-plus linear systems. These conditions have been formulated and proved under the assumption of strong non-anticipativeness of some of the systems involved in the problem. One can wonder if these conditions apply also when all the systems involved do not enjoy of the property of being strongly non-anticipative. The answer to this question has been anticipated in Remark 13 and is that the strong non-anticipativeness of the systems cannot be relaxed, unless different conditions of solvability are found. The objective of this section is to prove this assertion by means of appropriate counterexamples.

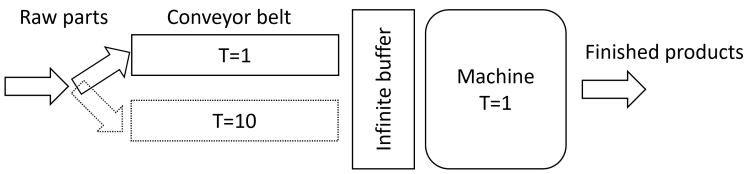

Figure 4.3: Scheme of the production plant.

4.10.1 Strong non-anticipativeness of the plant

This example proves that the strong non-anticipativeness of the plant is a necessary assumption for the validity of Theorem 7, Theorem 8, Theorem 9, and Theorem 10.

Let us consider a simple production plant, where, during standard operation, raw components, of a single type, after their entry into the system are loaded on a conveyor belt. The belt, in 1 time unit, brings them to an infinite buffer that acts as entry station for a production machine. This machine, in 1 time unit, transform one raw part in a finished product which exits immediately from the system.

In some circumstances, however, it is necessary to stop the conveyor belt to perform some maintenance activities. So, a switch and another conveyor belt have been added in order to continue the production during the maintenance periods. Such secondary conveyor belt is slower than the other one: it takes 10 time units to transport the raw parts to the input buffer of the production machine. The situation is graphically represented in Figure 4.3.

The plant can be modeled as a switching max-plus linear system with two modes of operation and initial state $x_P(0) = \epsilon$. The first mode models the system during standard operation, while the second one is relative to the maintenance periods. The only type of input event is “a raw component become available in the input of the system” and this is associated to the dater $u_P(.)$. The only internal event, that is “a finished part is completed”, is associated to $x_P(.)$. This event is also the only output event, so $x_P(.) = y_P(.)$. We get

$$\begin{array}{lcl} \Sigma_{P1} & \equiv & \left\{ \begin{array}{lcl} x_P(k) & = & 1x_P(k-1) \oplus 2u_P(k) \\ y_P(k) & = & x_P(k) \end{array} \right. \\ \Sigma_{P2} & \equiv & \left\{ \begin{array}{lcl} x_P(k) & = & 1x_P(k-1) \oplus 11u_P(k) \\ y_P(k) & = & x_P(k) \end{array} \right. \end{array} \tag{4.17}$$

As an example, we can consider the problem of tracking the output of the stationary model given by

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 1x_M(k-1) \oplus 11u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} \tag{4.18}$$

that has the same dynamics of the second mode of the plant Σ_{P2}.

Note that $A_{P2}B_{P1} \not\geq B_{P2}$, in fact $1 \otimes 2 = 3 \not\geq 11$, so the plant is not strongly non-anticipative.

It is easy to show that

$$\mathcal{K} = \mathcal{K}_1 = \mathcal{K}_2 = \text{Im}\begin{pmatrix} e \\ e \end{pmatrix} = \mathcal{V}_1^* = \mathcal{V}_2^* = \mathcal{V}^* \tag{4.19}$$

and that the conditions of Theorem 7, Theorem 8, Theorem 9, and Theorem 10 hold, except for the hypotesis of strong non-anticipativeness of the plant.

However, we can prove that the MMP is not solvable and this clearly implies that the FMMP is also unsolvable. In doing this, we consider the first two parts processed ($k \in \{1, 2\}$) and $u_M(1) = 0$, $u_M(2) = 1$ as an input to the model. We get $y_M(1) = 11$ and $y_M(2) = 12$. We consider the case in which, after processing the first part, a maintenance becomes necessary, and the second part is loaded on the emergency belt, so we have $\sigma(1) = 1$ and $\sigma(2) = 2$. For the mode Σ_{P1}, in order to get $y_P(1) = y_M(1) = 11$ we need to have $u_P(1) = 9$. However, for the second part, the switch makes the plant slower and the only way to get $y_P(2) = y_M(2) = 12$ would be to set $u_P(2) = 1$, but this would lead to a decreasing input signal, that is physically impossible.

4.10.2 Strong non-anticipativeness of the model

The following example proves that the strong non-anticipativeness of the model is a necessary assumption for the validity of Theorem 9 and Theorem 10.

Let us consider a plant $\Sigma_{P\sigma(k)}$ of the form (5.3) with initial state $x_P(0) = \epsilon$ and modes described by the equations

$$\Sigma_{P1} \equiv \begin{cases} x_P(k) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} x_P(k-1) \oplus \begin{pmatrix} \epsilon \\ e \end{pmatrix} u_P(k) \\ y_P(k) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} x_P(k) \end{cases} \tag{4.20}$$

$$\Sigma_{P2} \equiv \begin{cases} x_P(k) = \begin{pmatrix} e & e \\ \epsilon & e \end{pmatrix} x_P(k-1) \oplus \begin{pmatrix} e \\ e \end{pmatrix} u_P(k) \\ y_P(k) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} x_P(k) \end{cases} \tag{4.21}$$

and a model $\Sigma_{M\sigma(k)}$ of the form (5.4), with initial state $x_M(0) = \epsilon$ and modes described by the equations

$$\Sigma_{M1} \equiv \begin{cases} x_M(k) = \begin{pmatrix} e\,\epsilon \\ \epsilon\, e \end{pmatrix} x_M(k-1) \oplus \begin{pmatrix} \epsilon \\ e \end{pmatrix} u_M(k) \\ y_M(k) = \begin{pmatrix} e\,\epsilon \\ \epsilon\, e \end{pmatrix} x_M(k) \end{cases} \tag{4.22}$$

$$\Sigma_{M2} \equiv \begin{cases} x_M(k) = \begin{pmatrix} 1\,\epsilon \\ \epsilon\, 1 \end{pmatrix} x_M(k-1) \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_M(k) \\ y_M(k) = \begin{pmatrix} e\,\epsilon \\ \epsilon\, e \end{pmatrix} x_M(k) \end{cases} \tag{4.23}$$

The plant is strongly non-anticipative, but the model is not. The two modes of the extended system Σ_E of the form (5.6), are characterized by the matrices

$$A_{E1} = \begin{pmatrix} e & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \\ \epsilon & \epsilon & \epsilon & e \end{pmatrix} \quad B_{11} = \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ \epsilon \end{pmatrix} \quad B_{21} = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \end{pmatrix} \tag{4.24}$$

$$A_{E2} = \begin{pmatrix} e & e & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon & 1 \end{pmatrix} \quad B_{12} = \begin{pmatrix} e \\ e \\ \epsilon \\ \epsilon \end{pmatrix} \quad B_{22} = \begin{pmatrix} \epsilon \\ \epsilon \\ 1 \\ 1 \end{pmatrix} \tag{4.25}$$

The output equalizer family of semimodules $\mathcal{K}_\sigma$ is given by

$$\mathcal{K}_i = \mathrm{Im} \begin{pmatrix} e & \epsilon \\ \epsilon & e \\ e & \epsilon \\ \epsilon & e \end{pmatrix} \text{ for each } i \in \mathcal{I}$$

and the maximal $(A_\sigma, B_{1\sigma})$-invariant family of semimodules contained in $\mathcal{K}_\sigma$ is given by

$$\mathcal{V}_i^* = \mathrm{Im} \begin{pmatrix} e \\ e \\ e \\ e \end{pmatrix} = \mathcal{V}^* \text{for each } i \in \mathcal{I}.$$

The condition of Theorem 9 is not satisfied for $i = 1$, as we have

$$B_{21}u_M(1) \oplus B_{11}u_P(1) = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \end{pmatrix} u_M(1) \oplus \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ \epsilon \end{pmatrix} u_P(1) \notin \mathcal{V}^* \tag{4.26}$$

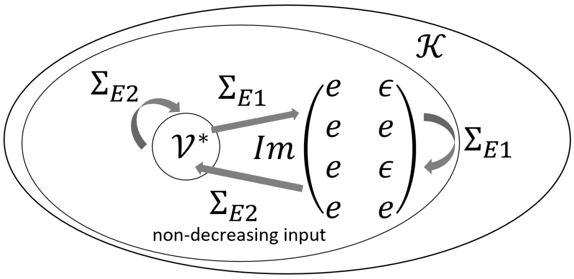

Figure 4.4: State semimodule of Σ_E and relevant semimodules.

for any $u_M(1) \neq \epsilon$ and for any $u_P(1) \in \mathbb{R}_{max}$. However, we will show that a solution for the FMMP of the form (5.5) exists, with matrices

$$\begin{array}{ll} F_1 = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon \end{pmatrix} & G_1 = e \\ F_2 = \begin{pmatrix} \epsilon & \epsilon & 1 & 1 \end{pmatrix} & G_2 = 1 \end{array} \tag{4.27}$$

so, not only the MMP is solvable, but also the FMMP.

Let us consider an event instance index $\bar{k} \in \mathbb{N}$ and a corresponding previous state $x_E(\bar{k}-1) \in \mathcal{V}^*$. We assume that the control law of the system $u_P(.)$ is of the form (4.7), with matrices from (4.27). If $\sigma(\bar{k}) = 2$, then it is easy to show that $x(\bar{k}) = 1x(\bar{k}-1) \in \mathcal{V}^*$. However, if $\sigma(\bar{k}) = 1$, or, more generally, $\sigma(\bar{k}+n) = 1$ for all $0 \leq n < \bar{n}$ and some $\bar{n} \in \mathbb{N}$, assuming non-decreasing input signals, we obtain

$$x(\bar{k}+n) = x(\bar{k}-1) \oplus \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ e \end{pmatrix} u_M(\bar{k}+n)$$

that is not inside $\mathcal{V}^*$ if $u_M(\bar{k}+n) \neq \epsilon$. However, if $\sigma(\bar{k}+\bar{n}) = 2$, we get

$$x(\bar{k}+\bar{n}) = 1x(\bar{k}) \oplus \begin{pmatrix} 1 \\ 1 \\ \epsilon \\ 1 \end{pmatrix} u_M(\bar{k}+\bar{n}-1) \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u_M(\bar{k}+\bar{n}) =$$

$$= 1x(\bar{k}) \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u_M(\bar{k}+\bar{n}) \in \mathcal{V}^*.$$

The situation is represented in Figure 4.4. The state of Σ_1 cannot be kept inside $\mathcal{V}^*$, however it can be kept inside $\mathcal{K}$. The system Σ_2 is such that, for non-decreasing input signals, the transition from Σ_1 to Σ_2 leads back the state inside $\mathcal{V}^*$, so the MMP and the FMMP are solvable.

Chapter 5

Max-plus periodic linear systems

Periodic max-plus linear system are event-varying max-plus linear systems whose structural variations are periodic of period ω, or ω-periodic, in the event domain. Such formalism is useful, for instance, to model max-plus linear systems where conflicts for shared resources are resolved using a periodic allocation scheme. A relevant use case is that of periodic scheduling of productive plants in order to produce, cyclically, the minimal part set (MPS) which is proportional to the production requirement [60]. For instance, if the objective is to produce 500 parts of type A, 500 parts of type B and 1000 parts of type C, we can consider the minimal part set as (A, B, C, C) and repeat its production for 500 times. When the production line has negligible setup times, such approach has important advantages including its simplicity, predictability, low work-in-process inventory and high machine utilization [103]. Effective techniques to compute efficient periodic schedules for production plants have been developed for a variety of applications, such as robotic cells [99] and flexible manufacturing systems [21]. Some of these techniques are based on the max-plus algebra [77].

5.1 State of the art and applications

Given the existence of a relevant practical interest in modeling discrete event systems subject to periodic structural variations, several authors worked on a possible formalization for the structure of such class of systems.

The first attempt in that direction is based on the theory of Petri nets and does not involve the max-plus algebra [59]. In that paper a suitable procedure is provided to transform such systems into equivalent Timed Event Graphs

(TEGs). Once this is done, one can easily model the obtained TEG as a max-plus linear system.

A modeling approach focused on the max-plus formalism is provided in [72] and extensively studied in [73]. The authors provide the tools to model repetitive manufacturing systems as periodic max-plus linear systems. Moreover, a procedure to transform these systems into equivalent stationary systems, by considering the evolution only in time instants that are multiple of the system period, is described. In [34] a method to transform periodic max-plus linear systems into Weight-Balanced Timed Event Graphs (WBTEG) is presented, together with new analysis tools for such class of systems.

In [33] and [106] some elementary operators are introduced in order to represent the input-output behaviour of event-varying and time-varying max-plus linear systems with a cyclic structure. These operators extend to the periodic case the theory of modeling and control of max-plus systems based on transfer functions. In particular, the authors show that event-varying systems with a cyclic structure can be decomposed in two subsystems: one with a stationary behaviour and the other containing the event-varying structure.

An alternative max-plus linear model for such class of systems is proposed in [77] and [98]. With the provided tools it is possible to model flexible manufacturing systems subject to a periodic schedule as max-plus linear systems. The linear model can be used to find schedules that are optimal for suitable criteria (e. g. minimum cycle time). The model is represented using max-plus linear equations, but its structure is pretty different from the classical one, described in [15]. The advantage of such alternative representation is the ability to model systems in which overtaking is possible [98], however there is a limitation in the fact that the results are only valid at steady-state.

It should be noted that the systems considered here are not equivalent to the max-plus systems with partial synchronization introduced in [35] and further studied, under the hypothesis of periodic synchronization, in [109]. Indeed, the systems considered by these authors have integer daters as input, output and internal variables. Also, more importantly, these systems are time-varying and not event-varying (the difference between the two properties is reported in the first paragraphs of Chapter 4).

In this chapter we define a formal structure for periodic max-plus linear systems, using the same approach followed in [72] and [73]. Then, we extend the geometric approach to such class of systems and we use it to solve the model matching problem. To the best of our knowledge, the research direction of the geometric approach applied to periodic max-plus linear system is new and has been initiated by us in [13].

5.2 System model

An event-varying, periodic max-plus linear system is a max-plus system in which the matrices that define the linear maps between the input, internal and output daters of the system can assume different values for different event instance indexes $k \in \mathbb{N}$. However, the variation in the system matrices must be periodic and known in advance.

Formally, a periodic linear max-plus system Σ of period $\omega \in \mathbb{N}$, also said ω-periodic, is a dynamical object whose evolution is defined by equations of the form

$$\Sigma \equiv \begin{cases} x(k) = A(k)x(k-1) \oplus B(k)u(k) \\ y(k) = C(k)x(k) \\ x(0) = \epsilon \end{cases} \tag{5.1}$$

where $k \in \mathbb{N}$ is the event instance index, $x(.) : \mathbb{N} \rightarrow \mathcal{X} = \mathbb{R}^n_{max}$ is the dater of internal events, $u(.) : \mathbb{N} \rightarrow \mathcal{U} = \mathbb{R}^m_{max}$ is the dater of input events, $y(.) : \mathbb{N} \rightarrow \mathcal{Y} = \mathbb{R}^p_{max}$ is the dater of output events and $A : \mathbb{N} \rightarrow \mathbb{R}^{n\times n}_{max}$, $B : \mathbb{N} \rightarrow \mathbb{R}^{n\times m}_{max}$, and $C : \mathbb{N} \rightarrow \mathbb{R}^{p\times n}_{max}$ are known ω-periodic functions. The semi-modules $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are, respectively, the state semimodule, the input semimodule and the output semimodule of the system.

The internal dater of the system $x(k)$, like all daters, must be non-decreasing in order to be physically realizable and the following definition provides a necessary requirement to obtain this result.

Definition 13 (Non-anticipative)**.** *We say that a periodic max-plus linear system of the form 5.1 is non-anticipative, if $A(k) \geq I_n$ for all $k \in \mathbb{N}$.*

Non-anticipative systems are such that, for each possible input sequence $\{u(k)\}_{k\in\mathbb{N}}$, the dater of internal events $x(k)$ is non-decreasing. Also in the case of periodic max-plus linear systems, the FIFO behaviour of the considered process is a necessary requirement, and the same considerations expressed in Remark 8 apply.

5.3 Example

In the example reported here, taken from our paper on the topic [13], we show how a manufacturing plant with a periodic schedule can be modeled as a max-plus linear system. The considered plant receives raw components in pairs, that, once separated, are fed into machine M1 and M2. The time required by machine M1 to perform its operation is 2 time units. The machine M2 performs an activity whose duration depends on the quality desired for

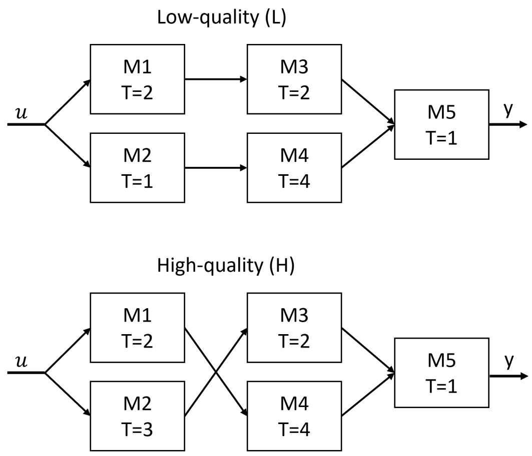

Figure 5.1: Representation of the plant

the product. Low-quality products are processed for 1 time unit and high-quality products for 3 time units. The semi-finished parts outgoing from the machines M1 and M2 need to be processed by either the machine M3 or M4. These machines perform the same activity, but M3 is faster than M4 as it require 2 time units, instead of 4. The part exiting from M1 can enter either M3 or M4 and the other piece of the original pair, after being processed in M2, enter the other machine (M4 or M3 respectively). The choice is performed through a switch operated in the following way: low-quality products exiting from M2 are sent to M4 while high-quality products exiting from M2 are sent to the the faster machine M3. Eventually, the pair of components is assembled into a single finished product by the machine M5. The process is graphically schematized in Figure 5.1.

The desired output for the plant is an alternation of high and low quality parts such that, every two low quality parts, a high quality part is produced. In other words, the plant must follow a periodic schedule whose minimal part set is (L, L, H).

We can model the plant as an ω-periodic linear max-plus system with $\omega = 3$ and two possible configurations:

L) The machine M2 requires 1 time unit and its output goes to M4, for $k = 1, 2$;

H) The machine M2 requires 3 time units and its output goes to M3, for $k = 3$;

The only type of event that is triggered by outside of the system is the "arrival of a pair to the machines M1 and M2", we refer to this as an input

$u(k)$	arrival time of the k-th pair of parts in the queue of M1 and M2
$x_1(k)$	completion time of the k-th cycle on the machine M1
$x_2(k)$	completion time of the k-th cycle on the machine M2
$x_3(k)$	completion time of the k-th cycle on the machine M3
$x_4(k)$	completion time of the k-th cycle on the machine M4
$\left.\begin{matrix} x_5(k) \\ y(k) \end{matrix}\right\}$	completion time of the k-th cycle on the machine M5

Table 5.1: Daters and events.

event. We consider as internal events the ones of type "completion of a cycle by the machine M1", "completion of a cycle by the machine M2", "completion of a cycle by the machine M3" and "completion of a cycle by the machine M4". The only type of event that is triggered internally by the system, and is visible from outside it, is the "completion of a part by the machine M5", we will classify this event both as an internal event and as an output event. We associate a dater function to each of this events, as reported in Table 5.1. Clearly, once a dater $u(k)$ has been provided by some external source, all the other daters can be determined using approriate rules.

In the following we illustrate how to find those rules, in the form of max-plus linear equations, for the system configuration which applies to the production of low-quality parts. Events of type "completion of a cycle by the machine M1" occur two time units after the same cycle has started. The starting time must be at least equal to the maximum between the instant in which the previous activity on M1 has been completed and the instant in which a new raw part has become available in the input queue of the machine. Assuming that all the activity starts as soon as possible, we can express this statement by the following equation:

$$\begin{aligned} x_1(k+1) &= \max\{x_1(k), u_1(k+1)\} + 2 = \\ &= \max\{x_1(k)+2, u_1(k+1)+2\} \end{aligned}$$

and, using similar arguments, we have

$$\begin{aligned} x_2(k+1) &= \max\{x_2(k)+1, u_1(k+1)+1\} \\ x_3(k+1) &= \max\{x_1(k+1)+2, x_3(k)+2\} \\ x_4(k+1) &= \max\{x_2(k+1)+4, x_4(k)+4\} \\ x_5(k+1) &= \max\{x_3(k+1)+1, x_4(k+1)+1, x_5(k)+1\} \\ y(k) &= x_5(k) \end{aligned}$$

These equations are linear in the max-plus algebra and can be written as

$$\begin{cases} x_1(k+1) &= 2 \otimes x_1(k) \oplus 2 \otimes u_1(k+1) \\ x_2(k+1) &= 1 \otimes x_2(k) \oplus 1 \otimes u_1(k+1) \\ x_3(k+1) &= 2 \otimes x_1(k+1) \oplus 2 \otimes x_3(k) \\ x_4(k+1) &= 4 \otimes x_2(k+1) \oplus 4 \otimes x_4(k) \\ x_5(k+1) &= 1 \otimes x_3(k+1) \oplus 1 \otimes x_4(k+1) \oplus 1 \otimes x_5(k) \\ y(k) &= x_5(k). \end{cases}$$

By substituting $x_1(k+1)$, $x_2(k+1)$, $x_3(k+1)$ and $x_4(k+1)$ at the second member of the third, fourth and fifth equations with their explicit expressions we get the system in standard form:

$$\begin{cases} x_1(k+1) &= 2 \otimes x_1(k) \oplus 2 \otimes u_1(k+1) \\ x_2(k+1) &= 1 \otimes x_2(k) \oplus 1 \otimes u_1(k+1) \\ x_3(k+1) &= 4 \otimes x_1(k) \oplus 2 \otimes x_3(k) \oplus 4 \otimes u_1(k+1) \\ x_4(k+1) &= 5 \otimes x_2(k) \oplus 4 \otimes x_4(k) \oplus 5 \otimes u_1(k+1) \\ x_5(k+1) &= 5 \otimes x_1(k) \oplus 6 \otimes x_2(k) \oplus 3 \otimes x_3(k) \oplus \\ & \quad \oplus 5 \otimes x_4(k) \oplus 1 \otimes x_5(k) \oplus 6 \otimes u_1(k+1) \\ y(k) &= x_5(k) \end{cases}$$

Similar arguments can be used to determine the linear equations associated to the situation in which the system produces high-quality parts. The linear equations relative to the different system configurations can be written in matrix form in order to obtain a max-plus periodic system of the form (5.1).

Considering the specific production schedule, whose MPS is (L, L, H), the plant can be modeled as an ω-periodic max-plus system of the form (5.1) with $\omega = 3$ and

$$A_P(1) = A_P(2) = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 1 & \epsilon & \epsilon & \epsilon \\ 4 & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & 5 & \epsilon & 4 & \epsilon \\ 5 & 6 & 3 & 5 & 1 \end{pmatrix} \quad B_P(1) = B_P(2) = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

$$A_P(3) = \begin{pmatrix} 2 & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & 3 & \epsilon & \epsilon & \epsilon \\ \epsilon & 5 & 2 & \epsilon & \epsilon \\ 6 & \epsilon & \epsilon & 4 & \epsilon \\ 7 & 6 & 3 & 5 & 1 \end{pmatrix} \quad B_P(3) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 6 \\ 7 \end{pmatrix}$$

$$C_P(k) = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix} \ \forall k \in \mathbb{N}$$

5.4 Geometric approach

The geometric approach, together with the concepts of invariance and controlled invariance, can be extended to the framework of periodic max-plus linear systems, as shown by us in a recent paper [13]. We do not report here in the details the meaning of the basic notions used in the geometric approach, as those illustrated in Section 3.1 apply also to the periodic case.

In this section we present an extension of the geometric approach to periodic max-plus linear systems, generalizing the results available for periodic systems over the conventional algebra [27], [53]. Periodic systems can be viewed as special cases of switched systems over digraphs and the generalization of the geometric approach to systems of such class was developed in [114], in the framework of the conventional algebra.

In the case of periodic systems it is not convenient to consider a single semimodule as an invariant for the system throughout its evolution. It is less conservative and more precise to introduce a periodic sequence of invariant semimodules, each associated to a specific event index. These considerations are the same that apply to switching max-plus linear systems, described in Section 4.5. In the case of periodic systems, this approach is coherently used also in the framework of systems over the conventional algebra [27], [53]. A formal description must begin with the following definition.

Definition 14. *An ω-periodic sequence of semimodules is a function $\mathcal{V}(.)$ that associates to each possible event instance index $k \in \mathbb{N}$ a subsemimodule $\mathcal{V}(k) \subseteq \mathbb{R}^n_{max}$ and that fulfil the property $\mathcal{V}(k) = \mathcal{V}(k+\omega)$ for all $k \in \mathbb{N}$.*

Because ω-periodic sequences of semimodules are widely used in the following, it is convenient to introduce some notation to improve the readability of the provided results. When no confusion can arise we refer to ω-periodic sequences of semimimodules simply as semimodules or ω-periodic semimodules. For instance we will refer to both $\mathcal{V}(.)$ and $\mathcal{X}$ as semimodules, but the first is a sequence of semimodules, while the latter is a proper semimodule.

We say that an ω-periodic semimodule $\mathcal{E}(.)$ is a subsemimodule of a semimodule $\mathcal{X}$, and denote such relation as $\mathcal{E}(.) \subseteq \mathcal{X}$, if $\mathcal{E}(k) \subseteq \mathcal{X}$ for each $k \in \mathbb{N}$. Moreover, given two ω-periodic subsemimodules $\mathcal{E}(.)$ and $\mathcal{I}(.)$ we say that:

- $\mathcal{E}(.)$ is finitely generated if, for all $k \in \mathbb{N}$, $\mathcal{E}(k)$ is finitely generated;
- $\mathcal{E}(.)$ is equal to $\mathcal{I}(.)$, or $\mathcal{E}(.) = \mathcal{I}(.)$, if $\mathcal{E}(k) = \mathcal{I}(k)$ for all $k \in \mathbb{N}$;
- $\mathcal{E}(.)$ is contained in $\mathcal{I}(.)$, or $\mathcal{E}(.) \subseteq \mathcal{I}(.)$, if $\mathcal{E}(k) \subseteq \mathcal{I}(k)$ for all $k \in \mathbb{N}$;
- an ω-periodic semimodule $\mathcal{S}(.)$ is the sum of $\mathcal{E}(.)$ and $\mathcal{I}(.)$, if $\mathcal{S}(k)$ is the sum of $\mathcal{E}(k)$ and $\mathcal{I}(k)$ for all $k \in \mathbb{N}$;

- an ω-periodic semimodule $\mathcal{R}(.)$ is the intersection of $\mathcal{E}(.)$ and $\mathcal{I}(.)$, or $\mathcal{R}(.) = \mathcal{E}(.) \cap \mathcal{I}(.)$, if $\mathcal{R}(k) = \mathcal{E}(k) \cap \mathcal{I}(k)$ for all $k \in \mathbb{N}$.

We can now introduce the geometric approach for periodic max-plus systems.

Definition 15 (Invariant semimodule). *Given an ω-periodic max-plus linear system Σ of the form (5.1), an ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an A-invariant semimodule, or, equivalently, an invariant semimodule for Σ, if for all $k \in \mathbb{N}$ and for all $v \in \mathcal{V}(k-1)$, $A(k)v$ belongs to $\mathcal{V}(k)$.*

Definition 16 (Controlled invariant semimodule). *Given an ω-periodic max-plus linear system Σ of the form (5.1), an ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an (A, B)-invariant semimodule, or, equivalently, a controlled invariant semimodule for Σ, if for all $k \in \mathbb{N}$ and for all $v \in \mathcal{V}(k-1)$ there exists $u \in \mathbb{R}^m_{max}$ such that $A(k)v \oplus B(k)u$ belongs to $\mathcal{V}(k)$.*

Given an ω-periodic linear max-plus system Σ of the form (5.1) and an ω-periodic subsemimodule $\mathcal{K}(.)$ contained in its state semimodule $\mathcal{X}$, the set of all the (A, B)-invariant semimodules contained in $\mathcal{K}(.)$ is a semi-lattice with respect to inclusion and sum of semimodules, so a maximum element of that set, denoted by $\mathcal{V}^*_{\mathcal{K}}(.)$, exists. The following algorithm allows to compute $\mathcal{V}^*_{\mathcal{K}}(.)$ under suitable hypotheses.

Theorem 11 ([13]). *Given an ω-periodic semimodule $\mathcal{K}(.) \subseteq \mathbb{R}^n_{max}$, the sequence of ω-periodic semimodules $\mathcal{V}_r(.)$ defined by*

$$\begin{array}{rcl} \mathcal{V}_0(k) & = & \mathcal{K}(k) \\ \mathcal{V}_r(k) & = & \mathcal{V}_{r-1}(k) \cap A^{-1}(k+1)(\mathcal{V}_{r-1}(k+1) \\ & & \ominus \mathrm{Im} B(k+1)) \\ k \in \mathbb{N} & & \end{array} \tag{5.2}$$

where $A^{-1}(k+1)(\mathcal{Y}) = \{v \in \mathbb{R}^n_{max}$, such that $A(k+1)v \in \mathcal{Y}\}$ and $\mathcal{V}_{r-1}(k+1) \ominus \mathrm{Im}\, B(k+1) = \{x \in \mathbb{R}^n_{max}$, for which there exists $u \in \mathbb{R}^m_{max}$ such that $x \oplus B(k+1)u \in \mathcal{V}_{r-1}(k+1)\}$ has the following properties:

1. *$\mathcal{V}_r(.) \subseteq \mathcal{V}_{r-1}(.)$ for all $r \in \mathbb{N}$;*

2. *letting $\mathcal{V}_\infty(.) = \lim_{r\to\infty} \mathcal{V}_r(.) = \bigcap_{r\in\mathbb{N}} \mathcal{V}_r(.)$, then every (A, B)-invariant ω-periodic semimodule contained in $\mathcal{K}(.)$ is also contained in $\mathcal{V}_\infty(.)$;*

3. *If $\mathcal{V}_r(.) = \mathcal{V}_{r-1}(.)$ then $\mathcal{V}_{r-1}(.)$ is an (A, B)-invariant ω-periodic semimodule and, in such case, $\mathcal{V}_\infty(.) = \mathcal{V}_{r-1}(.) = \mathcal{V}^*_{\mathcal{K}}(.)$.*

Proof. (1) The fact is a consequence of the definition of the sequence of ω-periodic semimodules.

(2) Let $\mathcal{P}(.) \subseteq \mathcal{K}(.) = \mathcal{V}_0(.)$ be an (A, B)-invariant ω-periodic semimodule and assume that, for some $r \in \mathbb{N}$, we have $\mathcal{P}(.) \subseteq \mathcal{V}_{r-1}(.)$. Then, since $\mathcal{P}(k) \subseteq A_i^{-1}(k+1)(\mathcal{P}(k+1) \ominus \mathrm{Im}B(k+1)) \subseteq A_i^{-1}(k+1)(\mathcal{V}_{r-1}(k+1) \ominus \mathrm{Im}B(k+1))$, we also have $\mathcal{P}(k) \subseteq \mathcal{V}_r(k)$ and the conclusion follows by induction.

(3) If $\mathcal{V}_r(.) = \mathcal{V}_{r-1}(.)$, the invaricance of $\mathcal{V}_r(.)$ is a direct consequence of equation (5.2). In this case the equality $\mathcal{V}_\infty(.) = \mathcal{V}_{r-1}(.)$ is obvious and $\mathcal{V}_\infty(.) = \mathcal{V}^*_\mathcal{K}(.)$ follows from (2). □

The practical implementation of Theorem 11 has the same issue analyzed for stationary and switching systems, in fact the sequence (5.2) does not necessarily converge in a finite number of steps and it cannot be considered a general algorithm for the computation of $\mathcal{V}^*_\mathcal{K}(.)$. Also, Remark 1 is still valid for the periodic case and we can assert that if $\mathcal{K}(.)$ is finitely generated, so are all the semimodules $\mathcal{V}_r$ in the sequence (5.2).

Definition 17. *Given an ω-periodic max-plus linear system Σ of the form (5.1), an ω-periodic semimodule $\mathcal{V}(.) \subseteq \mathcal{X}$ is said to be an (A, B)-invariant semimodule of feedback type for Σ if there exists an ω-periodic matrix $F(.) : \mathbb{N} \to \mathbb{R}^{m \times n}_{max}$ such that $(A(k) \oplus B(k)F(k))v$ belongs to $\mathcal{V}(k)$ for all $v \in \mathcal{V}(k-1)$, for all $k \in \mathbb{N}$.*

As happens in stationary and switching systems over the max-plus semiring, the property of controlled invariance of feedback type implies the property of controlled invariance, but the converse is not true.

5.5 Model matching problem

Similarly to the situation described in section 4.6, the few authors that have studied the problem of tracking the output of a given model with a periodic max-plus linear system, have formulated the model matching problem (or model reference control) in order to require that the output of the plant does not exceed the output of the model, possibly requiring to delay the control input as much as possible, in a just-in-time fashion. In particular, in [107] a theory on transfer behaviour of periodic max-plus linear systems is developed in order to provide useful algebraic tools for performance evaluation and controller synthesis. Such paper can be considered as an extension of [36], discussed in section 4.6, with a more specific focus on the periodic nature of the considered switching systems. In both these work the systems are directly represented in terms of transfer matrices.

As in the case of stationary and switching systems, we propose a formulation of the model matching problem whose objective is to control the input of a max-plus periodic plant to obtain an output equal to that of a given model of the same kind, for each possible input of the model. As always, the first formulation does not impose any constraints to the structure of the control sequence.

Problem 5 (Model Matching Problem [13]). *Given a non-anticipative ω-periodic max-plus linear system*

$$\Sigma_P \equiv \begin{cases} x_P(k) = A_P(k)x_P(k-1) \oplus B_P(k)u_P(k) \\ y_P(k) = C_P(k)x_P(k) \\ x_P(0) = \epsilon \end{cases} \tag{5.3}$$

of the form (5.1), called the plant, and a non-anticipative ω-periodic max-plus linear system

$$\Sigma_M \equiv \begin{cases} x_M(k) = A_M(k)x_M(k-1) \oplus B_M(k)u_M(k) \\ y_M(k) = C_M(k)x_M(k) \\ x_M(0) = \epsilon \end{cases} \tag{5.4}$$

of the form (5.1), called the model, with $x_P : \mathbb{N} \to \mathbb{R}^{n_P}_{max}$, $x_M : \mathbb{N} \to \mathbb{R}^{n_M}_{max}$, $u_P : \mathbb{N} \to \mathbb{R}^{m_P}_{max}$, $u_M : \mathbb{N} \to \mathbb{R}^{m_M}_{max}$ and $y_P, y_M : \mathbb{N} \to \mathbb{R}^{p}_{max}$, the Model Matching Problem (MMP) consists in finding, for all possible non-decreasing input sequence $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model, an appropriate non-decreasing control input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ for the plant, such that the output $\{y_P(k)\}_{k\in\mathbb{N}}$ of the plant equals the output $\{y_M(k)\}_{k\in\mathbb{N}}$ of the model, i.e. $y_P(k) = y_M(k)$ for all $k \in \mathbb{N}$.

If we require the control input $u_P(k)$ to be a linear function, with ω-periodic coefficients, of the state of the plant $x_P(k-1)$, the state of the model $x_M(k-1)$, and of the input of the model $u_M(k)$, we get a more restrictive formulation of the MMP. In this case, the control law consists of a state feedback term and a feed forward term as in the classical case of linear systems over a field.

Problem 6 (Feedback Model Matching Problem). *Given a plant of the form (5.3) and a model of the form (5.4), the Feedback Model Matching Problem (FMMP) consists in finding, for all possible non-decreasing input sequence $\{u_M(k)\}_{k\in\mathbb{N}}$ of the model, two ω-periodic matrices $F : \mathbb{N} \to \mathbb{R}^{m_P\times(n_P+n_M)}_{max}$*

and $G : \mathbb{N} \to \mathbb{R}_{max}^{m_P \times m_M}$*, such that the control input sequence defined by*

$$u_P(k) = \begin{cases} F(1)\begin{pmatrix} x_P(0) \\ x_M(0) \end{pmatrix} \oplus G(1)u_M(1) \text{ for } k=1 \\ F(k)\begin{pmatrix} x_P(k-1) \\ x_M(k-1) \end{pmatrix} \oplus G(k)u_M(k) \oplus u_P(k-1) \text{ for } k>1 \end{cases} \tag{5.5}$$

is a solution for the corresponding MMP.

Remark 14. *As in the case of switching systems (see Remark 10), the dynamic component* $u_P(k-1)$ *at the second member of equation (5.5) for* $k > 1$ *is needed to assure that the resulting control sequence is non-decreasing.*

As in the stationary and switching case, we do not require the causality of the feedback controller. The consequences are the same that are mentioned in Remark 3, in Section 3.2.

5.6 Solution

In this section we use the geometric approach to provide solvability conditions and procedures to compute a solution for the MMP and the FMMP in the framework of periodic max-plus linear systems. The approach is the same followed for stationary systems in Section 3.3, with the difference that in the case of periodic systems we rely on periodic sequences of semimodules and not proper semimodules as invariants for the system.

Given a plant Σ_P of the form (5.3) and a model Σ_M of the form (5.4), we can consider the extended ω-periodic system Σ_E described by

$$\Sigma_E \equiv \begin{cases} x_E(k) &= A_E(k)x_E(k-1) \oplus B_1(k)u_P(k) \oplus B_2(k)u_M(k) \\ x_E(0) &= \epsilon \end{cases} \tag{5.6}$$

where $x_E(.) = \begin{pmatrix} x_P(.) \\ x_M(.) \end{pmatrix} : \mathbb{N} \to \mathcal{X}_E = \mathbb{R}_{max}^{(n_P+n_M)}$ is the internal event dater, $A_E(k) = \begin{pmatrix} A_P(k) & \epsilon \\ \epsilon & A_M(k) \end{pmatrix}$, $B_1(k) = \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix}$, and $B_2(k) = \begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix}$.

Then, Problem 5 is equivalent to that of finding, for any input $\{u_M(k)\}_{k\in\mathbb{N}}$, a control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ that forces $x_E(k)$ to evolve inside the output equalizer ω-periodic semimodule $\mathcal{K}(.) \subseteq \mathcal{X}_E$ defined by

$$\mathcal{K}(k) = \{\begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathcal{X}_E \text{ s.t. } C_P(k)x_P = C_M(k)x_M\}. \tag{5.7}$$

As in the case of switching system, it is necessary to introduce the concept of strong non-anticipativeness.

Definition 18 (Strong non-anticipativeness [13])**.** *A periodic max-plus linear system Σ of the form (4.1) is said to be strongly non-anticipative if it is non-anticipative (i.e. $A(k) \geq I_n$ for all $k \in \mathbb{N}$) and*

$$A(k+1)B(k) \geq B(k+1) \text{ for all } k \in \mathbb{N} \tag{5.8}$$

For a complete discussion about the practical meaning and the implications of such a property, we refer the reader to Section 4.7, as the main concepts are identical for switching and periodic systems. In fact, in the following we prove the validity of theoretical results that are analogous to the ones provided for switching systems.

Lemma 4. *If a periodic linear max-plus system Σ of the form (5.1) is strongly non-anticipative and $u(k+1) = u(k)$ for some $k \in \mathbb{N}$, then the term $B(k+1)u(k+1)$ does not affect the evolution of the system.*

Proof. Given a periodic linear max-plus system Σ of the form (5.1), let $u(k+1) = u(k)$. Then, we have

$$\begin{aligned} x(k) &= A(k)x(k-1) \oplus B(k)u(k) \\ x(k+1) &= A(k+1)x(k) \oplus B(k+1)u(k+1) \\ &= A(k+1)x(k) \oplus B(k+1)u(k) \end{aligned}$$

and, by substitution, we get

$$x(k+1) = A(k+1)A(k)x(k-1) \oplus A(k+1)B(k)u(k) \oplus B(k+1)u(k)$$

By strong non-anticipativeness, it follows $A(k+1)B(k) \geq B(k+1)$, and hence

$$x(k+1) = A(k+1)A(k)x(k-1) \oplus A(k+1)B(k)u(k)$$

□

Remark 15. *A non-anticipative system with constant input matrix (i.e. such that $B(k) = \bar{B} \in \mathbb{R}_{max}^{n\times m}$ for all $k \in \mathbb{N}$) is strongly non-anticipative.*

We can now state a necessary and sufficient condition for the solvability of the MMP.

Theorem 12 ([13])**.** *Given a strongly non-anticipative ω-periodic plant Σ_P of the form (5.3) and a strongly non-anticipative ω-periodic model Σ_M of the form (5.4), consider the extended system Σ_E given by (5.6). Then, the related MMP is solvable if and only if for each $k \in \mathbb{N}$ and for each $x \in \operatorname{Im} B_2(k) = \operatorname{Im}\begin{pmatrix}\epsilon \\ B_M(k)\end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_1(k) = \operatorname{Im}\begin{pmatrix}B_P(k) \\ \epsilon\end{pmatrix} \subseteq \mathcal{X}_E$ such that $x \oplus z$ belongs to $\mathcal{V}^*(k) \subseteq \mathcal{X}_E$, where $\mathcal{V}^*(.)$ is the maximum (A_E, B_1)-invariant semimodule for Σ_E contained in the output equalizer semimodule $\mathcal{K}(.) \subseteq \mathcal{X}_E$ defined by (5.7).*

Proof. If. By the controlled invariance of $\mathcal{V}^*(.)$, it follows that for each $k \in \mathbb{N}$ and for each $x_E \in \mathcal{V}^*(k-1)$ there exists a vector $u_1(k) \in \mathbb{R}_{max}^{m_P}$ such that $A_E(k)x_E \oplus B_1(k)u_1(k)$ belongs to $\mathcal{V}^*(k)$. Moreover, by hypothesis, for each $k \in \mathbb{N}$ and each $u_M \in \mathbb{R}_{max}^{m_M}$ there exists $u_2(k) \in \mathbb{R}_{max}^{m_P}$ such that $B_2(k)u_M \oplus B_1(k)u_2(k) \in \mathcal{V}^*(k)$. Then, for each input $\{u_M(k)\}_{k\in\mathbb{N}}$, using the dynamics of Σ_E, we can construct recursively a control input for Σ_E as

$$u_P(k) = \begin{cases} u_2(1) \text{ for } k = 1 \\ u_1(k) \oplus u_2(k) \oplus u_P(k-1) \text{ for } k > 1 \end{cases}$$

and the corresponding state evolution, given by

$$x_E(k) = \begin{cases} B_1(k)u_2(k) \oplus B_2(k)u_M(k) \quad \text{for } k = 1 \\ \\ (A_E(k)x_E(k-1) \oplus B_1(k)u_1(k)) \oplus \\ (B_1(k)u_2(k) \oplus B_2(k)u_M(k)) \oplus B_1(k)u_P(k-1) \quad \text{for } k > 1 \end{cases}$$

and we can show by induction that $x_E(k)$ belongs to $\mathcal{V}^*(k)$ for all $k \in \mathbb{N}$. In fact, $x_E(1)$ belongs to $\mathcal{V}^*(1)$ by the definition of $u_2(.)$. For $k > 1$, we have, by the definition of $u_1(.)$, that $(A_E(k)x_E(k-1) \oplus B_1(k)u_1(k))$ belongs to $\mathcal{V}^*(k)$ if $x_E(k-1) \in \mathcal{V}^*(k-1)$ and, by the definition of $u_2(.)$, that $(B_1(k)u_2(k) \oplus B_2(k)u_M(k))$ also belongs to $\mathcal{V}^*(k)$. Moreover, since $A_E(k)x_E(k-1) \geq A_E(k)B_1(k-1)u_P(k-1) \geq B_1(k)u_P(k-1)$ due to the fact that the plant is strongly non-anticipative, the term $B_1(k)u_P(k-1)$ in $x_E(k)$ can be disregarded.

Only if. If the condition of the theorem does not hold, there exist $\bar{k} \in \mathbb{N}$ and $u_M(\bar{k}) = \bar{u}_M$ such that $B_2(\bar{k})u_M(\bar{k}) \oplus B_1(\bar{k})u_P \notin \mathcal{V}^*(\bar{k})$ for any $u_P \in \mathbb{R}_{max}^{m_P}$. In this case, the same condition holds for $u_M(\bar{k}) = \alpha\bar{u}_M$ with arbitrary $\alpha \in \mathbb{R}$. Then, taking the constant input sequence $u_M(k) = \alpha\bar{u}_M$, we have that we can choose α arbitrarily big in order to get that $x_E(\bar{k}) = A_E(\bar{k})x_E(\bar{k}-1) \oplus B_1(\bar{k})u_P(\bar{k}) \oplus B_2(\bar{k})\alpha\bar{u}_M$ does not belong to $\mathcal{V}^*(\bar{k})$ for any possible $u_P(\bar{k}) \in \mathbb{R}_{max}^{m_P}$. We can write, recursively, for $k > \bar{k}$, $x_E(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k) \oplus B_2(k)u_M(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k) \oplus B_2(k)\alpha\bar{u}_M$ and, thanks to the strong non-anticipativeness of the model, $x_E(k) = A_E(k)x_E(k-1) \oplus B_1(k)u_P(k)$. The fact that $x_E(\bar{k})$ does not belong to $\mathcal{V}^*(\bar{k})$ implies that for any input sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ there exists some $q \in \mathbb{Z}$ such that $x_E(q) \notin \mathcal{K}(q)$. In other words, $x_E(.)$ cannot be forced to evolve inside the semimodule $\mathcal{K}(.)$ and therefore the MMP cannot be solved. □

Remark 16. *If the plant is not strongly non-anticipative, the condition expressed in Theorem 12 is necessary but not sufficient. Dually, if the model* *is not strongly non-anticipative the condition is sufficient, but not necessary. In fact, these assumptions are used in the proof of the Theorem only in the corresponding sections. The fact that these assumptions are necessary for the Theorem to hold is proved by counterexamples in Section 5.8.*

The main result about the FMMP is stated in the following theorem.

Theorem 13 ([13]). *Given a strongly non-anticipative ω-periodic plant Σ_P of the form (5.3) and a strongly non-anticipative ω-periodic model Σ_M of the form (5.4), consider the extended system Σ_E given by (5.6). Then, the related FMMP is solvable if and only if there exists an (A_E, B_1)-invariant ω-periodic semimodule $\mathcal{V}(.)$ of feedback type contained in the output equalizer semimodule $\mathcal{K}(.)$ defined by (5.7) such that, for each $k \in \mathbb{N}$ and for each $x \in \operatorname{Im} B_2(k) = \operatorname{Im} \begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix} \subseteq \mathcal{X}_E$ there exists $z \in \operatorname{Im} B_1(k) = \operatorname{Im} \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix} \subseteq \mathcal{X}_E$ with $x \oplus z \in \mathcal{V}(k)$.*

Proof. If. Let $\mathcal{V}(.) \subseteq \mathcal{K}(.)$ be an (A_E, B_1)-invariant semimodule of feedback type for which the condition of the theorem holds. The controlled invariance of feedback type of $\mathcal{V}(.)$ implies the existence of an ω-periodic matrix $F(.)$ such that for each $k \in \mathbb{N}$ and $x_E(k-1) \in \mathcal{V}^*(k-1)$, $(A_E(k) \oplus B_1(k)F(k))x_E(k-1)$ belongs to $\mathcal{V}^*(k)$. The condition of the theorem implies the existence of an ω-periodic matrix $G(.)$ such that the columns of the matrix $\begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix} G(k) = \begin{pmatrix} B_P(k)G(k) \\ B_M(k) \end{pmatrix}$ belong to $\mathcal{V}(k)$ for each $k \in \mathbb{N}$. Then, applying a control law recursively defined as in equation (5.5), with the above defined matrices $F(.)$ and $G(.)$, we get the compensated dynamics

$$\begin{aligned} x_E(k) = {} & (A_E(k) \oplus B_1(k)F(k))x_E(k-1) \oplus \begin{pmatrix} B_P(k)G(k) \\ B_M(k) \end{pmatrix} u_M(k) \oplus \\ & \oplus B_1(k)u_P(k-1) \end{aligned} \tag{5.9}$$

where we consider $u_P(0) = \epsilon$. Since the plant is strongly non-anticipative, $A_E(k)x_E(k-1) \geq A_E(k)B_1(k-1)u_P(k-1) \geq B_1(k)u_P(k-1)$ holds and the last summand of the right-hand term of equation (5.9) does not affect the state of the system, that clearly evolves in $\mathcal{V}(.) \subseteq \mathcal{K}(.)$.

Only if. Assume that the FMMP is solved by a control law of the form (5.5). Then, for each $k \in \mathbb{N}$, the set of reachable states for the dynamics (5.9) at event instance k, is an (A_E, B_1)-invariant periodic semimodule of feedback type contained in $\mathcal{K}(k)$ that contains all the columns of the matrix $\begin{pmatrix} B_P(k)G(k) \\ B_M(k) \end{pmatrix} = \begin{pmatrix} \epsilon \\ B_M(k) \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P(k) \\ \epsilon \end{pmatrix} G(k)$. This clearly implies the condition of the theorem. □

Considerations about the computational aspects of the provided conditions and how to determine appropriate solutions are analogous to those already mentioned for stationary systems in Remark 5.

5.7 Examples

5.7.1 Example 1

We report here an example taken from our work on the topic [13]. The objective is to control the output of the production plant that we have modeled as a periodic max-plus linear system in Section 5.3. In the considered scenario the conditions of Theorem 12 and of Theorem 13 are satisfied, so the MMP and FMMP are solvable. A solution for the FMMP is computed and the regulated system is simulated in order to show the resulting evolution. We consider the following system of the form (5.4) as a model whose output should be tracked by the plant:

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 5x_M(k-1) \oplus 5u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} \tag{5.10}$$

where all the daters have image in $\mathbb{R}_{max}$. Both the plant and the model are strongly non-anticipative. The output equalizer semimodule is

$$\mathcal{K}(k) = \operatorname{Im} \begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & e \\ \epsilon & \epsilon & \epsilon & \epsilon & e \end{pmatrix} \text{ for all } k \in \mathbb{N}. \tag{5.11}$$

In this case, the output equalizer semimodule is constant for each possible value of k, because both the plant and the model have stationary output matrices. However, in general, this is not the case. We have implemented a Scicoslab procedure in order to compute the maximal (A_E, B_1)-invariant subsemimodule $\mathcal{V}^*(.)$ for the 3-periodic joint dynamics. The sequence of semimodules considered in Theorem 11 converges after two iterations (i.e.

$\mathcal{V}_1(.) = \mathcal{V}_2(.) = \mathcal{V}^*(.)$), and we get:

$$\mathcal{V}^*(1) = \mathcal{V}^*(3) = \text{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ e & 1 & e & e & e \\ e & 1 & e & e & e \end{pmatrix} \quad \mathcal{V}^*(2) = \text{Im}\begin{pmatrix} e & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 2 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & e & \epsilon \\ 2 & 1 & e & e & e \\ 2 & 1 & e & e & e \end{pmatrix}$$

The condition of Theorem 12 is satisfied, so the MMP is solvable. Moreover, $\mathcal{V}^*(.)$ is of feedback type, so the FMMP is also solvable by Theorem 13 and the control sequence $\{u_P(k)\}_{k\in\mathbb{N}}$ of the form (5.5), with

$$\begin{array}{ll} F(1) = F(2) = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & -1 & \epsilon \end{pmatrix} & G(1) = G(2) = -1 \\ F(3) = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & -2 \end{pmatrix} & G(3) = -2 \end{array} \tag{5.12}$$

solves the problem.

In order to check the correctness of the proposed solution, we have simulated the evolution of the plant and of the model with the designed feedback controller. The result of the simulation is graphically represented in Figure 5.2, with $y_P(k) = y_M(k)$ for all $k \in \{1, ..., 6\}$, as expected. Each one of the first five columns is associated to a different machine of the plant, and the last one is associated to the model, represented as a single machine that requires 5 time units in order to process the input. Time is represented on the vertical axis. Each rectangle is associated to a time interval in which the corresponding machine is busy. The different textures are used to distinguish low-quality parts from high-quality parts under production.

5.7.2 Example 2

In this example we show how a switching system subject to a periodic switching sequence can be modeled as a periodic max-plus linear system and we illustrate the possible advantages that can be obtained in doing so. In particular, we consider an instance of the model matching problem that is unsolvable in the domain of switching max-plus linear systems subject to arbitrary switching sequences, but is solvable for a specific periodic switching sequence.

We consider a production machine that is used for clothing production. The machine receives, as input, semi-finished products and can process them in two different ways. In the first operating mode some finishing touches are applied to the products in 1 time unit and, after that, the machine is immediately ready to accommodate another product to be processed. In the second operating mode, in addition to the normal activities, a sandblasting

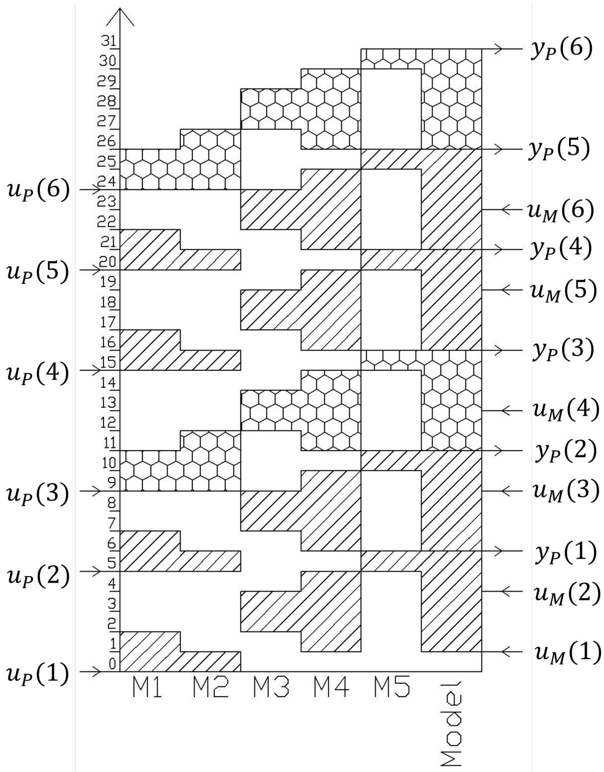

Figure 5.2: Timings of the example

phase is foreseen. In this case the machine requires 5 time units to process its input and another time unit to return operational after the completion of the last product.

Such machine can be modeled as a switching max-plus linear system whose input events associated to $u_P(.)$ are of type "a semi-finished product become available", the internal state $x_P(.)$ is associated to the event "the machine is ready to start a new processing" and the output dater $y_P(.)$ is associated to the event "a product has been completed". Such a switching system has clearly two modes ($\mathcal{I} = \{1, 2\}$) that are of the form

$$\Sigma_{P1} \equiv \begin{cases} x_P(k) &= 1x_P(k-1) \oplus 1u_P(k) \\ y_P(k) &= x_P(k) \\ x_P(0) &= \epsilon \end{cases}$$

$$\Sigma_{P2} \equiv \begin{cases} x_P(k) &= 6x_P(k-1) \oplus 6u_P(k) \\ y_P(k) &= -1x_P(k) \\ x_P(0) &= \epsilon \end{cases} .$$

We consider as a model whose output must be tracked by the plant, the following max-plus linear system

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 5x_M(k-1) \oplus 5u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} .$$

If we try to solve the problem in the framework of switching max-plus linear systems subject to arbitrary switching sequences, we find

$$\mathcal{K}_1 = Im \begin{pmatrix} e \\ e \end{pmatrix} \qquad \mathcal{K}_2 = Im \begin{pmatrix} 1 \\ e \end{pmatrix}$$

and the algorithm provided by Theorem 6 converges in two steps with

$$\mathcal{V}_i^1 = \mathcal{V}_i^2 = \{\epsilon\} \text{ for each } i \in \mathcal{I},$$

so the MMP and FMMP are unsolvable.

However, if we establish as an a priori defined ω-periodic switching sequence, with $\omega = 2$, the one of the form

$$\sigma(k) = \begin{cases} 1, & \text{for } k = 0 \text{ or k even} \\ 2, & \text{for k odd} \end{cases}$$

we can rewrite the switching max-plus linear plant as an ω-periodic max-plus linear plant of the form (5.3) with

$$A_P(1) = 1 \qquad B_P(1) = 1 \qquad C_P(1) = e$$

$$A_P(2) = 6 \qquad B_P(2) = 6 \qquad C_P(2) = -1$$

and find

$$\mathcal{K}(1) = Im \begin{pmatrix} e \\ e \end{pmatrix} \qquad \mathcal{K}(2) = Im \begin{pmatrix} 1 \\ e \end{pmatrix}.$$

The algorithm provided by Theorem 11 converges at the first iteration and we get

$$\mathcal{V}_1(1) = \mathcal{V}_0(1) = \mathcal{K}(1) = Im \begin{pmatrix} e \\ e \end{pmatrix} \qquad \mathcal{V}_1(2) = \mathcal{V}_0(2) = \mathcal{K}(2) = Im \begin{pmatrix} 1 \\ e \end{pmatrix}.$$

Such invariant ω-periodic sequence of semimodules is of feedback type, so the MMP and the FMMP are solvable. The control law of the form (5.5) with

$$\begin{array}{ll} F(1) = \begin{pmatrix} \epsilon & 4 \end{pmatrix} & g(1) = 4 \\ F(2) = \begin{pmatrix} \epsilon & e \end{pmatrix} & g(2) = e \end{array} \tag{5.13}$$

is a solution for the FMMP.

The reader can check that if we consider a different ω'-periodic switching sequence for the system, with $\omega' = 3$, for instance of the form

$$\sigma(k) = \begin{cases} 1, & \text{for } k = 0 \\ 2, & \text{for } k = 1, 2 \end{cases}$$

then we can rewrite the plant as an ω'-periodic max-plus linear plant of the form (5.3) with

$$A_P(1) = 1 \qquad B_P(1) = 1 \qquad C_P(1) = e$$

$$A_P(2) = A_P(3) = 6 \qquad B_P(2) = B_P(3) = 6 \qquad C_P(2) = C_P(3) = -1$$

and find

$$\mathcal{K}(1) = Im \begin{pmatrix} e \\ e \end{pmatrix} \qquad \mathcal{K}(2) = \mathcal{K}(3) = Im \begin{pmatrix} 1 \\ e \end{pmatrix}.$$

In this case, the algorithm provided by Theorem 11 converges at the fourth iteration and we get

$$\mathcal{V}_4(1) = \mathcal{V}_3(1) = \mathcal{V}_4(2) = \mathcal{V}_3(2) = \mathcal{V}_4(3) = \mathcal{V}_3(3) = \{\epsilon\}.$$

So, for this particular switching sequence the MMP is unsolvable.

5.8 Necessity of strong non-anticipativeness

In Section 5.6 we have provided some conditions for the solvability of the MMP and the FMMP for the class of periodic max-plus linear systems. These conditions have been formulated and proved under the assumption of strong non-anticipativeness of both the plant and the model. One can wonder if these conditions apply also when the systems involved do not enjoy of the property of being strongly non-anticipative. The answer to this question is that the strong non-anticipativeness of the systems cannot be relaxed, unless different conditions of solvability are found. The objective of this section is to prove this assertion by means of appropriate counterexamples.

5.8.1 Strong non-anticipativeness and sufficiency of the conditions

In this subsection we report a counterexample, taken from our work on the topic [13], that proves that the strong non-anticipativeness of the plant is a required assumption for the sufficiency of the conditions expressed in Theorem 12 and Theorem 13, as discussed in Remark 16.

Let us consider a simple production plant where raw samples of thermoplastic material, after their entry into the system, are loaded alternately inside two different presses that produce respectively left and right soles. The machine (M1) that produces left soles is relatively old and require 3 time units to perform its operations while the machine used for right parts (M2) was recently replaced and requires 1 time unit. Both M1 and M2 have multiple housings that allow them to work several parts in parallel. For simplicity we will assume that there is no maximum number of such parallel machining. When the soles are formed they are loaded into another machine (M3) whose role is to cool them before they exit the system. This last operation takes 1 time unit and cannot be done in parallel for multiple soles. The order in which the parts exit from the system must be the same in which the corresponding raw components arrived, in order to maintain alternation of left and right soles in the output of the system. An infinite buffer is available at the input of M3 to assure that such constraint is respected. The situation is graphically represented in Figure 5.3.

The plant can be modeled as an ω-periodic linear max-plus system of the form (5.3) with $\omega = 2$, initial state $x_P(0) = \epsilon$ and

$$\begin{array}{lll} A(1) = 1 & B_P(1) = 4 & C_P(1) = e \\ A(2) = 1 & B_P(2) = 2 & C_P(2) = e \end{array} \tag{5.14}$$

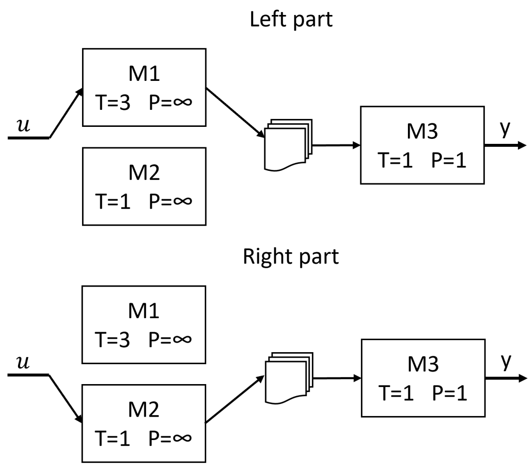

Figure 5.3: Scheme of the production plant.

As an example, we can consider the problem of tracking the output of the model given by

$$\Sigma_M \equiv \begin{cases} x_M(k) &= 1x_M(k-1) \oplus 4u_M(k) \\ y_M(k) &= x_M(k) \\ x_M(0) &= \epsilon \end{cases} \tag{5.15}$$

Such stationary max-plus linear system can be viewed as a special case of a periodic max-plus system where the period is equal to 1. We can consider $\omega = 2$ as the period of both the plant and the model inasmuch that is the minimum common multiple of the periods of the two considered systems. We notice that $A_P(3)B_P(2) \not\geq B_P(3)$, inasmuch $1 \otimes 2 = 3 \not\geq 4$. The plant is not strongly non-anticipative and this is the only unfulfilled assumption of Theorem 12.

Let us consider the first three parts processed ($k \in \{1, 2, 3\}$) with $u_M(1) = 0$, $u_M(2) = 6$ and $u_M(3) = 7$ as input of the model. We get $y_M(1) = 4$, $y_M(2) = 10$ and $y_M(3) = 11$. In order to get $y_P(1) = y_M(1) = 4$ we need to have $u_P(1) = 0$. For the second part, the objective $y_P(2) = y_M(2) = 10$ can be obtained with $u_P(2) = 8$. However, for the third part, the only way to get $y_P(3) = y_M(3) = 11$ would be to set $u_P(3) = 7$, but this would lead to a decreasing input signal, that is physically unfeasible. This shows that the MMP cannot be solved.

5.8.2 Strong non-anticipativeness and necessity of the conditions

In this subsection we report a counterexample, taken from our work on the topic [13], that proves that the strong non-anticipativeness of the plant is a required assumption for the necessity of the conditions expressed in Theorem 12 and Theorem 13, as discussed in Remark 16.

Let us consider a plant Σ_P of the form (5.3) with initial state $x_P(0) = \epsilon$, period $\omega = 2$, and

$$\begin{array}{ll} A_P(1) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} & B_P(1) = \begin{pmatrix} \epsilon \\ e \end{pmatrix} \\ A_P(2) = \begin{pmatrix} e & e \\ \epsilon & e \end{pmatrix} & B_P(2) = \begin{pmatrix} e \\ e \end{pmatrix} \end{array} \quad C_P(1) = C_P(2) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} \tag{5.16}$$

and a model Σ_M of the form (5.4), with initial state $x_M(0) = \epsilon$, period $\omega = 2$, and

$$\begin{array}{ll} A_M(1) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix} & B_M(1) = \begin{pmatrix} \epsilon \\ e \end{pmatrix} \\ A_M(2) = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} & B_M(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array} \quad C_M(1) = C_M(2) = \begin{pmatrix} e & \epsilon \\ \epsilon & e \end{pmatrix}. \tag{5.17}$$

The plant is strongly non-anticipative, but the model is not. The extended dynamics Σ_E of the form (5.6), is characterized by the matrices

$$A_E(1) = \begin{pmatrix} e & \epsilon & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & e & \epsilon \\ \epsilon & \epsilon & \epsilon & e \end{pmatrix} \; B_1(1) = \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ \epsilon \end{pmatrix} \; B_2(1) = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \end{pmatrix}$$

$$A_E(2) = \begin{pmatrix} e & e & \epsilon & \epsilon \\ \epsilon & e & \epsilon & \epsilon \\ \epsilon & \epsilon & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon & 1 \end{pmatrix} \; B_1(2) = \begin{pmatrix} e \\ e \\ \epsilon \\ \epsilon \end{pmatrix} \; B_2(2) = \begin{pmatrix} \epsilon \\ \epsilon \\ 1 \\ 1 \end{pmatrix}$$

The output equalizer semimodule is

$$\mathcal{K}(1) = \mathcal{K}(2) = \mathrm{Im} \begin{pmatrix} e & \epsilon \\ \epsilon & e \\ e & \epsilon \\ \epsilon & e \end{pmatrix} = \mathcal{K}$$

and the maximal (A_E, B_1)-invariant subspace contained in $\mathcal{K}(.)$ is given by

$$\mathcal{V}^*(1) = \mathcal{V}^*(2) = \mathrm{Im}\begin{pmatrix} e \\ e \\ e \\ e \end{pmatrix} = \mathcal{V}^*$$

The condition of Theorem 3 is not satisfied for $k = 1$, as

$$B_2(1)u_M(1) \oplus B_1(1)u_P(1) = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \end{pmatrix} u_M(1) \oplus \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ \epsilon \end{pmatrix} u_P(1)$$

is not in $\mathcal{V}^*$ for any $u_M(1) \neq \epsilon$ and for any $u_P(1) \in \mathbb{R}_{max}$. However, a feedback solution of the MMP of the form (5.5) exists, with matrices

$$\begin{array}{ll} F(1) = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon \end{pmatrix} & G(1) = e \\ F(2) = \begin{pmatrix} \epsilon & \epsilon & 1 & 1 \end{pmatrix} & G(2) = 1 \end{array} \tag{5.18}$$

so, not only the MMP is solvable, but also the FMMP.

If the control law of the system $u_P(.)$ is of the form (5.5), with matrices from (5.18), it is easy to show that, assuming a non-decreasing input sequence for the model, we obtain

$$x_E(1) = x_E(0) \oplus \begin{pmatrix} \epsilon \\ e \\ \epsilon \\ e \end{pmatrix} u_M(1)$$

that is not inside $\mathcal{V}^*$ if $u_M(1) \neq \epsilon$, but it is always inside $\mathcal{K}$. Moreover,

$$x(2) = 1x(1) \oplus \begin{pmatrix} 1 \\ 1 \\ \epsilon \\ 1 \end{pmatrix} u_M(1) \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u_M(2) = 1x(1) \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u_M(2) \in \mathcal{V}^*.$$

The situation is represented in Figure 5.4. The state of Σ_E cannot be kept inside $\mathcal{V}^*$, however it can be kept inside $\mathcal{K}$. The mode $\Sigma_E(1)$ has the property for which every initial state in $\mathcal{V}^*$ is mapped in a new state in $\mathcal{K}$ and, for non-decreasing input sequences, the mode $\Sigma_E(2)$ maps every state in $\mathcal{K}$ to a new state inside $\mathcal{V}^*$, so the MMP and the FMMP are solvable.

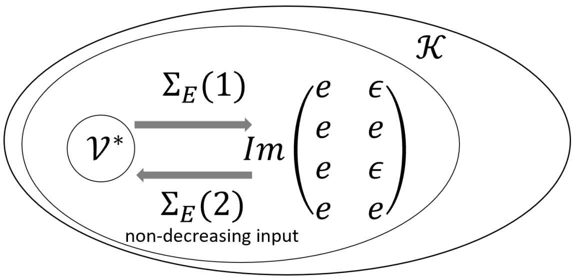

Figure 5.4: Relevant subsemimodules in the state semimodule of Σ_E

Chapter 6

Conclusion and future developments

We have addressed the problem of controlling a max-plus linear system to synchronize its output with the one produced by a model of the same kind, subject to an exogenous input. Stationary, switching and periodic max-plus systems have been considered. In all these cases, we have found new necessary and sufficient conditions for the solvability of the problem, based on the geometric approach. The basic notions required by the geometric approach were available for stationary systems [65], but not for switching and periodic max-plus systems. In the case of stationary systems, only a few authors have considered the possible applications to the model matching problem, and it remained largely open.

The provided methodology can be easily extended to the case in which the output of the model has to be interpreted as a deadline and exact synchronization is not required. We are working on such a generalization and we will publish it once fully developed.

As discussed in Section 3.2, another interesting future development could be to generalize the theory presented here to the case of arbitrary initial states for both the plant and the model, by combining in an appropriate way the results provided in [88], with the one described here.

The methodology provided by this thesis to solve the model matching problem is based on the geometric approach. These results can be placed on a parallel research direction with respect to others that have been considered in the literature, such as the residuation theory on rational series. It would be interesting, as a future work, to consider the possible analogies and differences in the results that arise from the application of these different approaches to the solution of problems of the same kind, possibly obtaining a general theory that can inherit the valuable properties of different approaches.

Appendix A

Precedence graphs of max-plus matrices

A directed graph, or digraph, $\mathcal{G}$ is defined as a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of elements called nodes and where $\mathcal{E}$ is a set whose elements are ordered pairs of nodes, called arcs. Nodes and arcs can also be referred to as vertices and edges, respectively. The symbols $\mathcal{V}$ and $\mathcal{E}$ are used because they are the first letters of the latter two names.

Definition 19 (Precedence graph [15])**.** *The precedence graph of a square matrix $A \in \mathbb{R}_{max}^{n \times n}$ is a weighted digraph with n nodes and all the arcs of the form (j, i) where $A_{ij} \neq \epsilon$, with a weight of the arc equal to the numerical value of A_{ij}. The precedence graph is denoted $\mathcal{G}(A)$.*

It is easy to note that any weighted digraph is the precedence graph of an appropriately defined square matrix. Some basic concepts of graph theory are reported in the following.

Definition 20 (Predecessor)**.** *If in a digraph $(\mathcal{V}, \mathcal{E})$, the arc (i, j) is contained in $\mathcal{E}$, then the node i is a predecessor of the node j. The set of all predecessors of j is denoted as $\pi(j)$.*

Definition 21 (Path)**.** *A path is a sequence of nodes $(i_1, i_2, ..., i_p)$, with $p > 1$, such that $i_j \in \pi(i_{j+1})$ for each $j = 1, ..., p-1$.*

Definition 22 (Circuit)**.** *A circuit is a path whose initial and final nodes coincide.*

Definition 23 (Weight of a path)**.** *The weight of a path is the sum (+) of the weights of the arcs that are part of the considered path.*

These basic concepts are the one that are required for a complete understanding of this thesis. For further information on the topic of graphs associated to max-plus matrices we refer the reader to [15].

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