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BRUHAT INTERVALS AND PARABOLIC COSETS IN ARBITRARY COXETER GROUPS

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ABSTRACT. In [Journal of Pure and Applied Algebra 224 (2020), no 12, 106449], V. Mazorchuk and R. Mrden (with some help by A. Hultman) prove that, given a Weyl group, the intersection of a Bruhat interval with a parabolic coset has a unique maximal element and a unique minimal element. We show that such intersections are actually Bruhat intervals also in the case of an arbitrary Coxeter group.

1. INTRODUCTION

Let W be a Weyl group and S be a set of Coxeter generators of W . Consider the Bruhat order with respect to S . Let J be a subset of S and $x, y, u \in W$. In [11], V. Mazorchuk and R. Mrden prove that, if the intersection of the Bruhat interval $[x, y]$ with the parabolic coset uW_J is nonempty, then it has a unique maximal element (see [11, Lemma 3], for which the authors acknowledge help by A. Hultman) and a unique minimal element (see [11, Lemma 5]). The proof for the existence of a unique maximal element works also in the case of an arbitrary Coxeter group. On the other hand, the proof for the existence of a unique minimal element makes use of the longest element of W , which does not exist in infinite Coxeter groups. In this short note, we give an alternative proof that does not assume the finiteness and works for all Coxeter groups.

2. NOTATION AND PRELIMINARIES

This section reviews the background material that is needed in the proof of Theorem 3.1.

Let (W, S) be an arbitrary Coxeter system. The group W , under Bruhat order (see, e.g., [1, §2.1] or [8, §5.9]), sometimes also called Bruhat-Chevalley order, is a graded partially ordered set having the length function ℓ as its rank function. This means that W has a minimum, which is the identity element e , and the function ℓ satisfies $\ell(e) = 0$ and $\ell(y) = \ell(x) + 1$ for all $x, y \in W$ with $x \triangleleft y$. Here $x \triangleleft y$, as well as $y \triangleright x$, means that the Bruhat interval $[x, y]$ coincides with $\{x, y\}$.

Given $w \in W$, we let $D_R(w)$ denote the right descent set $\{s \in S : \ell(ws) < \ell(w)\}$ of w . Given a subset J of S , we let W_J denote the parabolic subgroup of W generated by J and W^J denote the set $\{w \in W : D_R(w) \subseteq S \setminus J\}$ of minimal left coset representatives. For $x \in W$, we let $W_{\leq x} = \{w \in W : w \leq x\}$ and $W_{\geq x} = \{w \in W : w \geq x\}$.

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The following results are well known (see, e.g., [1, Proposition 2.2.7] or [8, Proposition 5.9] for the first one, [1, §2.4] or [8, §1.10] for the second one, and [1, §3.2] for the lattice property of the weak Bruhat order implying the third one).

Lemma 2.1 (Lifting Property). *Let $s \in S$ and $u, w \in W$, $u \leq w$. Then*

- (i) *if $s \in D_R(w)$ and $s \in D_R(u)$ then $us \leq ws$,*
- (ii) *if $s \notin D_R(w)$ and $s \notin D_R(u)$ then $us \leq ws$,*
- (iii) *if $s \in D_R(w)$ and $s \notin D_R(u)$ then $us \leq w$ and $u \leq ws$.*

Proposition 2.2. *Let $J \subseteq S$. Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ with $w^J \in W^J$ and $w_J \in W_J$; for this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$.*

Proposition 2.3. *Let $x \in W$. If $s_1, s_2 \in D_R(x)$, then the order of the product $s_1 s_2$ is finite.*

Symmetrically, left versions of Lemma 2.1 and Proposition 2.2 hold, as well as of the following well-known (and immediate to prove) result:

$$(2.1) \quad v \leq w \implies v^J \leq w^J$$

3. ARBITRARY COXETER GROUPS

Theorem 3.1. *Let (W, S) be an arbitrary Coxeter system. The intersection of a Bruhat interval with a parabolic coset is a Bruhat interval.*

Proof. By (2.1), it is sufficient to prove that the intersection of a Bruhat interval with a parabolic coset, if nonempty, has a unique maximal element and a unique minimal element. The proof in [11, Lemma 3] for the existence of a unique maximal element in Weyl groups works also for arbitrary Coxeter groups.

Let us prove the existence of a unique minimal element in the case of a left coset (the mirrored argument works for a right coset). It is sufficient to prove the following claim: given $x, u \in W$ and a subset J of S , the intersection $W_{\geq x} \cap uW_J$, if nonempty, has a unique minimal element.

Let x, u , and J be as in the claim and suppose $W_{\geq x} \cap uW_J \neq \emptyset$. We may also suppose $u \in W^J$. We use induction on $\ell(x)$. If $\ell(x) = 0$, then $x = e$ and $W_{\geq e} \cap uW_J$ has a unique minimal element, which is u .

Suppose $\ell(x) > 0$ and, towards a contradiction, suppose that m_1 and m_2 are two distinct minimal elements of $W_{\geq x} \cap uW_J$.

Fix $i \in \{1, 2\}$. Clearly $m_i \neq u$. Hence, there exists $s_i \in D_R(m_i) \cap J$. The minimality of m_i implies $xs_i \triangleleft x$ since otherwise, by Lemma 2.1(iii), we would have $m_i s_i \in W_{\geq x} \cap uW_J$. Let $m^i = \min(W_{\geq xs_i} \cap uW_J)$, which exists by the induction hypothesis and satisfies $m^i \leq m_1$ and $m^i \leq m_2$ since m_1 and m_2 both belong to $W_{\geq xs_i} \cap uW_J$. Furthermore, $m^i s_i \triangleright m^i$ since otherwise, by Lemma 2.1(iii), we would have $m^i \in W_{\geq x} \cap uW_J$, against the minimality of m_1 and m_2 . Again Lemma 2.1(iii) implies $m^i s_i \leq m_i$ while Lemma 2.1(ii) implies $m^i s_i \in W_{\geq x} \cap uW_J$; by the minimality of m_i , we have $m^i s_i = m_i$. Furthermore, if we let $\bar{i}$ be the element of the singleton $\{1, 2\} \setminus \{i\}$, then we have $m_{\bar{i}} s_i \triangleright m_{\bar{i}}$ since otherwise the same argument would imply that also $m_{\bar{i}}$ coincides with $m^i s_i$, but $m_1 \neq m_2$.

The four relations $m_1 s_1 \triangleleft m_1$, $m_1 s_1 \leq m_2$, $m_2 s_2 \triangleleft m_2$, $m_2 s_2 \leq m_1$ imply $\ell(m_1) = \ell(m_2)$, $m_1 s_1 \triangleleft m_2$, and $m_2 s_2 \triangleleft m_1$. By a repeated use of Lemma 2.1(i), we conclude that there exists $w \in W^{\{s_1, s_2\}}$ such that m_1 and m_2 belong to the coset $wW_{\{s_1, s_2\}}$.

Notice that $xs_1 \triangleleft x$ and $xs_2 \triangleleft x$; hence $W_{\{s_1, s_2\}}$ is finite by Lemma 2.3, and x is the top element of the coset $xW_{\{s_1, s_2\}}$.

Lemma 2.1(i) implies

$$w(\underbrace{\cdots s_i s_i s_i}_{h \text{ terms}}) \geq x \iff w(\underbrace{\cdots s_i s_i}_{h-1 \text{ terms}}) \geq xs_i \iff \cdots \iff w \geq x(\underbrace{s_i s_i s_i}_{h \text{ terms}})$$

for each $h \in \mathbb{N}$ smaller than, or equal to, the rank of $W_{\{s_1, s_2\}}$.

Hence, by the four relations $m_1 \geq x$, $m_2 \geq x$, $m_1 s_1 \not\geq x$, and $m_2 s_2 \not\geq x$, we conclude that the intersection $W_{\leq w} \cap xW_{\{s_1, s_2\}}$ has two distinct maximal elements, which is a contradiction since we know that $W_{\leq w} \cap xW_{\{s_1, s_2\}}$ has a unique maximal element. $\square$

- Remark 3.2.** (1) Prior to [11], the special case of the existence of a unique maximal element in $W_{\geq x} \cap W_J$, for all $x \in W$, is proved in [12, Lemma 7].
- (2) The proof of Theorem 3.1 provides another evidence of the fundamental role of (parabolic) dihedral subgroups and dihedral intervals in understanding the combinatorial properties of Coxeter groups (see, for example, [2], [3], [4], [5], [6], [7], [9], [10]).
- (3) Fix a subset J of S . Let $x \in W$ and $u_1, u_2 \in W^J$. While $\min(W_{\geq x} \cap u_1 W_J) \leq \min(W_{\geq x} \cap u_2 W_J)$, as well as $\max(W_{\geq x} \cap u_1 W_J) \leq \max(W_{\geq x} \cap u_2 W_J)$, clearly implies $u_1 \leq u_2$ by (2.1), the converse does not hold in general, as one may see already in type A_2 .

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