On the fractional relativistic Schrödinger operator

ON THE FRACTIONAL RELATIVISTIC SCHRÖDINGER OPERATOR

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Abstract. We collect some interesting results for equations driven by the fractional relativistic Schrödinger operator $(-\Delta+m^2)^s$ with $s\in(0,1)$ and $m>0$. More precisely, for the linear theory, we prove Hölder-Schauder-Zygmund regularity results and a Kato's inequality. For the nonlinear theory, we obtain L^∞-regularity, exponential decay, a Pohozaev-type identity, and a symmetry result for solutions of certain nonlinear fractional problems.

1. Introduction

In this work we deal with the relativistic Schrödinger operator

$$(-\Delta+m^2)^s \quad \text{with } m>0 \text{ and } s\in(0,1), \tag{1.1}$$

which may be defined via Fourier transform (see [29, 30]) by

$$(-\Delta+m^2)^s u(x)=\mathcal{F}^{-1}((|\xi|^2+m^2)^s\mathcal{F}u(\xi))(x), \quad x\in\mathbb{R}^N, \tag{1.2}$$

for any $u:\mathbb{R}^N\to\mathbb{R}$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decaying functions, or equivalently via singular integrals (see [22, 36]) as

$$(-\Delta+m^2)^s u(x)=m^{2s}u(x)+C(N,s)m^{\frac{N+2s}{2}}P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dy, \quad x\in\mathbb{R}^N, \tag{1.3}$$

where $P.V.$ stands for the Cauchy principal value, K_ν is the modified Bessel function of third kind of order ν (see [6, 51]), which satisfies the following well-known asymptotic formulas:

$$K_\nu(r)\sim\frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu} \text{ as } r\to 0, \text{ for } \nu>0, \tag{1.4}$$

$$K_\nu(r)\sim\sqrt{\frac{\pi}{2}}r^{-\frac{1}{2}}e^{-r} \text{ as } r\to\infty, \text{ for } \nu\in\mathbb{R}, \tag{1.5}$$

and $C(N,s)$ is a positive constant whose exact value is

$$C(N,s):=2^{1-\frac{N-2s}{2}}\pi^{-\frac{N}{2}}\frac{s(1-s)}{\Gamma(2-s)}.$$

When $s=\frac{1}{2}$, the operator (1.1) was considered in [52, 53] for spectral problems and has a clear meaning in quantum mechanics. Indeed the Hamiltonian for the motion of a free relativistic particle of mass m and momentum p is given by

$$\mathcal{H}:=\sqrt{p^2c^2+m^2c^4},$$

where c is the speed of the light. With the usual quantization rule $p\mapsto -\imath\nabla$, we get the so called relativistic Hamiltonian operator

$$\widehat{\mathcal{H}}:=\sqrt{-c^2\Delta+m^2c^4}-mc^2.$$

The point of the subtraction of the constant mc^2 is to make sure that the spectrum of the operator $\widehat{\mathcal{H}}$ is $[0,\infty)$ and this explains the terminology of relativistic Schrödinger operators for the operators of the form $\widehat{\mathcal{H}}+V(x)$, where $V(x)$ is a potential. Equations involving $\widehat{\mathcal{H}}$ arise in the study of the following time-dependent Schrödinger equation

$$\imath\hbar\frac{\partial\Phi}{\partial t}=\widehat{\mathcal{H}}\Phi-f(x,|\Phi|^2)\Phi, \quad (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where $\Phi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{C}$ is a wave function and $f:\mathbb{R}^N\times[0,\infty)\to\mathbb{R}$ is a continuous function, which describes the behaviour of bosons, spin-0 particles in relativistic fields. Physical models related to $\widehat{\mathcal{H}}$ have been widely studied over the past 30 years and there exists an important literature on the spectral properties of relativistic Hamiltonians, most of it has been strongly influenced by the works of Lieb on the stability of relativistic matter; see for instance [36, 37, 52]. On the other hand, the operator (1.1) is strictly connected to the theory

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of stochastic processes. More precisely, $m^{2s}-(-\Delta+m^2)^s$ is the infinitesimal generator of a Lévy process $X_t^{2s,m}$ called relativistic $2s$-stable process having the following characteristic function

$$E^0 e^{\imath\xi\cdot X_t^{2s,m}}=e^{-t[(|\xi|^2+m^2)^s-m^{2s}]},\quad \xi\in\mathbb{R}^N;$$

see [9, 15, 27, 42] for more details.

When $m\to 0$, then (1.1) reduces to the fractional Laplacian operator $(-\Delta)^s$ defined, for $u\in\mathcal{S}(\mathbb{R}^N)$, via Fourier transform by

$$(-\Delta)^s u(x)=\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi))(x),\quad x\in\mathbb{R}^N,$$

or via singular integrals by

$$(-\Delta)^s u(x)=C_{N,s}P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,\quad x\in\mathbb{R}^N. \tag{1.6}$$

This operator has gained tremendous popularity during the last two decades thanks to its applications in different fields, such as, among others, phase transition phenomena, crystal dislocation, population dynamics, anomalous diffusion, flame propagation, chemical reactions of liquids, conservation laws, quasi-geostrophic flows, water waves. Moreover, from a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a (rotationally) symmetric $2s$-stable Lévy process. We refer to [5, 8, 18, 21] for a very nice introduction on this subject. Note that the most important difference between the operators $(-\Delta)^s$ and $(-\Delta+m^2)^s$ is that the first one is homogeneous in scaling whereas the second one is inhomogeneous as should be clear from the presence of the Bessel function K_ν in (1.3).

Our paper is motivated by some recent investigations concerning the following fractional relativistic Schrödinger equation

$$(-\Delta+m^2)^s u=g(x,u)\text{ in }\mathbb{R}^N, \tag{1.7}$$

for which many interesting existence and multiplicity results have been established via suitable variational techniques; see [3, 4, 19, 20, 24, 33, 44]. In this work we focus our attention on the regularity, decay and symmetry of solutions for (1.7). Such questions have been extensively analyzed for $(-\Delta)^s$, see for instance [5, 10, 18, 23, 45], so our purpose is to go further in this direction by considering (1.1).

Firstly, we show how (1.1) interacts with Hölder-Zygmund spaces. These facts are essentially known in the theory of Besov spaces [40, 43, 50] but here we prefer to give an alternative proof by following the approach in [46]. We recall that Hölder estimates for $(-\Delta)^s$ were considered in [45] by taking advantage of the pointwise formula (1.6) and combining a localization trick with Schauder estimates for the classical Poisson equation. Later, in [13, 47] the authors used a semigroup approach to deduce Schauder-Zygmund and Schauder-Hölder-Zygmund estimates for $(-\Delta)^s$. Here we extend these results for (1.1) by following the treatment of the Lipschitz spaces Λ_α, with $\alpha>0$, as in [46, 49]. We emphasize that one could follow the arguments in [45] because (1.3) and asymptotic estimates (1.4) and (1.5) allow us to deduce the corresponding Propositions 2.8 and 2.9 in [45]. However, this way does not permit the case of Hölder-Zygmund spaces; see Remark 3.3.

Secondly, we provide a nonlocal version of Kato's inequality [34] for the solutions of a linear problem involving (1.1), that is, if $u\in H^s_m(\mathbb{R}^N,\mathbb{C})$ solves $(-\Delta+m^2)^s u=f\in L^1_{loc}$ then the inequality $(-\Delta+m^2)^s|u|\leqslant\Re(\text{sign}(\bar{u})f)$ holds in the distributional sense. As pointed out in [12], for the case $m=0$, the distributional Kato inequality for $(-\Delta)^s$ is essentially a consequence of the Córdoba-Córdoba inequality $(-\Delta)^s\varphi(u)\leqslant\varphi'(u)(-\Delta)^s u$, where $\varphi\in C^2(\mathbb{R})$ is convex, and a standard approximation argument. We stress that in [12], to prove the Kato inequality for $(-\Delta)^s$, the authors invoked the extension technique [11] and the Hopf lemma in [10]. When $m>0$ and $u\in\mathcal{S}(\mathbb{R}^N)$, we can exploit the integral representation formula for (1.1) (see Theorem 2.2) to infer a Kato's inequality for (1.1) in the pointwise sense; see Remark 3.4. The desired distributional inequality deserves more effort. To our knowledge, a distributional Kato's inequality has been obtained in [28] for the magnetic relativistic Schrödinger operator $(\sqrt{(-\imath\nabla-A(x))^2+m^2}\,)^\alpha$, where $\alpha\in(0,1]$ and $m>0$, or $\alpha=1$ and $m=0$, with $A\in L^2_{loc}(\mathbb{R}^N,\mathbb{R}^N)$, by following the original approach in [34] and using some commutator estimates. Anyway, these arguments do not work in our context because some estimates fail. Therefore we use a different strategy based on the choice of a suitable test function and some convenient estimates which permit us to pass to the limit in the weak formulation of the linear problem and achieve our goal. Note that our result is valid for all $m>0$ and $s\in(0,1)$.

Next, we study the boundedness of solutions for (1.7) under appropriate growth assumptions on the nonlinearity g, by combining a Brezis-Kato argument [7] with a Moser-iteration scheme [39]. After that, we show that the decay of solutions to (1.7) is of exponential type, contrary to the case $m=0$ for which is well-known that the decay of solutions is of power-type; see [23]. In order to achieve our aim, inspired by [4], we construct a suitable comparison function and we carry out some refined estimates which take care of an adequate estimate concerning $2s$-stable relativistic density with parameter m found in [27], and the asymptotic estimate (1.5). We recall that exponential estimates for positive solutions of (1.7) appeared in [19] in which $s = \frac{1}{2}$ and $g(x,u) = \mu u + |u|^{p-2}u + \sigma(W * u^2)u$ with $\mu < m$, $N \geqslant 3$, $p \in (2, \frac{2N}{N-1})$, $\nu, \sigma \geqslant 0$ (but not equal 0 booth), $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $r > \frac{N}{2}$, $W \geqslant 0$, $W(x) = W(|x|) \to 0$ as $|x| \to \infty$, and in [24] where $s \in (0,1)$ and $g(x,t) \geqslant 0$ is continuous, $g(x,t) \leqslant C_2|t|^p$ for some $p \in (1, \frac{N+2s}{N-2s})$, $t \mapsto \frac{g(x,t)}{t}$ is increasing for $t > 0$ and all $x \in \mathbb{R}^N$, g satisfies the Ambrosetti-Rabinowitz condition, and $0 \leqslant g(x,t) - \bar{g}(t) \leqslant C(|t| + |t|^p)$ for some continuous functions $\bar{g}$ and $a(x) \to 0$ as $|x| \to \infty$ such that the Lebesgue measure of $\{g(\cdot,t) > \bar{g}(t) \text{ for all } t > 0\}$ is positive. Nevertheless, our approach is totally different from these papers and holds for any $s \in (0,1)$ and more general nonlinearities.

The exponential decay plays an important role in the proof of Theorem 4.2 to deduce a Pohozaev-type identity for (1.1); see [24, 26, 33, 44] for related results on this identity. For completeness, we recall that Pohozaev-type identities for solutions of fractional elliptic equations of the form $(-\Delta)^s u = g(u)$ in $\mathbb{R}^N$ can be found in [5, 16, 26, 41].

Finally, when we consider positive solutions and the nonlinearity g in (1.7) does not depend on x, we prove the radial symmetry and the monotonicity under the assumption $g'(t) \leqslant m^{2s}$ by using the moving plane method. For the case $m = 0$ one can see [5, 17, 18, 23]. When $m > 0$, in [38] the authors used a Hardy-Littlewood-Sobolev type inequality for the Bessel potentials to establish the radial symmetry and monotonicity results for (1.7) with $g(x,u) = u^p$ with $p > 1$. Their approach does not work with more general nonlinearities so we develop a different strategy which combines a maximum principle for anti-symmetric functions and a key boundary estimate as in [17], taking into account the properties of the function $r \in (0,\infty) \mapsto K_{\frac{N+2s}{2}}(mr)r^{-\frac{N+2s}{2}}$; see Theorem 4.4. We also mention that the radial symmetry has been considered in [3] in which the author studied (1.7) with $g(x,u) = (m^{2s} - \mu)u + |u|^{p-2}u$, with $\mu > 0$ and $p \in (2, 2^*_s)$, via a variant of the extension procedure [11] given in [19, 22, 48] and applying the moving plane method.

We highlight that some of the results established in this work, such as, the L^∞-estimate, the Pohozaev identity and the radial symmetry of positive solutions, could be also proved by using the extension technique for (1.1). However, we prefer to work directly with (1.1) by making use of fractional Sobolev spaces and properties of Bessel functions that permit us to underline the nonlocal character of the involved operator.

The paper is organized as follows. In Section 2 we fix the notations and we collect some results of independent interest. In Section 3 we study linear theory for (1.1). First, we treat with Hölder-Schauder-Zygmund regularity results. Second, we give the proof of Kato's inequality for (1.1) and we present an application. In Section 4 we deal with nonlinear theory for (1.1). We start by showing an L^∞-estimate for the solutions of (1.7). Then we study the exponential decay of solutions and prove a Pohozaev-type identity. Finally, we consider the question related to the radial symmetry and monotonicity of positive solutions to (1.7) in the autonomous case $g(x,t) \equiv g(t)$.

2. Notations and first results

Let $p \in [1,\infty]$ and $A \subset \mathbb{R}^N$ be a measurable set. We denote by $L^p(A)$ the set of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{L^p(A)} := \begin{cases} \left(\int_A |u|^p \, dx\right)^{1/p} < \infty & \text{if } p < \infty, \\ \operatorname{esssup}_{x\in A}|u(x)| & \text{if } p = \infty. \end{cases}$$

Fix $s \in (0,1)$ and $m > 0$. Let $H^s_m(\mathbb{R}^N)$ be the fractional Sobolev space defined as the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{H^s_m(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}.$$

We recall that, when $N > 2s$, $H^s_m(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*_s)$, where $2^*_s := \frac{2N}{N-2s}$ is the fractional critical exponent, and compactly in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^*_s)$; see [1, 6, 14, 50].

Consider the Bessel kernel in the upper-half space $\mathbb{R}^{N+1}_+ := \{(x,y) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\, y > 0\}$ (see formula (110) in [22])

$$P_{s,m}(x,y) := C'_{N,s}\, y^{2s} m^{\frac{N+2s}{2}} |(x,y)|^{-\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(m|(x,y)|),$$

where

$$C'_{N,s} := p_{N,s} \frac{2^{\frac{N+2s}{2}-1}}{\Gamma(\frac{N+2s}{2})},$$

and $p_{N,s}$ is the constant for the Poisson kernel in $\mathbb{R}^{N+1}_+$ (see [11, 46, 48]), i.e.

$$P_s(x,y) := p_{N,s} \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{N+2s}{2}}}, \quad p_{N,s} := \frac{\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(s)}.$$

We recall that $P_{s,m}$ is the Fourier transform of $\xi \mapsto \theta_s(\sqrt{|\xi|^2+m^2})$ and that

$$\int_{\mathbb{R}^N} P_{s,m}(x,y)\,dx = \theta_s(my), \tag{2.1}$$

where $\theta_s \in H^1(\mathbb{R}_+, y^{1-2s})$ solves the following problem

$$\begin{cases} \theta_s'' + \frac{1-2s}{y}\theta_s' - \theta_s = 0 & \text{in } (0,\infty), \\ \theta_s(0)=1, \quad \lim_{y\to\infty}\theta_s(y)=0. \end{cases} \tag{2.2}$$

Note that θ_s is given by (see [22, 51])

$$\theta_s(y) = \frac{2^{1-s}}{\Gamma(s)}\, y^s K_s(y),$$

(observe that in the case $s=\frac{1}{2}$ we have $\theta_{\frac{1}{2}}(y)=e^{-y}$) and that satisfies the following properties:

- $\theta_s \in C^0([0,\infty)) \cap C^\infty((0,\infty))$. This fact is obvious in the case $s=\frac{1}{2}$, whereas for $s\neq\frac{1}{2}$ we use the smoothness of $K_s(y)$ for $y>0$ (see [6, 51]) and the formula (1.4).
- θ_s is decreasing in $[0,\infty)$. This is evident if $s=\frac{1}{2}$, while in the case $s\neq\frac{1}{2}$ we use $K_{-\nu}(y)=K_\nu(y)$ for $\nu\in\mathbb{R}$ (see formula (8) at pag.79 in [51]), $(y^\nu K_\nu(y))' = -y^\nu K_{\nu-1}(y)$ for $\nu\in\mathbb{R}$ (see formula (5) at pag.79 in [51]), $K_\nu(y)>0$ for $y>0$ and $\nu>-\frac{1}{2}$ (see formula (4) at pag.172 in [51]) and $s\in(0,1)$, to deduce that
$$\theta_s'(y) = \frac{2^{1-s}}{\Gamma(s)}(y^sK_s(y))' = -\frac{2^{1-s}}{\Gamma(s)}y^sK_{s-1}(y) = -\frac{2^{1-s}}{\Gamma(s)}y^sK_{1-s}(y)<0 \quad \text{for all } y>0.$$
Since $\theta_s(0)=1$, we also see that $0\leqslant\theta_s(y)\leqslant 1$ for all $y\geqslant 0$.

Finally, by using (2.2), an integration by parts, the above expression of $\theta_s'(y)$ and (1.4), we have

$$\int_0^\infty y^{1-2s}(|\theta_s(y)|^2+|\theta_s'(y)|^2)\,dy = -\lim_{y\to 0} y^{1-2s}\theta_s'(y) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)} =: \kappa_s. \tag{2.3}$$

With the previous notations, we can prove the following result motivated by Theorem 7.12 in [36] with $s=\frac{1}{2}$.

Theorem 2.1. *A function u is in $H^s_m(\mathbb{R}^N)$ if and only if it is in $L^2(\mathbb{R}^N)$ and*

$$I^t_s(u) := \frac{1}{t^{2s}}\left[(u,u)_{L^2(\mathbb{R}^N)} - (u, e^{-t(-\Delta+m^2)^s}u)_{L^2(\mathbb{R}^N)}\right]$$

is uniformly bounded in $t>0$, in which case

$$\sup_{t>0} I^t_s(u) = \lim_{t\to 0} I^t_s(u) = \frac{\kappa_s}{2s}\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi.$$

Here $(\cdot,\cdot)_{L^2(\mathbb{R}^N)}$ is the inner product in $L^2(\mathbb{R}^N)$, $e^{-t(\Delta+m^2)^s}(x,y) := P_{s,m}(x-y,t)$ and

$$\mathcal{F}(e^{-t(-\Delta+m^2)^s}u)(\xi) = \theta_s(t\sqrt{|\xi|^2+m^2})\mathcal{F}u(\xi). \tag{2.4}$$

Furthermore,

$$\int_{\mathbb{R}^N}\left[|(-\Delta+m^2)^{\frac{s}{2}}u|^2 - m^{2s}u^2\right]dx = \frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy \tag{2.5}$$

for all $u\in H^s_m(\mathbb{R}^N)$. Consequently,

$$\|u\|^2_{H^s_m(\mathbb{R}^N)} = m^{2s}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy. \tag{2.6}$$

Proof. The proof of formula (2.5) is contained in Proposition 6 in [22] see (*iii*) and (*iv*) of Theorem 7.12 in [36] for the case $s=\frac{1}{2}$). Combining (2.5) and the definition of H^s_m-norm, we deduce that (2.6) is satisfied. Now, we focus our attention on the first statement. It is sufficient to show that $u\in L^2(\mathbb{R}^N)$ and $I^t_s(u)$ is uniformly bounded in t if and only if

$$\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi < \infty.$$

Note that by Plancherel's theorem

$$I^t_s(u) = \frac{1}{t^{2s}}\int_{\mathbb{R}^N}[1-\theta_s(t\sqrt{|\xi|^2+m^2})]|\mathcal{F}u(\xi)|^2\,d\xi. \tag{2.7}$$

Define $\psi_s(y) := \frac{1-\theta_s(y)}{y^{2s}} \in C^\infty((0,\infty))$, and we show that $\psi_s(y)$ is decreasing for $y>0$. When $s=\frac{1}{2}$, then $\psi'_{\frac{1}{2}}(y) = \frac{y+1-e^y}{y^2 e^y} < 0$ for all $y>0$ thanks to $e^y > y+1$ for all $y>0$, and the claim is verified. Consider the case $s \neq \frac{1}{2}$. Since $(y^{-s}K_s(y))' = -y^{-s}K_{s+1}(y)$ (see formula (6) at pag.79 in [51]), we have

$$\psi'_s(y) = y^{-2s-1}\left[-2s + \frac{2^{1-s}}{\Gamma(s)} y^{s+1} K_{s+1}(y)\right].$$

Now, we observe that the function $g_s(y) := y^{s+1}K_{s+1}(y)$ is decreasing for $y>0$ because $(y^{-\nu}K_\nu(y))' = -y^{-\nu}K_{\nu+1}(y)$ for $\nu \in \mathbb{R}$ (see formula (5) at pag.79 in [51]) and $K_\nu(y) > 0$ for $\nu > -\frac{1}{2}$ (see formula (4) at pag.172 in [51]), imply that $g'_s(y) = -y^{s+1}K_s(y) < 0$ for all $y>0$, and that g_s satisfies the following limit

$$\lim_{y\to 0} g_s(y) = \Gamma(s+1)2^s = s\Gamma(s)2^s,$$

where we exploited (1.4) and the property $\Gamma(s+1) = s\Gamma(s)$. Therefore, $0 < g_s(y) < s\Gamma(s)2^s$ for all $y>0$, and this implies that

$$\psi'_s(y) = y^{-2s-1}\left[-2s + \frac{2^{1-s}}{\Gamma(s)} g_s(y)\right] < 0 \quad \text{for all } y>0.$$

Hence the desired monotonicity of ψ_s is proved. On the other hand, by $\theta_s(0)=1$ and (2.3), we get

$$\lim_{y\to 0}\psi_s(y) = -\frac{1}{2s}\lim_{y\to 0} y^{1-2s}\theta'_s(y) = \frac{\kappa_s}{2s}.$$

Then we can see that

$$\frac{1-\theta_s(t\sqrt{|\xi|^2+m^2})}{t^{2s}} = \psi_s(t\sqrt{|\xi|^2+m^2})(|\xi|^2+m^2)^s$$

and that it converges monotonically to $\frac{\kappa_s}{2s}(|\xi|^2+m^2)^s$ as $t\to 0$. Thus if we suppose $u \in H^s_m(\mathbb{R}^N)$, $I^t_s(u)$ is uniformly bounded in t. Conversely, if $I^t_s(u)$ is uniformly bounded in t, the monotone convergence theorem yields

$$\frac{\kappa_s}{2s}\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi = \lim_{t\to 0} I^t_s(u) = \sup_{t>0} I^t_s(u) < \infty,$$

that is $u \in H^s_m(\mathbb{R}^N)$. This concludes the proof of Theorem 2.1. □

Remark 2.1. *Since $H^s_m(\mathbb{R}^N)$ is a Hilbert space with the inner product*

$$(u,v)_{H^s_m(\mathbb{R}^N)} := \int_{\mathbb{R}^N}(|\xi|^2+m^2)^s\mathcal{F}u(\xi)\mathcal{F}v(\xi)\,d\xi, \ \ \textit{for all } u,v \in H^s_m(\mathbb{R}^N),$$

it follows from (2.6) and

$$\|u-v\|^2_{H^s_m(\mathbb{R}^N)} = \|u\|^2_{H^s_m(\mathbb{R}^N)} + \|v\|^2_{H^s_m(\mathbb{R}^N)} - 2(u,v)_{H^s_m(\mathbb{R}^N)}$$

that

$$(u,v)_{H^s_m(\mathbb{R}^N)} = m^{2s}(u,v)_{L^2(\mathbb{R}^N)} + \frac{C(N,s)}{2} m^{\frac{N+2s}{2}} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy.$$

Remark 2.2. *By (2.6), it is easily seen that if $u \in H^s_m(\mathbb{R}^N)$ and $\varphi:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function with Lipschitz constant $L>0$ and such that $\varphi(0)=0$, then $\varphi(u) \in H^s_m(\mathbb{R}^N)$. Indeed, we know that $|\varphi(t_1)-\varphi(t_2)| \leqslant L|t_1-t_2|$ for all $t_1,t_2 \in \mathbb{R}$. Choosing $t_1 = u(x)$ and $t_2 = 0$, we deduce that $u \in L^2(\mathbb{R}^N)$. Taking $t_1 = u(x)$ and $t_2 = u(y)$, and using (2.6), we conclude that $\|\varphi(u)\|_{H^s_m(\mathbb{R}^N)} \leqslant L\|u\|_{H^s_m(\mathbb{R}^N)}$.*

As byproduct of Theorem 2.1, we can prove the following Pólya-Szegö-type inequality which shows that the norm in $H^s_m(\mathbb{R}^N)$ does not increase under rearrangement (see also [2, 25] for the case of $W^{s,p}(\mathbb{R}^N)$-norm, with $s \in (0,1)$ and $p \in [1,\infty)$).

Proposition 2.1. *(Pólya-Szegö-type inequality) Let $u:\mathbb{R}^N \to \mathbb{R}$ be a measurable function that vanishes at infinity, that is $|\{x \in \mathbb{R}^N : |u(x)| > t\}| < \infty$ for all $t>0$, and let u^* denote its symmetric-decreasing rearrangement. Assume that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy < \infty.$$

Then,

$$\iint_{\mathbb{R}^{2N}} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy \leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy. \tag{2.8}$$

In particular, if $u \in H^s_m(\mathbb{R}^N)$ then

$$\|u^*\|_{H^s_m(\mathbb{R}^N)} \leqslant \|u\|_{H^s_m(\mathbb{R}^N)}. \tag{2.9}$$

The inequality in (2.8) *is strict unless u is the translate of a symmetric-decreasing function.*

Proof. The proof is inspired by Lemma 7.17 in [36] where (2.9) is stated without proof in the case $s=\frac{1}{2}$ in Remark (3). Without loss of generality, we can replace u by $|u|$, which does not change u^* and only decreases the right hand side of (2.8) because $||u(x)|-|u(y)||\leqslant |u(x)-u(y)|$ for all $x,y\in\mathbb{R}^N$. Thus we may suppose that $u\geqslant 0$. We first show that it suffices to prove (2.8) for functions in $L^2(\mathbb{R}^N)$. Take $c\in(0,1)$ and define

$$u_c(x):=\min\left\{\max\{u(x)-c,0\},\frac{1}{c}\right\}.$$

It follows from the definition of the rearrangement that $(u_c)^*=(u^*)_c$. Since u vanishes at infinity, $u_c\in L^2(\mathbb{R}^N)$. Now we observe that $|u_c(x)-u_c(y)|\leqslant|u(x)-u(y)|$ for all $x,y\in\mathbb{R}^N$. By the monotone convergence theorem we have that

$$\lim_{c\to 0}\iint_{\mathbb{R}^{2N}}\frac{|u_c(x)-u_c(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy,$$

and the same holds for u^*.

Now we assume that $u\in L^2(\mathbb{R}^N)$. Since $u\in H^s_m(\mathbb{R}^N)$, we can see that Theorem 2.1 yields

$$\lim_{t\to 0}I^t_s(u)=\frac{\kappa_s}{2s}\|u\|^2_{H^s_m(\mathbb{R}^N)}.$$

Recalling that $\mathcal{F}(f*g)=(2\pi)^{\frac{N}{2}}\mathcal{F}(f)\mathcal{F}(g)$ and using (2.4), it follows from the Parseval identity and (2.7) that

$$I^t_s(u)=\frac{1}{t^{2s}}\left[\int_{\mathbb{R}^N}u^2(x)\,dx-\iint_{\mathbb{R}^{2N}}u(x)P_{s,m}(x-y,t)u(y)\,dxdy\right].\tag{2.10}$$

The $L^2(\mathbb{R}^N)$ norm of u does not change under rearrangements and the second term on the right-hand side in (2.10) increases by Riesz's rearrangement inequality (see Theorem 3.7 in [36]). Thus, $I^t_s(u^*)\leqslant I^t_s(u)$ and by using Theorem 2.1 we deduce that $I^t_s(u^*)$ converges to $\frac{\kappa_s}{2s}\|u^*\|^2_{H^s_m(\mathbb{R}^N)}$. Consequently, (2.9) is true, and using (2.6) and $\|u^*\|_{L^2(\mathbb{R}^N)}=\|u\|_{L^2(\mathbb{R}^N)}$, we obtain (2.8). The strict inequality in (2.8) is a consequence of Lemma $A.2$ in [25] with $J(t)=t^2$ that is a non-negative strictly convex function on $\mathbb{R}$ such that $J(0)=0$, and $k(x)=K_{\frac{N+2s}{2}}(m|x|)|x|^{-\frac{N+2s}{2}}$ which is a L^1 symmetric strictly-decreasing function. □

Remark 2.3. *When $m=1$ and $u\in H^s_1(\mathbb{R}^N)$, the proof of the inequality* (2.9) *is essentially contained in Proposition* 4 *in [44] but we kept it for the sake of completeness.*

We also have the following integral representation formula for (1.1) (see Theorem 4.5.2 in [5] for a proof).

Theorem 2.2. *Let $s\in(0,1)$ and $m>0$. Then, for all $u\in\mathcal{S}(\mathbb{R}^N)$,*

$$(-\Delta+m^2)^s u(x)=m^{2s}u(x)+\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\int_{\mathbb{R}^N}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|y|)\,dy.$$

Let us observe that one can define (1.1) on spaces of functions with weaker regularity. Indeed, as emphasized in [45], the natural space to work with the fractional Laplacian $(-\Delta)^s$ (that is $m=0$) is given by

$$L_s:=\left\{u:\mathbb{R}^N\to\mathbb{R}:\int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2s}}\,dx<\infty\right\}.$$

Concerning the case $m>0$, taking into account the asymptotic behaviors (1.4) and (1.5), we can see that

$$\frac{K_{\frac{N+2s}{2}}(t)}{t^{\frac{N+2s}{2}}}\sim\Gamma\left(\frac{N+2s}{2}\right)2^{-(\frac{N+2s-2}{2})}t^{-(N+2s)}\qquad\text{as } t\to 0,$$

and

$$\frac{K_{\frac{N+2s}{2}}(t)}{t^{\frac{N+2s}{2}}}\sim\sqrt{\frac{\pi}{2}}\frac{e^{-t}}{t^{\frac{N+1+2s}{2}}}\qquad\text{as } t\to\infty.$$

Therefore it is natural to introduce the next space to work with (1.1)

$$L^{\exp}_s:=\left\{u:\mathbb{R}^N\to\mathbb{R}\text{ measurable }:\int_{\mathbb{R}^N}\frac{|u(x)|e^{-|x|}}{1+|x|^{\frac{N+2s+1}{2}}}\,dx<\infty\right\}$$

endowed with the norm

$$\|u\|_{L^{\exp}_s}:=\int_{\mathbb{R}^N}\frac{|u(x)|e^{-|x|}}{1+|x|^{\frac{N+2s+1}{2}}}\,dx.$$

Equivalently, $u \in L_s^{\exp}$ if and only if $u \in L^1(\mathbb{R}^N, d\mu)$ where $d\mu(x) := \frac{e^{-|x|}}{1+|x|^{\frac{N+2s+1}{2}}}$. Then one can check that for all $u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap L_s^{\exp}$

$$P.V. \int_{\mathbb{R}^N} \frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)dy < \infty$$

and hence the definition (1.3) makes sense for this class of functions.

Finally, we introduce the Bessel potentials and Bessel potential spaces; see [1,6,14,46,49] for more details. The Bessel potential of order $\alpha > 0$ of $u \in \mathcal{S}(\mathbb{R}^N)$ is defined as

$$\mathcal{J}_\alpha u(x) := (-\Delta+1)^{-\frac{\alpha}{2}} u(x) = (\mathcal{G}_\alpha * u)(x) = \int_{\mathbb{R}^N} \mathcal{G}_\alpha(x-y)u(y)\, dy,$$

where $\mathcal{G}_\alpha$ given through the Fourier transform

$$\mathcal{F}\mathcal{G}_\alpha(\xi) := (2\pi)^{-\frac{N}{2}}(|\xi|^2+1)^{-\frac{\alpha}{2}}$$

is the so-called Bessel kernel. If $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$), one can define the Bessel potential of a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^N)$; see [1,14].

We list some interesting formulas and properties of $\mathcal{G}_\alpha$ with $\alpha > 0$ (see section 4 in [6], chapter 4 in [49] and chapter 5 in [46]):

- The kernel $\mathcal{G}_\alpha$ is given by
$$\mathcal{G}_\alpha(x) = \frac{1}{2^{\frac{N+\alpha-2}{2}} \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})} K_{\frac{N-\alpha}{2}}(|x|)|x|^{\frac{\alpha-N}{2}}.$$
- The kernel $\mathcal{G}_\alpha(x)$ is everywhere positive, decreasing function of $|x|$, analytic except at $x = 0$, and for $x \in \mathbb{R}^N \setminus \{0\}$, $\mathcal{G}_\alpha(x)$ is an entire function of α.
- From (1.4) and (1.5), we obtain
$$\mathcal{G}_\alpha(x) \sim \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})} |x|^{\alpha-N} \text{ as } |x| \to 0, \text{ if } 0 < \alpha < N,$$
$$\mathcal{G}_\alpha(x) \sim \frac{1}{2^{\frac{N+\alpha-1}{2}} \pi^{\frac{N-1}{2}} \Gamma(\frac{\alpha}{2})} |x|^{\frac{\alpha-N-1}{2}} e^{-|x|} \text{ as } |x| \to \infty.$$
- $\mathcal{G}_\alpha \in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \mathcal{G}_\alpha(x)\, dx = 1$.
- We have the composition formula $\mathcal{G}_{\alpha+\beta} = \mathcal{G}_\alpha * \mathcal{G}_\beta$ for all $\alpha, \beta > 0$.
- The following integral formula holds:
$$\mathcal{G}_\alpha(x) = \frac{1}{(4\pi)^{\frac{N}{2}}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty e^{-\frac{|x|^2}{4\delta}} e^{-\delta} \delta^{\frac{\alpha-N}{2}} \frac{d\delta}{\delta}.$$

For $p \in [1, \infty]$ and $\alpha \in \mathbb{R}$, we define the Bessel potential space

$$\mathscr{L}^p_\alpha := \mathcal{J}_\alpha(L^p(\mathbb{R}^N)) = \{u \in \mathcal{S}'(\mathbb{R}^N) : u = \mathcal{J}_\alpha f, f \in L^p(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{\mathscr{L}^p_\alpha} := \|f\|_{L^p(\mathbb{R}^N)} \quad \text{ if } u = \mathcal{J}_\alpha f.$$

With respect to this norm, $\mathscr{L}^p_\alpha$ is a Banach space. Clearly, $\mathscr{L}^p_\alpha$ is a subspace of $L^p(\mathbb{R}^N)$ for all $\alpha \geqslant 0$. We summarize some of its properties.

Theorem 2.3. *[1,14,46]*

(i) If $\alpha \geqslant 0$ and $1 \leqslant p < \infty$, then $\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathscr{L}^p_\alpha$.
(ii) If $1 < p < \infty$ and p' its conjugate exponent, then the dual of $\mathscr{L}^p_\alpha$ is isometrically isomorphic to $\mathscr{L}^{p'}_{-\alpha}$.
(iii) If $\beta < \alpha$, then $\mathscr{L}^p_\alpha$ is continuously embedded in $\mathscr{L}^p_\beta$.
(iv) If $\beta \leqslant \alpha$ and if either $1 < p \leqslant q \leqslant \frac{Np}{N-(\alpha-\beta)p} < \infty$ or $p = 1$ and $1 \leqslant q < \frac{N}{N-\alpha+\beta}$, then $\mathscr{L}^p_\alpha$ is continuously embedded in $\mathscr{L}^q_\beta$.
(v) If $0 < \mu \leqslant \alpha - \frac{N}{p} < 1$, then $\mathscr{L}^p_\alpha$ is continuously embedded in $C^{0,\mu}(\mathbb{R}^N)$.
(vi) $\mathscr{L}^p_k = W^{k,p}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$ and $1 < p < \infty$, $\mathscr{L}^2_\alpha = W^{\alpha,2}(\mathbb{R}^N)$ for any α.
(vii) If $1 < p < \infty$ and $\varepsilon > 0$, then for every α we have the following continuous embeddings:

$$\mathscr{L}^p_{\alpha+\varepsilon} \subset W^{\alpha,p}(\mathbb{R}^N) \subset \mathscr{L}^p_{\alpha-\varepsilon}.$$

3. Linear Theory for (1.1)

3.1. **Regularity results.** We collect some regularity results for linear equations involving (1.1). First, we fix some notations. For $k \in \mathbb{N} \cup \{0\}$, we denote by $C_b^k(\mathbb{R}^N)$ the set of all $C^k(\mathbb{R}^N)$ functions whose derivatives up to order k are bounded. Let $\alpha \in (0,1]$ and $k \in \mathbb{N} \cup \{0\}$. We define the Hölder space $C^{k,\alpha}(\mathbb{R}^N)$ as

$$C^{0,\alpha}(\mathbb{R}^N) := \left\{ u \in C_b^0(\mathbb{R}^N) : [u]_{C^{0,\alpha}(\mathbb{R}^N)} := \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\} \text{ if } k=0,$$
$$C^{k,\alpha}(\mathbb{R}^N) := \left\{ u \in C_b^k(\mathbb{R}^N) : D^\gamma u \in C^{0,\alpha}(\mathbb{R}^N) \text{ for all } |\gamma| = k \right\} \text{ if } k \geqslant 1,$$

endowed with the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{R}^N)} := \sum_{|\gamma| \leqslant k} \|D^\gamma u\|_{L^\infty(\mathbb{R}^N)} + \sum_{|\gamma|=k} [D^\gamma u]_{C^{0,\alpha}(\mathbb{R}^N)}.$$

Let us now introduce the Lipschitz spaces (also called Hölder-Zygmund spaces) Λ_α with $\alpha > 0$; see [14, 30, 35, 46, 49, 50, 54]. When $\alpha \in (0,1)$, we set $\Lambda_\alpha := C^{0,\alpha}(\mathbb{R}^N)$. If $\alpha = 1$, we define the Zygmund space (see [54])

$$\Lambda_1 := \left\{ u \in C_b^0(\mathbb{R}^N) : \sup_{|h|>0} \frac{\|u(\cdot+h)+u(\cdot-h)-2u(\cdot)\|_{L^\infty(\mathbb{R}^N)}}{|h|} < \infty \right\},$$

equipped with the norm

$$\|u\|_{\Lambda_1} := \|u\|_{L^\infty(\mathbb{R}^N)} + \sup_{|h|>0} \frac{\|u(\cdot+h)+u(\cdot-h)-2u(\cdot)\|_{L^\infty(\mathbb{R}^N)}}{|h|}.$$

If $\alpha > 1$, we put

$$\Lambda_\alpha := \left\{ u \in C_b^0(\mathbb{R}^N) : \frac{\partial u}{\partial x_j} \in \Lambda_{\alpha-1} \text{ for all } j = 1, \dots, N \right\},$$

endowed with the norm

$$\|u\|_{\Lambda_\alpha} := \|u\|_{L^\infty(\mathbb{R}^N)} + \sum_{j=1}^k \left\| \frac{\partial u}{\partial x_j} \right\|_{\Lambda_{\alpha-1}}.$$

Note that $\Lambda_\beta \subset \Lambda_\alpha$ if $0 < \alpha < \beta$, Λ_α is complete, and $\Lambda_\alpha = C^{[\alpha],\alpha-[\alpha]}(\mathbb{R}^N)$ if $\alpha > 0$ and $\alpha \notin \mathbb{N}$, where $[x] := \max\{k \in \mathbb{Z} : k \leqslant x\}$ for $x \in \mathbb{R}$. Moreover, $C_b^1(\mathbb{R}^N) \subset C^{0,1}(\mathbb{R}^N) \subset \Lambda_1$ and that these inclusions are strict; see [35, 46, 49]. We recall the following useful result (see Theorem 4 at pag.149 in [46], Proposition 8.6.6 in [30] and Theorem 5 in [49]):

Theorem 3.1. *[30, 46, 49] Let $\alpha \geqslant 0$ and $\beta > 0$. Then $\mathcal{J}_{2\alpha} = (-\Delta+1)^{-\alpha}$ maps Λ_β isomorphically onto $\Lambda_{\beta+2\alpha}$.*

Remark 3.1. *For $s > 0$ and $m > 0$, set $\mathcal{J}_{2s,m}u(x) := (-\Delta+m^2)^{-s}u(x) = (\mathcal{G}_{2s,m} * u)(x)$, where $\mathcal{G}_{2s,m}$ is defined by $\mathcal{F}\mathcal{G}_{2s,m}(\xi) := (2\pi)^{-\frac{N}{2}}(|\xi|^2+m^2)^{-s}$. Clearly, $\mathcal{J}_{2s,1} = \mathcal{J}_{2s}$ and $\mathcal{G}_{2s,1} = \mathcal{G}_{2s}$. By the scaling property of the Fourier transform, it follows that $\mathcal{G}_{2s,m}(x) = m^{N-2s}\mathcal{G}_{2s}(mx)$. Then, for $\beta > 0$, $\mathcal{J}_{2s,m} = (-\Delta+m^2)^{-s}$ maps Λ_β isomorphically onto $\Lambda_{\beta+2s}$.*

As a consequence of Theorem 3.1 and the definition of Λ_α, we easily deduce the following Schauder-Zygmund regularity result (see Proposition 5.1 in [13] and Proposition 2.8 in [45] for the case $m = 0$).

Corollary 3.1. *(Schauder-Zygmund regularity) Let $s \in (0,1)$, $m > 0$ and $\alpha \in (0,1]$. Assume that $f \in C^{0,\alpha}(\mathbb{R}^N)$ and let $u := \mathcal{J}_{2s,m}f$. Then $u \in \Lambda_{\alpha+2s}$. In particular:*

- *If $\alpha+2s < 1$, then $u \in C^{0,\alpha+2s}(\mathbb{R}^N)$.*
- *If $1 < \alpha+2s < 2$, then $u \in C^{1,\alpha+2s-1}(\mathbb{R}^N)$.*
- *If $2 < \alpha+2s < 3$, then $u \in C^{2,\alpha+2s-2}(\mathbb{R}^N)$.*
- *If $\alpha+2s = k \in \{1,2\}$, then $u \in \Lambda_k$.*

Next we give a Schauder-Hölder-Zygmund regularity result for $(-\Delta+m^2)^s$ with $m > 0$ (see Proposition 5.2 in [13] and Proposition 2.9 in [45] for the case $m = 0$)

Theorem 3.2. *(Schauder-Hölder-Zygmund regularity) Let $s \in (0,1)$ and $m > 0$. Assume that $f \in L^\infty(\mathbb{R}^N)$ and let $u := \mathcal{J}_{2s,m}f$. Then $u \in \Lambda_{2s}$. In particular:*

- *If $2s < 1$, then $u \in C^{0,2s}(\mathbb{R}^N)$.*
- *If $2s = 1$, then $u \in \Lambda_1$.*
- *If $2s > 1$, then $u \in C^{1,2s-1}(\mathbb{R}^N)$.*

Proof. This result is essentially known in the literature. Indeed the scale of spaces $\Lambda_t = C^t_* = B^t_{\infty,\infty}$ has been defined for all $t \in \mathbb{R}$ long ago. It is a case of Besov spaces $B^t_{p,q}$ (see [43, 50]) (denoted by $\Lambda^{p,q}_t$ in [46]). The isomorphism property of $\mathcal{J}_r = (1-\Delta)^{-r/2}$ from C^t_* to C^{t+r}_* for all real t and r is proved, for instance, in Proposition 8.6.6 in [30]; see also []. Since L^∞ is a subset of C^0_* (recalled e.g. Proposition 2.1 in [43]), the fact that the solution operator $\mathcal{J}_{2s}$ maps C^0_* into C^{2s}_* implies immediately that L^∞ is mapped into C^{2s}_*. Nevertheless, we prefer to give a proof of this fact by following the approach in Stein's book [46]. In view of Remark 3.1, we can take $m = 1$. We show that $u = \mathcal{G}_{2s} * f \in \Lambda_{2s}$. For this purpose, we use the following characterization for Lipschitz spaces Λ_α, with $\alpha > 0$, in terms of the Poisson integral given in [46, 49]:

$$u \in \Lambda_\alpha \Longleftrightarrow u \in L^\infty(\mathbb{R}^N) \text{ and } \left\| \frac{\partial^k U}{\partial y^k}(\cdot, y) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant C y^{-k+\alpha}, \quad y > 0,$$

where $k := [\alpha] + 1$ and $U(x,y) := (P_{\frac{1}{2}}(\cdot, y) * u)(x)$ is the so-called Poisson integral of u. In the above characterization of Λ_α, one can replace k by any integer greater than α. Therefore, using the fact that $s \in (0,1)$,

$$u \in \Lambda_{2s} \Longleftrightarrow u \in L^\infty(\mathbb{R}^N) \text{ and } \left\| \frac{\partial^2 U}{\partial y^2}(\cdot, y) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant C y^{-2+2s}, \quad y > 0.$$

Since $\mathcal{G}_{2s} \in L^1(\mathbb{R}^N)$, with $\|\mathcal{G}_{2s}\|_{L^1(\mathbb{R}^N)} = 1$, and $f \in L^\infty(\mathbb{R}^N)$, by Young's inequality we deduce that $u \in C^0_b(\mathbb{R}^N)$ and $\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant \|f\|_{L^\infty(\mathbb{R}^N)}$. Now, we observe that

$$U(x,y) = [P(\cdot, y) * (\mathcal{G}_{2s} * f)](x) = (G_{2s}(\cdot, y) * f)(x),$$

where $G_{2s}(x,y)$ is the Poisson integral of $\mathcal{G}_{2s}$. We recall that for all integer ℓ greater than α it holds (see formula (59) at pag. 149 in [46])

$$\left\| \frac{\partial^\ell G_\alpha}{\partial y^\ell}(\cdot, y) \right\|_{L^1(\mathbb{R}^N)} \leqslant C y^{-\ell+\alpha}, \quad y > 0.$$

On the other hand, we can see that $\mathcal{G}_{2s} * f \in \Lambda_s$ because $\|\mathcal{G}_s(\cdot + h) - \mathcal{G}_s(\cdot)\|_{L^1(\mathbb{R}^N)} \leqslant C|h|^s$ (see at pag. 158 in [46]), $f \in L^\infty(\mathbb{R}^N)$ and Young's inequality imply

$$\|\mathcal{G}_s(\cdot + h) - \mathcal{G}_s(\cdot)\|_{L^\infty(\mathbb{R}^N)} \leqslant C|h|^s \|f\|_{L^\infty(\mathbb{R}^N)},$$

and that by differentiating the identity $G_{2s}(\cdot, y) = G_{2s}(\cdot, y_1) * G_{2s}(\cdot, y_2)$, with $y = y_1 + y_2$, we find

$$\frac{\partial^2 U}{\partial y^2}(\cdot, y) = \frac{\partial G_s}{\partial y_1}(\cdot, y_1) * \frac{\partial}{\partial y_2}(G_s(\cdot, y_2) * f), \text{ with } y = y_1 + y_2.$$

Consequently, by using Young's inequality, and choosing $y_1 = y_2 = \frac{y}{2}$, we get

$$\begin{aligned} \left\| \frac{\partial^2 U}{\partial y^2}(\cdot, y) \right\|_{L^\infty(\mathbb{R}^N)} &\leqslant \left\| \frac{\partial G_s}{\partial y_1}(\cdot, y_1) \Big|_{y_1 = \frac{y}{2}} \right\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial}{\partial y_2}(G_s(\cdot, y_2) * f) \Big|_{y_2 = \frac{y}{2}} \right\|_{L^\infty(\mathbb{R}^N)} \\ &\leqslant C y^{-2+2s} \|f\|_{L^\infty(\mathbb{R}^N)}, \quad y > 0. \end{aligned}$$

The proof of Theorem 3.2 is now complete. □

Remark 3.2. *If $u \in L^\infty(\mathbb{R}^N)$ is such that $f := (-\Delta + m^2)^s u \in L^\infty(\mathbb{R}^N)$ (respectively, $f \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1]$), then we can argue as in [13, 47] to obtain the estimate $\|u\|_{\Lambda_{2s}} \leqslant C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(\mathbb{R}^N)})$ (respectively, $\|u\|_{\Lambda_{\alpha+2s}} \leqslant C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{0,\alpha}(\mathbb{R}^N)})$) for some constant $C > 0$. We only give the details when $f \in L^\infty(\mathbb{R}^N)$ because the case $f \in C^{0,\alpha}(\mathbb{R}^N)$ can be studied in a similar way. Let $f \in L^\infty(\mathbb{R}^N)$ with compact support and $u = (-\Delta + m^2)^{-s} f$. Set $e^{t\Delta} u(x) := (W(\cdot, t) * u)(x)$, where $W(x,t) := \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}}$ is the Gauss-Weierstrass kernel. Note that the expression below*

$$\|u\|_{L^\infty(\mathbb{R}^N)} + [u]_{\Lambda_{2s}} := \|u\|_{L^\infty(\mathbb{R}^N)} + \sup_{t>0} t^{1-s} \left\| \partial_t e^{t\Delta} u(\cdot) \right\|_{L^\infty(\mathbb{R}^N)}$$

is a norm in Λ_{2s} equivalent to $\|\cdot\|_{\Lambda_\beta}$ (see [49]). Using standard estimates for W and its derivatives, it is easy to check that $|\partial_t e^{t\Delta} f(x)| \leqslant \frac{c}{t} \|f\|_{L^\infty(\mathbb{R}^N)}$ for $x \in \mathbb{R}^N$ and $t > 0$, Now, by the gamma function identity

$$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-r\lambda} \frac{dr}{r^{1-\alpha}}, \quad \textit{for all } \lambda > 0,$$

we have

$$u(x) = (-\Delta + m^2)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-m^2 r} e^{r\Delta} f(x) \frac{dr}{r^{1-s}}.$$

Hence,

$$e^{t\Delta}u(x)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-m^2r}e^{t\Delta}(e^{r\Delta}f)(x)\,\frac{dr}{r^{1-s}},$$

and recalling the semigroup property $e^{t\Delta}e^{r\Delta}f=e^{(t+r)\Delta}f$, *we get*

$$\begin{aligned}
t^{1-s}\left|\partial_t e^{t\Delta}u(x)\right| &\leqslant Ct^{1-s}\int_0^\infty e^{-m^2r}\left|(\partial_w e^{w\Delta}f(x))_{|w=r+t}\right|\frac{dr}{r^{1-s}}\\
&\leqslant Ct^{1-s}\|f\|_{L^\infty(\mathbb{R}^N)}\int_0^\infty \frac{e^{-m^2r}}{(t+r)^k}\,\frac{dr}{r^{1-s}}\\
&\leqslant Ct^{1-s}\|f\|_{L^\infty(\mathbb{R}^N)}\left[t^{-1}\int_0^t r^{s-1}\,dr+e^{-m^2t}\int_t^\infty r^{s-2}\,dr\right]\\
&\leqslant C\|f\|_{L^\infty(\mathbb{R}^N)}
\end{aligned}$$

for $x\in\mathbb{R}^N$ *and* $t>0$, *which implies that* $u\in\Lambda_{2s}$ *and the required estimate is true. When* $f\in L^\infty(\mathbb{R}^N)$ *ha son compact support we only give a sketch of the proof; see [45,47] for more details. Take* $\eta\in C_c^\infty(\mathbb{R}^N)$ *such that* $0\leqslant\eta\leqslant1$, $\mathrm{supp}(\eta)\subset B_2(0)$, $\eta=1$ *in* $B_1(0)$, *and write* $f=f_1+f_2$, *where* $f_1:=\eta f$ *and* $f_2:=(1-\eta)f$, *and set* $u:=u_1+u_2$, *where* u_1 *is the solution to* $(-\Delta+m^2)^s u_1=f_1$ *in* $\mathbb{R}^N$. *Since* f_1 *has compact support, we can apply the above estimate to deduce that* $\|u_1\|_{\Lambda_{2s}}\leqslant C(\|u_1\|_{L^\infty(\mathbb{R}^N)}+\|f_1\|_{L^\infty(\mathbb{R}^N)})$. *On the other hand,* $u_2\in L^\infty(\mathbb{R}^N)$ *solves* $(-\Delta+m^2)^s u_2=0$ *in* $B_1(0)$, *so* u_2 *is smooth in* $B_{\frac12}(0)$ *(by hypoellipticity of* $(-\Delta+m^2)^s$ *(see [29]) or by using Proposition 4 in [22]) and all its derivatives in* $B_{\frac12}(0)$ *are bounded by*

$$C\|u_2\|_{L^\infty(\mathbb{R}^N)}=C\|u-u_1\|_{L^\infty(\mathbb{R}^N)}\leqslant C(\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(\mathbb{R}^N)}).$$

Putting together the estimates for u_1 *and* u_2, *we get the desired estimate for* u.

Remark 3.3. *In view of* (1.3), *and the fact that* (1.4) *and* (1.5) *imply the following useful estimate:*

$$0<t^{-\frac{N+2s}{2}}K_{\frac{N+2s}{2}}(mt)\leqslant Ct^{-(N+2s)}\quad \text{for all } t>0, \tag{3.1}$$

for some $C=C(N,m,s)>0$, *one could proceed as in [45] to deduce some of the regularity results above exposed (more precisely, the corresponding Propositions* 2.8 *and* 2.9 *in [45]). Anyway, here we use a different approach in order to obtain sharp regularity when* $\alpha+2s\in\mathbb{N}$ *in Corollary 3.1, and* $2s=1$ *in Theorem 3.2, which are not included in [45].*

We conclude this subsection by proving a simple L^∞-regularity result for a linear problem that will be used later.

Lemma 3.1. *Let* $\mu\in(0,m^{2s})$ *and* $f\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. *Let* $u\in H^s_m(\mathbb{R}^N)$ *be the unique weak solution of*

$$(-\Delta+m^2)^s u-\mu u=f \ \text{ in } \mathbb{R}^N, \tag{3.2}$$

that is u *satisfies* $(u,v)_{H^s_m(\mathbb{R}^N)}=(f,v)_{L^2(\mathbb{R}^N)}$ *for all* $v\in H^s_m(\mathbb{R}^N)$. *Then,* $u\in L^\infty(\mathbb{R}^N)$ *and*

$$\|u\|_{L^\infty(\mathbb{R}^N)}\leqslant\frac{1}{m^{2s}-\mu}\|f\|_{L^\infty(\mathbb{R}^N)}. \tag{3.3}$$

Proof. We first note that the existence and uniqueness of u is guaranteed by the Lax-Milgram theorem. Indeed,

$$\|u\|_e:=\sqrt{\|u\|^2_{H^s_m(\mathbb{R}^N)}-\mu\|u\|^2_{L^2(\mathbb{R}^N)}}$$

is a norm in $H^s_m(\mathbb{R}^N)$ equivalent to $\|\cdot\|_{H^s_m(\mathbb{R}^N)}$ due to the fact that

$$\mu\|u\|^2_{L^2(\mathbb{R}^N)}=\frac{\mu}{m^{2s}}(m^{2s}\|u\|^2_{L^2(\mathbb{R}^N)})\leqslant\frac{\mu}{m^{2s}}\|u\|^2_{H^s_m(\mathbb{R}^N)}$$

and thus

$$\left(1-\frac{\mu}{m^{2s}}\right)\|u\|^2_{H^s_m(\mathbb{R}^N)}\leqslant\|u\|_e^2\leqslant\|u\|^2_{H^s_m(\mathbb{R}^N)}.$$

Now we follow an argument found in the proof of Lemma 3.5 in [32]. Take $r>\frac{1}{m^{2s}-\mu}\|f\|_{L^\infty(\mathbb{R}^N)}$ and consider the Lipschitz function $\varphi:[0,\infty)\to[0,\infty)$ given by

$$\varphi(t):=\begin{cases}0 & \text{if } 0\leqslant t\leqslant r,\\ \frac{t-r}{t} & \text{if } t>r.\end{cases}$$

Note that $0\leqslant\varphi<1$ in $[0,\infty)$, and $t_1|\varphi(t_1)-\varphi(t_2)|\leqslant t_2|\varphi(t_1)-\varphi(t_2)|\leqslant|t_1-t_2|$ for all $0\leqslant t_1\leqslant t_2$. By using these properties and that $|u|\in H^s_m(\mathbb{R}^N)$ (by Remark 2.2), we see that $u\varphi(|u|)\in H^s_m(\mathbb{R}^N)$. Indeed, $|u\varphi(|u|)|^2\leqslant|u|^2$, and

$$|u(x)\varphi(|u(x)|)-u(y)\varphi(|u(y)|)|^2\leqslant[|u(x)-u(y)|\varphi(|u(x)|)+|\varphi(|u(x)|)-\varphi(|u(x)|)||u(y)|]^2$$

$$\leqslant 2[|u(x)-u(y)|^2 + ||u(x)|-|u(y)||^2] \leqslant 4|u(x)-u(y)|^2,$$

which combined with (2.6) implies the claim. Then, testing (3.2) with $u\varphi(|u|)$, we have

$$(u, u\varphi(|u|))_{H^s_m(\mathbb{R}^N)} - \mu(u, u\varphi(|u|))_{L^2(\mathbb{R}^N)} = (f, u\varphi(|u|))_{L^2(\mathbb{R}^N)},$$

which can be written as

$$\big[(u, u\varphi(|u|))_{H^s_m(\mathbb{R}^N)} - m^{2s}(u, u\varphi(|u|))_{L^2(\mathbb{R}^N)}\big] + (m^{2s}-\mu)(u, u\varphi(|u|))_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} fu\varphi(|u|)\,dx.$$

Now we observe that the term $[\dots]$ in the above identity is nonnegative because, by using the fact that φ is nondecreasing,

$$\begin{aligned}&(u(x)-u(y))(u(x)\varphi(|u(x)|)-u(y)\varphi(|u(y)|))\\ &= \frac{1}{2}(u(x)-u(y))^2(\varphi(|u(x)|)+\varphi(|u(y)|)) + \frac{1}{2}(|u(x)|^2-|u(y)|^2)(\varphi(|u(x)|)-\varphi(|u(y)|)) \geqslant 0.\end{aligned}$$

Therefore,

$$(m^{2s}-\mu)\int_{\mathbb{R}^N} u^2\varphi(|u|)\,dx \leqslant \int_{\mathbb{R}^N} fu\varphi(|u|)\,dx,$$

and recalling that $\varphi(t)=0$ for $0\leqslant t\leqslant r$, we obtain

$$\int_{\{|u|>r\}} |u|\varphi(|u|)\left(|u| - \frac{1}{m^{2s}-\mu}\|f\|_{L^\infty(\mathbb{R}^N)}\right) dx \leqslant 0.$$

Since $r > \frac{1}{m^{2s}-\mu}\|f\|_{L^\infty(\mathbb{R}^N)}$ and $\varphi(t)>0$ for $t>r$, we achieve a contradiction unless $\{|u|>r\}$ is of measure zero, that is $|u|\leqslant r$ a.e. in $\mathbb{R}^N$. By the arbitrariness of $r > \frac{1}{m^{2s}-\mu}\|f\|_{L^\infty(\mathbb{R}^N)}$, we conclude that (3.3) is true. □

3.2. **A Kato's inequality.** We present a Kato's inequality [34] for (1.1). Along this subsection, we use the notation $H^s_m(\mathbb{R}^N,\mathbb{K})$ with $\mathbb{K}=\mathbb{R},\mathbb{C}$, to emphasize when we consider functions $u:\mathbb{R}^N\to\mathbb{R}$ or $u:\mathbb{R}^N\to\mathbb{C}$. For $z\in\mathbb{C}$, we denote by $\Re(z)$ its real part and $\bar{z}$ is its conjugate. The proof of the following useful inequality is immediate.

Lemma 3.2. *(Diamagnetic inequality) Let $u\in H^s_m(\mathbb{R}^N,\mathbb{C})$. Then $|u|\in H^s_m(\mathbb{R}^N,\mathbb{R})$ and*

$$\||u|\|_{H^s_m(\mathbb{R}^N)} \leqslant \|u\|_{H^s_m(\mathbb{R}^N)}.$$

Proof. This is a direct consequence of the inequality $||z|-|w||\leqslant|z-w|$ for all $z,w\in\mathbb{C}$ and the validity of (2.6). □

We now prove the main result of this subsection.

Theorem 3.3. *(Kato's inequality) Let $u\in H^s_m(\mathbb{R}^N,\mathbb{C})$ and $f\in L^1_{loc}(\mathbb{R}^N,\mathbb{C})$ be such that*

$$\begin{aligned}&\Re\left(\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\overline{(\psi(x)-\psi(y))}\,dxdy + m^{2s}\int_{\mathbb{R}^N} u\bar{\psi}\,dx\right)\\ &= \Re\left(\int_{\mathbb{R}^N} f\bar{\psi}\,dx\right)\end{aligned} \tag{3.4}$$

for all $\psi:\mathbb{R}^N\to\mathbb{C}$ measurable with compact support and such that

$$\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy < \infty.$$

Then it holds $(-\Delta+m^2)^s|u| \leqslant \Re(\mathrm{sign}(\bar{u})f)$ in the distributional sense, that is

$$\begin{aligned}&\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}} \frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy + m^{2s}\int_{\mathbb{R}^N}|u|\varphi\,dx\\ &\qquad\leqslant \Re\left(\int_{\mathbb{R}^N}\mathrm{sign}(\bar{u})f\varphi\,dx\right)\end{aligned} \tag{3.5}$$

for all $\varphi\in C^\infty_c(\mathbb{R}^N,\mathbb{R})$ such that $\varphi\geqslant 0$ in $\mathbb{R}^N$, where

$$\mathrm{sign}(\bar{u})(x) := \begin{cases} \frac{\overline{u(x)}}{|u(x)|} & \text{if } u(x)\neq 0,\\ 0 & \text{if } u(x)=0.\end{cases}$$

Proof. We follow the approach used in the proof of Theorem 17.3.5 in [5]. Take $\varphi \in C^{\infty}_{c}(\mathbb{R}^{N},\mathbb{R})$ such that $\varphi \geqslant 0$. For $\delta>0$, we set $u_{\delta}(x):=\sqrt{|u(x)|^{2}+\delta^{2}}$, and we test equation (3.4) by

$$\omega_{\delta}(x):=\frac{u(x)}{u_{\delta}(x)}\varphi(x).$$

Firstly, we show that ω_{δ} is admissible as test function. It is clear that ω_{δ} has compact support. On the other hand, we can observe

$$\begin{aligned}\omega_{\delta}(x)-\omega_{\delta}(y)&=\left(\frac{u(x)}{u_{\delta}(x)}\right)\varphi(x)-\left(\frac{u(y)}{u_{\delta}(y)}\right)\varphi(y)\\&=[u(x)-u(y)]\frac{\varphi(x)}{u_{\delta}(x)}+\left[\frac{\varphi(x)}{u_{\delta}(x)}-\frac{\varphi(y)}{u_{\delta}(y)}\right]u(y)\\&=[u(x)-u(y)]\frac{\varphi(x)}{u_{\delta}(x)}+\left[\frac{1}{u_{\delta}(x)}-\frac{1}{u_{\delta}(y)}\right]\varphi(x)u(y)+[\varphi(x)-\varphi(y)]\frac{u(y)}{u_{\delta}(y)}\end{aligned}$$

which implies that

$$\begin{aligned}&|\omega_{\delta}(x)-\omega_{\delta}(y)|^{2}\\&\leqslant\frac{4}{\delta^{2}}|u(x)-u(y)|^{2}\|\varphi\|^{2}_{L^{\infty}(\mathbb{R}^{N})}+4\left|\frac{u(y)}{u_{\delta}(y)}\right|^{2}\frac{1}{|u_{\delta}(x)|^{2}}\|\varphi\|^{2}_{L^{\infty}(\mathbb{R}^{N})}|u_{\delta}(y)-u_{\delta}(x)|^{2}+4|\varphi(x)-\varphi(y)|^{2}\\&\leqslant\frac{4}{\delta^{2}}|u(x)-u(y)|^{2}\|\varphi\|^{2}_{L^{\infty}(\mathbb{R}^{N})}+\frac{4}{\delta^{2}}||u(x)|-|u(y)||^{2}\|\varphi\|^{2}_{L^{\infty}(\mathbb{R}^{N})}+4|\varphi(x)-\varphi(y)|^{2},\end{aligned}$$

where we used the following elementary inequalities

$$\begin{aligned}&|z+w+k|^{2}\leqslant 4(|z|^{2}+|w|^{2}+|k|^{2})\quad\text{for all } z,w,k\in\mathbb{C},\\&|\sqrt{|z|^{2}+\delta^{2}}-\sqrt{|w|^{2}+\delta^{2}}|\leqslant||z|-|w||\quad\text{for all } z,w\in\mathbb{C},\end{aligned}$$

and that $|e^{\imath t}|=1$ for all $t\in\mathbb{R}$, $u_{\delta}\geqslant\delta$, $|\frac{u}{u_{\delta}}|\leqslant 1$. Since $u\in H^{s}_{m}(\mathbb{R}^{N},\mathbb{C})$, $|u|\in H^{s}_{m}(\mathbb{R}^{N},\mathbb{R})$ (by Lemma 3.2) and $\varphi\in C^{\infty}_{c}(\mathbb{R}^{N},\mathbb{R})$, we deduce that

$$\iint_{\mathbb{R}^{2N}}\frac{|\omega_{\delta}(x)-\omega_{\delta}(y)|^{2}}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy<\infty.$$

Then we have

$$\begin{aligned}&\Re\left[\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}}\left(\frac{\overline{u(x)}}{u_{\delta}(x)}\varphi(x)-\frac{\overline{u(y)}}{u_{\delta}(y)}\varphi(y)\right)K_{\frac{N+2s}{2}}(m|x-y|)dxdy+m^{2s}\int_{\mathbb{R}^{N}}u\frac{\overline{u}}{u_{\delta}}\varphi\,dx\right]\\&=\Re\left(\int_{\mathbb{R}^{N}}f\frac{\overline{u}}{u_{\delta}}\varphi\,dx\right).\end{aligned}\tag{3.6}$$

Since $\Re(z)\leqslant|z|$ for all $z\in\mathbb{C}$, we see that

$$\begin{aligned}&\Re\left[(u(x)-u(y))\left(\frac{\overline{u(x)}}{u_{\delta}(x)}\varphi(x)-\frac{\overline{u(y)}}{u_{\delta}(y)}\varphi(y)\right)\right]\\&=\Re\left[\frac{|u(x)|^{2}}{u_{\delta}(x)}\varphi(x)+\frac{|u(y)|^{2}}{u_{\delta}(y)}\varphi(y)-\frac{u(x)\overline{u(y)}}{u_{\delta}(y)}\varphi(y)-\frac{u(y)\overline{u(x)}}{u_{\delta}(x)}\varphi(x)\right]\\&\geqslant\left[\frac{|u(x)|^{2}}{u_{\delta}(x)}\varphi(x)+\frac{|u(y)|^{2}}{u_{\delta}(y)}\varphi(y)-|u(x)|\frac{|u(y)|}{u_{\delta}(y)}\varphi(y)-|u(y)|\frac{|u(x)|}{u_{\delta}(x)}\varphi(x)\right].\end{aligned}\tag{3.7}$$

Now, we note that

$$\begin{aligned}&\frac{|u(x)|^{2}}{u_{\delta}(x)}\varphi(x)+\frac{|u(y)|^{2}}{u_{\delta}(y)}\varphi(y)-|u(x)|\frac{|u(y)|}{u_{\delta}(y)}\varphi(y)-|u(y)|\frac{|u(x)|}{u_{\delta}(x)}\varphi(x)\\&=\frac{|u(x)|}{u_{\delta}(x)}(|u(x)|-|u(y)|)\varphi(x)-\frac{|u(y)|}{u_{\delta}(y)}(|u(x)|-|u(y)|)\varphi(y)\\&=\left[\frac{|u(x)|}{u_{\delta}(x)}(|u(x)|-|u(y)|)\varphi(x)-\frac{|u(x)|}{u_{\delta}(x)}(|u(x)|-|u(y)|)\varphi(y)\right]+\left(\frac{|u(x)|}{u_{\delta}(x)}-\frac{|u(y)|}{u_{\delta}(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\\&=\frac{|u(x)|}{u_{\delta}(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))+\left(\frac{|u(x)|}{u_{\delta}(x)}-\frac{|u(y)|}{u_{\delta}(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\\&\geqslant\frac{|u(x)|}{u_{\delta}(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))\end{aligned}\tag{3.8}$$

where in the last inequality we used the fact that

$$\left(\frac{|u(x)|}{u_\delta(x)}-\frac{|u(y)|}{u_\delta(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\geqslant 0$$

because $h(t):=\frac{t}{\sqrt{t^2+\delta^2}}$ is increasing for $t\geqslant 0$ and $\varphi\geqslant 0$ in $\mathbb{R}^N$. Recalling (3.1), we have

$$\begin{aligned}&\frac{|\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))|}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\\&\leqslant \frac{||u(x)|-|u(y)||\,|\varphi(x)-\varphi(y)|}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\\&\leqslant C\frac{||u(x)|-|u(y)||}{|x-y|^{\frac{N+2s}{2}}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\frac{N+2s}{2}}}\in L^1(\mathbb{R}^{2N},\mathbb{R}).\end{aligned}$$

Since $\frac{|u(x)|}{u_\delta(x)}\to 1$ a.e. in $\mathbb{R}^N$ as $\delta\to 0$, we can use (3.7), (3.8) and the dominated convergence theorem to deduce that

$$\begin{aligned}&\liminf_{\delta\to 0}\Re\left[\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}}\left(\frac{\overline{u(x)}}{u_\delta(x)}\varphi(x)-\frac{\overline{u_n(y)}}{u_\delta(y)}\varphi(y)\right)K_{\frac{N+2s}{2}}(m|x-y|)dxdy+m^{2s}\int_{\mathbb{R}^N}u\frac{\overline{u}}{u_\delta}\varphi\,dx\right]\\&\geqslant\liminf_{\delta\to 0}\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}}\frac{|u(x)|}{u_\delta(x)}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy+\lim_{\delta\to 0}m^{2s}\int_{\mathbb{R}^N}\frac{|u|^2}{u_\delta}\varphi\,dx\\&\geqslant\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy+m^{2s}\int_{\mathbb{R}^N}|u|\varphi\,dx.\end{aligned}\tag{3.9}$$

On the other hand, observing that $|f\frac{\overline{u}}{u_\delta}\varphi|\leqslant|f\varphi|\in L^1(\mathbb{R}^N,\mathbb{R})$ and $f\frac{\overline{u}}{u_\delta}\varphi\to f\mathrm{sign}(\bar{u})\varphi$ a.e. in $\mathbb{R}^N$ as $\delta\to 0$, we can apply the dominated convergence theorem to infer that as $\delta\to 0$

$$\Re\left(\int_{\mathbb{R}^N}f\,\frac{\overline{u}}{u_\delta}\varphi\,dx\right)\to\Re\left(\int_{\mathbb{R}^N}f\,\mathrm{sign}(\bar{u})\varphi\,dx\right).\tag{3.10}$$

Combining (3.6), (3.9) and (3.10), we see that (3.5) is valid. □

Remark 3.4. *If $u\in\mathcal{S}(\mathbb{R}^N,\mathbb{R})$, then we can obtain a pointwise Kato's inequality. Indeed, if $\varphi\in C^1(\mathbb{R},\mathbb{R})$ is a convex function, by using Theorem 2.2 and the fact that a convex function is above its tangent line, we deduce the following (pointwise) Córdoba-Córdoba type inequality*

$$[(-\Delta+m^2)^s-m^{2s}]\varphi(u(x))\leqslant\varphi'(u(x))[(-\Delta+m^2)^s-m^{2s}]u(x).\tag{3.11}$$

Taking $\varphi_\varepsilon(t):=\sqrt{t^2+\varepsilon^2}$, with $\varepsilon>0$, in (3.11) and sending $\varepsilon\to 0$ we find

$$[(-\Delta+m^2)^s-m^{2s}]|u(x)|\leqslant\mathrm{sign}(u(x))[(-\Delta+m^2)^s-m^{2s}]u(x),$$

or equivalently

$$(-\Delta+m^2)^s|u(x)|\leqslant\mathrm{sign}(u(x))(-\Delta+m^2)^su(x).$$

Finally we give a simple application of Theorem 3.3 to nonnegative potentials.

Corollary 3.2. *Let $V\in L^2_{loc}(\mathbb{R}^N,\mathbb{R})$ be such that $V\geqslant 0$ a.e. in $\mathbb{R}^N$. If $u\in H^s_m(\mathbb{R}^N,\mathbb{C})$ satisfies*

$$(-\Delta+m^2)^su=-(V(x)+1)u\ \text{ in }\mathbb{R}^N,$$

in the following sense

$$\begin{aligned}&\Re\left(\frac{C(N,s)}{2}\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\overline{(\psi(x)-\psi(y))}\,dxdy+m^{2s}\int_{\mathbb{R}^N}u\bar{\psi}\,dx\right)\\&\quad=-\Re\left(\int_{\mathbb{R}^N}(V(x)+1)u\bar{\psi}\,dx\right)\end{aligned}$$

for all $\psi:\mathbb{R}^N\to\mathbb{C}$ measurable with compact support and such that

$$\iint_{\mathbb{R}^{2N}}\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy<\infty,$$

then $u\equiv 0$.

Proof. Note that $f := -(V+1)u \in L^1_{loc}(\mathbb{R}^N,\mathbb{C})$ because of $V \in L^2_{loc}(\mathbb{R}^N,\mathbb{R})$ and $u \in L^2(\mathbb{R}^N,\mathbb{C})$. Then, using Theorem 3.3 and $V \geqslant 0$, we deduce that, in the distributional sense,

$$(-\Delta+m^2)^s|u| \leqslant -(V(x)+1)|u| \leqslant 0 \text{ in } \mathbb{R}^N.$$

Since $|u| \in H^s_m(\mathbb{R}^N,\mathbb{R})$ and $C^\infty_c(\mathbb{R}^N,\mathbb{R})$ is dense in $H^s_m(\mathbb{R}^N,\mathbb{R})$, we can find a sequence $(\varphi_n) \subset C^\infty_c(\mathbb{R}^N,\mathbb{R})$, $\varphi_n \geqslant 0$ for all $n\in\mathbb{N}$ such that $\varphi_n \to |u|$ in $H^s_m(\mathbb{R}^N,\mathbb{R})$. Hence, $(|u|,\varphi_n)_{H^s_m(\mathbb{R}^N)} \leqslant 0$, and by passing to the limit as $n\to\infty$ we find $\||u|\|^2_{H^s_m(\mathbb{R}^N)} \leqslant 0$, from which $u\equiv 0$ in $\mathbb{R}^N$. This ends the proof of the corollary. □

4. Nonlinear theory for (1.1)

4.1. **An L^∞-estimate.** This subsection is devoted to establish the boundedness of solutions for subcritical or critical nonlinear problems driven by (1.1). We combine a Brezis-Kato type argument [7] with a Moser iteration scheme [39].

Lemma 4.1. *(L^∞-estimate) Let $s\in(0,1)$, $m>0$ and $N>2s$. Let $u\in H^s_m(\mathbb{R}^N)$ be a weak solution to (1.7), where $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $|g(x,t)| \leqslant C_0(|t|+|t|^p)$, for some $p\in[1,2^*_s-1]$ and $C_0>0$. Then, $u\in L^q(\mathbb{R}^N)$ for all $q\in[2,\infty]$.*

Proof. We argue as in the proof of Proposition 3.2.14 in [5]. Assume that $u\not\equiv 0$ (otherwise, there is nothing to prove) and that $\{|u|>1\}$ has positive measure (otherwise, $|u|\leqslant 1$ a.e. in $\mathbb{R}^N$, that is $u\in L^\infty(\mathbb{R}^N)$, and using the fact that $u\in L^2(\mathbb{R}^N)$ we deduce, by interpolation, that $u\in L^q(\mathbb{R}^N)$ for all $q\in[2,\infty]$). For any $L>0$ and $\beta>0$, we consider the Lipschitz function $t\in\mathbb{R}\mapsto tt_L^{2\beta}$, where $t_L := \min\{|t|,L\}$. We recall the following elementary inequality (see Lemma 3.1 in [31]):

$$(a-b)(aa_L^{2\beta}-bb_L^{2\beta}) \geqslant \frac{2\beta+1}{(\beta+1)^2}(aa_L^\beta-bb_L^\beta)^2 \quad \text{for all } a,b\in\mathbb{R}. \tag{4.1}$$

Taking $uu_L^{2\beta}\in H^s_m(\mathbb{R}^N)$ as test function in the weak formulation of (1.7) and using Remark 2.1, we have

$$\begin{aligned}&\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}}((uu_L^{2\beta})(x)-(uu_L^{2\beta})(y))K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy\\&+m^{2s}\int_{\mathbb{R}^N}u^2u_L^{2\beta}\,dx=\int_{\mathbb{R}^N}g(x,u)uu_L^{2\beta}\,dx\end{aligned}$$

which combined with (4.1) gives

$$\begin{aligned}&\frac{C_1}{\beta+1}\left[\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{|w_L(x)-w_L(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy+m^{2s}\int_{\mathbb{R}^N}w_L^2\,dx\right]\\&\leqslant\int_{\mathbb{R}^N}g(x,u)uu_L^{2\beta}\,dx,\end{aligned}\tag{4.2}$$

where $w_L := uu_L^\beta$ and $C_1>0$ is a constant independent of L and β. Now, by using (2.5), the inequality $(a^2+b^2)^s \geqslant 2^{s-1}a^{2s}$ for all $a,b\geqslant 0$, Propositions 3.4 and 6.5 in [21], we see that

$$\begin{aligned}&\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^{2N}}\frac{|w_L(x)-w_L(y)|^2}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy+m^{2s}\int_{\mathbb{R}^N}w_L^2\,dx\\&=\int_{\mathbb{R}^N}|(-\Delta+m^2)^{\frac{s}{2}}w_L|^2\,dx\\&=\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}w_L(\xi)|^2\,d\xi\\&\geqslant 2^{s-1}\int_{\mathbb{R}^N}|\xi|^{2s}|\mathcal{F}w_L(\xi)|^2\,d\xi\\&=D(N,s)\iint_{\mathbb{R}^{2N}}\frac{|w_L(x)-w_L(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&\geqslant E(N,s)\|w_L\|^2_{L^{2^*_s}(\mathbb{R}^N)},\end{aligned}\tag{4.3}$$

where $D(N,s)$ and $E(N,s)$ are positive constants depending only on N and s. From (4.2), (4.3), and the growth assumption on g, we deduce that,

$$\|w_L\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C_2(\beta+1)\int_{\mathbb{R}^N}(u^2u_L^{2\beta}+|u|^{p+1}u_L^{2\beta})\,dx,\tag{4.4}$$

where $C_2 := C_0(C_1E(N,s))^{-1} > 0$. Now, we prove that there exist a constant $c > 0$ and a function $h \in L^{N/2s}(\mathbb{R}^N)$, $h \geqslant 0$ and independent of L and β, such that

$$u^2u_L^{2\beta} + |u|^{p+1}u_L^{2\beta} \leqslant (c+h)u^2u_L^{2\beta} \quad \text{on } \mathbb{R}^N. \tag{4.5}$$

Firstly, we notice that

$$u^2u_L^{2\beta} + |u|^{p+1}u_L^{2\beta} = u^2u_L^{2\beta} + |u|^{p-1}u^2u_L^{2\beta} \quad \text{on } \mathbb{R}^N.$$

Moreover,

$$|u|^{p-1} \leqslant 1+h \quad \text{on } \mathbb{R}^N,$$

for some $h \in L^{N/2s}(\mathbb{R}^N)$. Indeed,

$$|u|^{p-1} = \chi_{\{|u|\leqslant 1\}}|u|^{p-1} + \chi_{\{|u|>1\}}|u|^{p-1} \leqslant 1 + \chi_{\{|u|>1\}}|u|^{p-1} \quad \text{on } \mathbb{R}^N,$$

and if $(p-1)\frac{N}{2s} < 2$ then

$$\int_{\mathbb{R}^N} \chi_{\{|u|>1\}}|u|^{\frac{N}{2s}(p-1)}\,dx \leqslant \int_{\mathbb{R}^N} \chi_{\{|u|>1\}}|u|^2\,dx < \infty,$$

while if $2 \leqslant (p-1)\frac{N}{2s}$ we deduce that $\frac{N}{2s}(p-1) \in [2, 2_s^*]$. Therefore, $h := \chi_{\{|u|>1\}}|u|^{p-1}$ satisfies the desired properties. Taking into account (4.4) and (4.5), we obtain that

$$\|w_L\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant C_2(\beta+1)\int_{\mathbb{R}^N}(c+h(x))u^2u_L^{2\beta}\,dx,$$

and, by Fatou's lemma and the monotone convergence theorem, we can pass to the limit as $L \to \infty$ to infer that

$$\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant C_2c(\beta+1)\int_{\mathbb{R}^N}|u|^{2(\beta+1)}\,dx + C_2(\beta+1)\int_{\mathbb{R}^N}h(x)|u|^{2(\beta+1)}\,dx. \tag{4.6}$$

Fix $M > 0$ and let $A_1 := \{h \leqslant M\}$ and $A_2 := \{h > M\}$. Then,

$$\int_{\mathbb{R}^N} h(x)|u|^{2(\beta+1)}\,dx \leqslant M\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)} + \varepsilon(M)\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \tag{4.7}$$

where

$$\varepsilon(M) := \left(\int_{A_2} h^{N/2s}\,dx\right)^{\frac{2s}{N}} \to 0 \text{ as } M \to \infty.$$

In view of (4.6) and (4.7), we get

$$\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant C_2(\beta+1)(c+M)\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)} + C_2(\beta+1)\varepsilon(M)\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)}. \tag{4.8}$$

Choosing $M > 0$ sufficiently large so that

$$C_2(\beta+1)\varepsilon(M) < \frac{1}{2},$$

and using (4.8) we obtain

$$\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant 2C_2(\beta+1)(c+M)\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)}. \tag{4.9}$$

Then we can start a bootstrap argument: since $u \in L^{2_s^*}(\mathbb{R}^N)$ we can apply (4.9) with $\beta_1+1 = \frac{N}{N-2s}$ to deduce that $u \in L^{\frac{(\beta_1+1)2N}{N-2s}}(\mathbb{R}^N) = L^{\frac{2N^2}{(N-2s)^2}}(\mathbb{R}^N)$. Applying again (4.9), after k iterations, we find $u \in L^{\frac{2N^k}{(N-2s)^k}}(\mathbb{R}^N)$, and so $u \in L^q(\mathbb{R}^N)$ for all $q \in [2,\infty)$.

Now we prove that $u \in L^\infty(\mathbb{R}^N)$. Since $u \in L^q(\mathbb{R}^N)$ for all $q \in [2,\infty)$ we have that $h \in L^{\frac{N}{s}}(\mathbb{R}^N)$. By the generalized Hölder inequality, we can see that for all $\lambda > 0$

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)|u|^{2(\beta+1)}\,dx &\leqslant \|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)}\||u|^{\beta+1}\|_{L^2(\mathbb{R}^N)}\||u|^{\beta+1}\|_{L^{2_s^*}(\mathbb{R}^N)} \\ &\leqslant \|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)}\left(\lambda\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\lambda}\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)}\right).\end{aligned}$$

Then, using (4.6), we deduce that

$$\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant C_2(\beta+1)(c+\lambda\|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)})\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)} + \frac{C_2(\beta+1)\|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)}}{\lambda}\||u|^{\beta+1}\|^2_{L^{2_s^*}(\mathbb{R}^N)}. \tag{4.10}$$

Taking $\lambda > 0$ such that

$$\frac{C_2(\beta+1)\|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)}}{\lambda} = \frac{1}{2}$$

we find that

$$\||u|^{\beta+1}\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leqslant 2C_2(\beta+1)(c+\lambda\|h\|_{L^{\frac{N}{s}}(\mathbb{R}^N)})\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)} =: M_\beta\||u|^{\beta+1}\|^2_{L^2(\mathbb{R}^N)}.$$

and the advantage with respect to (4.9) is that now we control the dependence on β of the constant M_β. Indeed, recalling our choice of λ, for some constant $M_0>0$ independent of β it holds

$$M_\beta \leqslant \left(2C_2c+4C_2^2\|h\|^2_{L^{\frac{N}{s}}(\mathbb{R}^N)}\right)(1+\beta)^2 \leqslant M_0^2 e^{2\sqrt{\beta+1}},$$

which implies that

$$\|u\|_{L^{2^*_s(\beta+1)}(\mathbb{R}^N)} \leqslant M_0^{\frac{1}{\beta+1}} e^{\frac{1}{\sqrt{\beta+1}}}\|u\|_{L^{2(\beta+1)}(\mathbb{R}^N)}.$$

Iterating this last relation and choosing $\beta_0=0$ and $2(\beta_{n+1}+1)=2^*_s(\beta_n+1)$, we get

$$\|u\|_{L^{2^*_s(\beta_n+1)}(\mathbb{R}^N)} \leqslant M_0^{\sum_{i=0}^n \frac{1}{\beta_i+1}} e^{\sum_{i=0}^n \frac{1}{\sqrt{\beta_i+1}}}\|u\|_{L^{2(\beta_0+1)}(\mathbb{R}^N)}.$$

Since $1+\beta_n=(\frac{N}{N-2s})^n$, we have that

$$\sum_{i=0}^\infty \frac{1}{\beta_i+1}<\infty \quad \text{and} \quad \sum_{i=0}^\infty \frac{1}{\sqrt{\beta_i+1}}<\infty$$

and from this we deduce that

$$\|u\|_{L^\infty(\mathbb{R}^N)} = \lim_{n\to\infty}\|u\|_{L^{2^*_s(\beta_n+1)}(\mathbb{R}^N)}<\infty.$$

This completes the proof of Lemma 4.1. □

4.2. **Exponential decay.** We focus our attention on the exponential decay of solutions to (1.7). We recall that in the case $m=0$ (see [23]), under the assumption $\lim_{t\to 0}\frac{g(x,t)}{t}=0$ uniformly in $x\in\mathbb{R}^N$, every classical positive solution u to $(-\Delta)^s u = g(x,u)$ in $\mathbb{R}^N$, has a power-type decay at infinity, more precisely

$$0<u(x)\leqslant C|x|^{-(N+2s)} \quad \text{for all } |x|\geqslant 1.$$

Firstly, we recall the following comparison principle for $(-\Delta+m^2)^s$ established in [4].

Lemma 4.2. *(Comparison principle) Let $\Omega\subset\mathbb{R}^N$ be an open set, $\gamma<m^{2s}$, $u_1,u_2\in H^s_m(\mathbb{R}^N)$ be such that $u_1\leqslant u_2$ in $\mathbb{R}^N\setminus\Omega$ and $(-\Delta+m^2)^s u_1-\gamma u_1\leqslant(-\Delta+m^2)^s u_2-\gamma u_2$ in Ω, that is*

$$(m^{2s}-\gamma)\int_{\mathbb{R}^N} u_1(x)v(x)\,dx+\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^N}\frac{(u_1(x)-u_1(y))(v(x)-v(y))}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy$$
$$\leqslant (m^{2s}-\gamma)\int_{\mathbb{R}^N} u_2(x)v(x)\,dx+\frac{C(N,s)}{2}m^{\frac{N+2s}{2}}\iint_{\mathbb{R}^N}\frac{(u_2(x)-u_2(y))(v(x)-v(y))}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dxdy$$

for all $v\in H^s_m(\mathbb{R}^N)$ such that $v\geqslant 0$ in $\mathbb{R}^N$ and $v=0$ in $\mathbb{R}^N\setminus\Omega$. Then $u_1\leqslant u_2$ in $\mathbb{R}^N$.

Remark 4.1. *In the case $\Omega=\mathbb{R}^N$, the previous result reads as follows: if $\gamma<m^{2s}$ and $u_1,u_2\in H^s_m(\mathbb{R}^N)$ are such that $(-\Delta+m^2)^s u_1-\gamma u_1\leqslant(-\Delta+m^2)^s u_2-\gamma u_2$ in $\mathbb{R}^N$ then $u_1\leqslant u_2$ in $\mathbb{R}^N$.*

Now we prove the following key lemma.

Lemma 4.3. *Let $\mu\in(0,m^{2s})$ and $\phi\in L^2(\mathbb{R}^N)$. Then there exists a unique function $\bar{w}\in H^s_m(\mathbb{R}^N)$ which solves*

$$(-\Delta+m^2)^s\bar{w}-\mu\bar{w}=\phi \quad \text{in } \mathbb{R}^N. \tag{4.11}$$

If in addition $\phi\in L^\infty(\mathbb{R}^N)$, $\mathrm{supp}(\phi)$ is compact, $\phi\geqslant 0$ and $\phi\not\equiv 0$, then $\bar{w}\in C^{0,\alpha}(\mathbb{R}^N)$, $\bar{w}>0$ in $\mathbb{R}^N$ and $\bar{w}$ has exponential decay, that is there exist $c,C>0$ such that

$$0<\bar{w}(x)\leqslant Ce^{-c|x|} \quad \text{for all } x\in\mathbb{R}^N. \tag{4.12}$$

Proof. The existence and uniqueness of $\bar{w}$ is guaranteed by the Lax-Milgram theorem. Now, taking the Fourier transform in (4.11), we have $\bar{w}=\mathcal{B}_{2s,m}*\phi$, where

$$\mathcal{B}_{2s,m}(x):=(2\pi)^{-\frac{N}{2}}\mathcal{F}^{-1}([(|\xi|^2+m^2)^s-\mu]^{-1}).$$

Assume in addition that $\phi\in L^\infty(\mathbb{R}^N)$, $\mathrm{supp}(\phi)$ is compact, $\phi\geqslant 0$, and $\phi\not\equiv 0$. By comparison (see Remark 4.1 with $u_1=0$, $u_2=\bar{w}$ and $\gamma=\mu$), we see that $\bar{w}\geqslant 0$ in $\mathbb{R}^N$. Combining Lemma 3.1 and Theorem 3.2, we deduce that $\bar{w}\in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$. Since $\bar{w}\in C^{0,\alpha}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, we get $\bar{w}(x)\to 0$ as $|x|\to\infty$. Note that $\bar{w}=\mathcal{G}_{2s,m}*(\mu\bar{w}+\phi)$, where $\mathcal{G}_{2s,m}$ is defined as in Remark 3.1. Clearly, $\|\mathcal{G}_{2s,m}\|_{L^1(\mathbb{R}^N)}=(2\pi)^{\frac{N}{2}}\mathcal{F}\mathcal{G}_{2s,m}(0)=m^{-2s}$. Let us show that $\bar{w}>0$ in $\mathbb{R}^N$. In fact, if there exists $x_0\in\mathbb{R}^N$

such that $\bar{w}(x_0) = \min_{\mathbb{R}^N} \bar{w}$, observing that $\mathcal{G}_{2s,m} * \phi > 0$ (since $\mathcal{G}_{2s,m}$ is everywhere positive and $\phi \geqslant 0$ but $\not\equiv 0$), we have

$$\begin{aligned}\bar{w}(x_0) &= \int_{\mathbb{R}^N} \mathcal{G}_{2s,m}(y)[\mu\bar{w}(x_0-y) + \phi(x_0-y)]\,dy\\ &> \mu\int_{\mathbb{R}^N} \mathcal{G}_{2s,m}(y)\bar{w}(x_0-y)\,dy\\ &\geqslant \mu\bar{w}(x_0)\|\mathcal{G}_{2s,m}\|_{L^1(\mathbb{R}^N)} = \frac{\mu}{m^{2s}}\bar{w}(x_0),\end{aligned}$$

which combined with $\mu \in (0, m^{2s})$ gives $\bar{w}(x_0) > 0$ and this is impossible because $\bar{w}(x) \to 0$ as $|x| \to \infty$. Hence, $\bar{w}$ does not have any global minimum on $\mathbb{R}^N$ and thus $\bar{w} > 0$ in $\mathbb{R}^N$. Since ϕ has compact support, the exponential decay of $\bar{w}$ at infinity follows if we show the exponential decay of $\mathcal{B}_{2s,m}(x)$ for big values of $|x|$. After that, due to the fact that $\bar{w}$ is continuous in $\mathbb{R}^N$, we can deduce the exponential decay of $\bar{w}$ in the whole of $\mathbb{R}^N$. Next we prove the exponential decay of $\mathcal{B}_{2s,m}(x)$ for $|x|$ large. We argue as in the proof of Theorem 1.1 in [4]. Then we have

$$\begin{aligned}\mathcal{B}_{2s,m}(x) &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{\imath\xi\cdot x}\frac{1}{[(|\xi|^2+m^2)^s - \mu]}\,d\xi\\ &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{\imath\xi\cdot x}\left(\int_0^\infty e^{-t[(|\xi|^2+m^2)^s-\mu]}\,dt\right)d\xi\\ &= \int_0^\infty e^{-\gamma t}\left(\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{\imath\xi\cdot x}e^{-t[(|\xi|^2+m^2)^s-m^{2s}]}\,d\xi\right)dt\\ &= \int_0^\infty e^{-\gamma t}\,p_{s,m}(x,t)\,dt \end{aligned} \tag{4.13}$$

where $\gamma := m^{2s} - \mu > 0$, and

$$p_{s,m}(x,t) := e^{m^{2s}t}\int_0^\infty \frac{1}{(4\pi z)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4z}}e^{-m^2 z}\vartheta_s(t,z)\,dz$$

is the transition density function of the relativistic $2s$-stable process with parameter m (see formula (7) in [42], and formula (2.12) and Lemma 2.2 in [9]), and $\vartheta_s(t,z)$ is the density function of the strictly s-stable process whose Laplace transform is $e^{-t\lambda^s}$ (see pag.3 in [42]). Using the scaling property $p_{s,m}(x,t) = m^N p_{s,1}(mx, m^{2s}t)$ (see formula (2.15) in [9]) and Lemma 2.2 in [27], we can see that that for some constant $C > 0$ depending only on N, s, m,

$$p_{s,m}(x,t) \leqslant C\left(g_{m^{2s}t}\left(\frac{mx}{\sqrt{2}}\right) + t\nu^1\left(\frac{mx}{\sqrt{2}}\right)\right) \qquad \text{for all } x\in\mathbb{R}^N, t>0, \tag{4.14}$$

where

$$g_t(x) := \frac{1}{(4\pi t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4t}},$$

and $\nu^{\mathfrak{m}}$ is the density function of the Lévy measure of the relativistic $2s$-stable process with parameter $\mathfrak{m} > 0$ (see Lemma 2 in [42] and formula (2.17) in [9]) given by

$$\nu^{\mathfrak{m}}(x) := \frac{2s2^{\frac{2s-N}{2}}}{\pi^{\frac{N}{2}}\Gamma(1-s)}\left(\frac{\mathfrak{m}}{|x|}\right)^{\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(\mathfrak{m}|x|).$$

Therefore, (4.13) and (4.14) yield

$$\mathcal{B}_{2s,m}(x) \leqslant C\int_0^\infty e^{-\gamma t}g_{m^{2s}t}\left(\frac{mx}{\sqrt{2}}\right)dt + C\int_0^\infty e^{-\gamma t}\,t\nu^1\left(\frac{mx}{\sqrt{2}}\right)dt =: I_1(x) + I_2(x). \tag{4.15}$$

We start with the estimate of $I_1(x)$ for $|x| \geqslant 2$. Observing that

$$\gamma t + \frac{m^{2-2s}}{8t}|x|^2 \geqslant \gamma t + \frac{m^{2-2s}}{2t} \qquad \text{for all } |x|\geqslant 2, t>0,$$

and that $ab \leqslant \epsilon a^2 + \frac{1}{4\epsilon}b^2$ for all $a, b \geqslant 0$ and $\epsilon > 0$ gives

$$\gamma t + \frac{m^{2-2s}}{8t}|x|^2 \geqslant \frac{m^{1-s}}{\sqrt{2}}|x|\sqrt{\gamma} \qquad \text{for all } x\in\mathbb{R}^N, t>0,$$

we deduce that for all $|x| \geqslant 2$ and $t > 0$

$$\gamma t + \frac{m^{2-2s}}{8t}|x|^2 \geqslant \gamma\frac{t}{2} + \frac{m^{2-2s}}{4t} + \frac{m^{1-s}}{2\sqrt{2}}|x|\sqrt{\gamma}.$$

Thus, using the definition of g_t, we can see that for all $|x|\geqslant 2$

$$I_1(x)\leqslant C_1\int_0^\infty \frac{e^{-\gamma\frac{t}{2}}}{t^{\frac{N}{2}}}e^{-\frac{m^{2-2s}}{4t}}e^{-\frac{m^{1-s}}{2\sqrt{2}}|x|\sqrt{\gamma}}\,dt$$
$$=C_1e^{-C_2|x|}\int_0^\infty \frac{e^{-\gamma\frac{t}{2}}}{t^{\frac{N}{2}}}e^{-\frac{m^{2-2s}}{4t}}\,dt\leqslant C_3e^{-C_2|x|}, \tag{4.16}$$

where we used the fact that

$$\int_0^\infty \frac{e^{-\alpha t}}{t^p}e^{-\frac{\beta}{t}}\,dt<\infty \quad \text{for all } \alpha,\beta,p>0.$$

Now we estimate $I_2(x)$ for large values of $|x|$. Recalling formula (1.5) concerning the asymptotic behavior of K_ν at infinity, we deduce that there exists $r_0>0$ such that

$$\frac{K_\nu(r)}{r^\nu}\leqslant C_4\frac{e^{-r}}{r^{\nu+\frac{1}{2}}} \quad \text{for all } r\geqslant r_0,$$

and then

$$\frac{K_{\frac{N+2s}{2}}(\frac{m}{\sqrt{2}}|x|)}{(\frac{m}{\sqrt{2}}|x|)^{\frac{N+2s}{2}}}\leqslant C_4\frac{e^{-\frac{m}{\sqrt{2}}|x|}}{|x|^{\frac{N+2s+1}{2}}} \quad \text{for all } |x|\geqslant \frac{\sqrt{2}}{m}r_0=:r_0'.$$

Consequently, using the definition of ν^1, for all $|x|\geqslant r_0'$ we get

$$I_2(x)\leqslant C_5\frac{e^{-\frac{m}{\sqrt{2}}|x|}}{|x|^{\frac{N+2s+1}{2}}}\int_0^\infty t\,e^{-\gamma t}\,dt\leqslant C_6\frac{e^{-C_7|x|}}{|x|^{\frac{N+2s+1}{2}}}. \tag{4.17}$$

Gathering (4.13), (4.15), (4.16), (4.17), we find that for any $|x|\geqslant \max\{r_0',2\}$

$$\mathcal{B}_{2s,m}(x)\leqslant C_3e^{-C_2|x|}+C_6\frac{e^{-C_7|x|}}{|x|^{\frac{N+2s+1}{2}}}\leqslant C_8e^{-C_9|x|}.$$

This completes the proof of Lemma 4.3. □

Remark 4.2. *When $s=\frac{1}{2}$, $p_{\frac{1}{2},m}(x,t)$ can be calculated explicitly (see pag.185 in [36]) and is given by*

$$p_{\frac{1}{2},m}(x,t)=2\left(\frac{m}{2\pi}\right)^{\frac{N+1}{2}}t\,e^{mt}(|x|^2+t^2)^{-\frac{N+1}{4}}K_{\frac{N+1}{2}}(m\sqrt{|x|^2+t^2}).$$

Remark 4.3. *By the definitions of $\mathcal{B}_{2s,m}$ and $p_{s,m}$, it follows that $\mathcal{B}_{2s,m}$ is radial, positive, decreasing in $|x|$, and smooth on $\mathbb{R}^N\setminus\{0\}$.*

With the help of Lemma 4.3, we establish the exponential decay of solutions to (1.7).

Theorem 4.1. *(exponential decay) Let $s\in(0,1)$, $m>0$ and $N\geqslant 2$. Let $u\in H^s_m(\mathbb{R}^N)$ be a weak solution to (1.7), where $g\in C^0(\mathbb{R}^N\times\mathbb{R})$ is such that*

$$|g(x,t)|\leqslant C_0(|t|+|t|^{2^*_s-1}) \quad \textit{for all } (x,t)\in\mathbb{R}^N\times\mathbb{R}, \tag{4.18}$$

for some constant $C_0>0$, and

$$\limsup_{t\to 0}\sup_{x\in\mathbb{R}^N}\frac{g(x,t)}{t}\in(-\infty,m^{2s}). \tag{4.19}$$

Then there exist $C,c>0$ such that $|u(x)|\leqslant Ce^{-c|x|}$ for all $x\in\mathbb{R}^N$.

Proof. From (4.18) and Lemma 4.1, we have that $u\in L^q(\mathbb{R}^N)$ for all $q\in[2,\infty]$. Hence, $g(x,u)\in L^\infty(\mathbb{R}^N)$ and applying Theorem 3.2 we deduce that $u\in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$. Since $u\in L^2(\mathbb{R}^N)\cap C^{0,\alpha}(\mathbb{R}^N)$, we get $|u(x)|\to 0$ as $|x|\to\infty$. From (4.19), we can find $\ell\in(0,m^{2s})$ and $t_0>0$ such that

$$\frac{g(x,t)}{t}<m^{2s}-\ell \quad \text{for all } x\in\mathbb{R}^N,\ 0<|t|\leqslant t_0.$$

Since $|u(x)|\to 0$ as $|x|\to\infty$, there exists $R_1>0$ such that $|u(x)|\leqslant t_0$ for all $|x|\geqslant R_1$ and so

$$\frac{g(x,u(x))}{u(x)}\leqslant m^{2s}-\ell \quad \text{for all } |x|\geqslant R_1:\ u(x)\neq 0. \tag{4.20}$$

By using Theorem 3.3 and (4.20), we can see that

$$(-\Delta+m^2)^s|u|\leqslant(m^{2s}-\ell)|u| \quad \text{in } \mathbb{R}^N\setminus\overline{B_{R_1}}(0).$$

On the other hand, by the proof of Lemma 4.3, we know that there exist $R_2>0$ and a positive continuous function $\bar{w}\in H^s_m(\mathbb{R}^N)$ satisfying $(-\Delta+m^2)^s\bar{w}=(m^{2s}-\ell)\bar{w}$ in $\mathbb{R}^N\setminus\overline{B_{R_2}}(0)$ and $\bar{w}$ has exponential decay. Let $R:=\max\{R_1,R_2\}$. Define $\sigma:=\|u\|_{L^\infty(\mathbb{R}^N)}(\min_{\overline{B_R}(0)}\bar{w})^{-1}>0$ and note that $|u(x)|\leqslant\sigma\bar{w}(x)$ for all

$|x| \leqslant R$. Since $z := |u| - \sigma \bar{w}$ solves $(-\Delta + m^2)^s z \leqslant (m^{2s} - \ell) z$ in $\mathbb{R}^N \setminus \overline{B_R}(0)$, by Lemma 4.2 we deduce that $z \leqslant 0$ in $\mathbb{R}^N$ which implies the thesis. □

Remark 4.4. *Under the assumptions of Theorem 4.1 in [33] the author proved that u decays faster than any polynomial. Here we improve this result by using Lemma 4.3.*

4.3. **Pohozaev identity for (1.1).** Thanks to Lemma 4.3, we prove a Pohozaev-type identity for (1.7). When $m = 0$, in [16] the authors used the Caffarelli-Silvestre extension method [11] to prove that any weak u solution to

$$(-\Delta)^s u = h(u) \text{ in } \mathbb{R}^N,$$

where $N \geqslant 2$, $s \in (0,1)$, $h \in C^1(\mathbb{R})$ fulfills $h(0) = 0$, $-\infty < \liminf_{t\to 0} \frac{h(t)}{t} \leqslant \limsup_{t\to 0} \frac{h(t)}{t} < 0$, $\lim_{|t|\to\infty} \frac{|h(t)|}{|t|^{2^*_s - 1}} = 0$, $H(t_0) > 0$ for some $t_0 > 0$, with $H(t) := \int_0^t h(\tau)\, d\tau$, satisfies the following Pohozaev identity:

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi = N \int_{\mathbb{R}^N} H(u(x))\, dx.$$

Due to the lack of scaling of (1.1), we will see that an identity of different structure holds in the case $m > 0$; see [24, 33, 44] for related results in $\mathbb{R}^N$, and [26] for the case of bounded domains. We start by proving a simple useful technical lemma.

Lemma 4.4. *Let $\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho(\frac{x}{\varepsilon})$, with $\varepsilon > 0$, be a sequence of mollifiers such that $\mathrm{supp}(\rho_\varepsilon) \subset \overline{B_\varepsilon(0)}$ and $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = 1$. If $|u(x)| \leqslant C_1 e^{-c_1|x|}$ for all $x \in \mathbb{R}^N$, with $C_1, c_1 > 0$, then $|(\rho_\varepsilon * u)(x)| \leqslant C_2 e^{-c_2|x|}$ for all $x \in \mathbb{R}^N$ and $\varepsilon \in (0,1)$, for some constants $C_2, c_2 > 0$ independent of ε.*

Proof. By using the Young inequality and $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = 1$, we see that

$$\|\rho_\varepsilon * u\|_{L^\infty(\mathbb{R}^N)} \leqslant \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|u\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C_1. \tag{4.21}$$

Fix $\varepsilon \in (0,1)$ and $|x| \geqslant 2$. Note that

$$|x| \geqslant 2 \geqslant 2\varepsilon \geqslant 2|z| \quad \text{ for all } |z| \leqslant \varepsilon \tag{4.22}$$

and

$$|x+z| \geqslant |x| - |z| \geqslant \frac{|x|}{2} \quad \text{ for all } |z| \leqslant \frac{|x|}{2}. \tag{4.23}$$

Then, using $\mathrm{supp}(\rho_\varepsilon) \subset \overline{B_\varepsilon(0)}$, the exponential decay of u, (4.22) and (4.23), we obtain that

$$\begin{aligned} |(\rho_\varepsilon * u)(x)| &= \left| \int_{\mathbb{R}^N} \rho_\varepsilon(z) u(x+z)\, dz \right| \\ &\leqslant C_1 \int_{|z|\leqslant\varepsilon} \rho_\varepsilon(z) e^{-c_1|x+z|}\, dz \\ &\leqslant C_1 \int_{|z|\leqslant\frac{|x|}{2}} \rho_\varepsilon(z) e^{-c_1|x+z|}\, dz \\ &\leqslant C_1 \int_{|z|\leqslant\frac{|x|}{2}} \rho_\varepsilon(z) e^{-\frac{c_1}{2}|x|}\, dz \\ &\leqslant C_1 e^{-\frac{c_1}{2}|x|} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\, dz = C_1 e^{-\frac{c_1}{2}|x|}. \end{aligned} \tag{4.24}$$

Combining (4.21) with (4.24), we get the thesis. □

Theorem 4.2. *(Pohozaev identity) Let $s \in (0,1)$, $m > 0$ and $N \geqslant 2$. Assume that $g \in C^0(\mathbb{R}^N \times \mathbb{R})$ satisfies (4.18) and (4.19). When $s \in (0, \frac{1}{2}]$, we also assume that $g \in C^1_b(\mathbb{R}^N \times \mathbb{R})$. Let $u \in H^s_m(\mathbb{R}^N)$ be a weak solution to (1.7). Then u satisfies the following Pohozaev-type identity:*

$$\begin{aligned} &\frac{N-2s}{2} \int_{\mathbb{R}^N} |\mathcal{F}u(\xi)|^2 (|\xi|^2 + m^2)^s \, d\xi + s m^2 \int_{\mathbb{R}^N} |\mathcal{F}u(\xi)|^2 (|\xi|^2 + m^2)^{s-1}\, d\xi \\ &= N \int_{\mathbb{R}^N} G(x, u(x))\, dx + \int_{\mathbb{R}^N} (x \cdot \nabla_x G)(x,u)\, dx \end{aligned}$$

where $G(x,t) := \int_0^t g(x,\tau)\, d\tau$.

Proof. By Theorem 4.1, we know that u has exponential decay. Moreover, in the light of Corollary 3.1 and Theorem 3.2, we can apply a standard bootstrap argument (see for instance Lemma 4.4 in [10]) to obtain that $u \in C^1_b(\mathbb{R}^N)$. Now we follow an argument given in the proof of Theorem 1.3 in [24] (see also [33, 44]) with some appropriate modifications. Using the exponential decay of u, we have that $u_\varepsilon := \rho_\varepsilon * u \in \mathcal{S}(\mathbb{R}^N)$. Note that

$$(-\Delta+m^2)^s u_\varepsilon = (-\Delta+m^2)^s(\rho_\varepsilon * u) = \rho_\varepsilon * (-\Delta+m^2)^s u = \rho_\varepsilon * g(x,u), \tag{4.25}$$

and since $x\cdot\nabla u_\varepsilon \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (-\Delta+m^2)^s u(x)\,(x\cdot\nabla u_\varepsilon(x))\,dx = \int_{\mathbb{R}^N} g(x,u(x))(x\cdot\nabla u_\varepsilon(x))\,dx. \tag{4.26}$$

Arguing as in the proof Proposition 5.1 in [24], we see that

$$\begin{aligned}(-\Delta+m^2)^s u(x)\,(x\cdot\nabla u_\varepsilon(x)) &= x\cdot\nabla[(-\Delta+m^2)^s u_\varepsilon(x)] \\ &\quad + 2s(-\Delta+m^2)^s u_\varepsilon(x) - 2sm^2(-\Delta+m^2)^{s-1}u_\varepsilon(x)\end{aligned}$$

which combined with (4.25) yields

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta+m^2)^s u(x)\,(x\cdot\nabla u_\varepsilon(x))\,dx &= \int_{\mathbb{R}^N} u(x)(-\Delta+m^2)^s(x\cdot\nabla u_\varepsilon(x))\,dx \\ &= \int_{\mathbb{R}^N} u(x)x\cdot\nabla(\rho_\varepsilon * g(\cdot,u))(x)\,dx + 2s\int_{\mathbb{R}^N} u(x)(\rho_\varepsilon * g(\cdot,u))(x)\,dx \\ &\quad - 2sm^2\int_{\mathbb{R}^N} u(x)(-\Delta+m^2)^{s-1}u_\varepsilon(x)\,dx \\ &=: I_\varepsilon + II_\varepsilon + III_\varepsilon.\end{aligned} \tag{4.27}$$

We start by considering the term I_ε. Since g satisfies (4.18) and u has exponential decay, we have that $|g(x,u(x))| \leqslant C_1 e^{-C_2|x|}$ for all $x\in\mathbb{R}^N$. Therefore, $g(\cdot,u)\in L^r(\mathbb{R}^N)$ for all $r\in[1,\infty]$ and we get $\rho_\varepsilon * g(\cdot,u) \to g(\cdot,u)$ in $L^p(\mathbb{R}^N)$ as $\varepsilon\to 0$ for all $p\in[1,\infty)$. Then, by $u\in L^2(\mathbb{R}^N)$, we find

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} u(x)(\rho_\varepsilon * g(\cdot,u))(x)\,dx = \int_{\mathbb{R}^N} u(x)g(x,u(x))\,dx. \tag{4.28}$$

Now, by Lemma 4.4, we see that $\rho_\varepsilon * g(x,u)$ has exponential decay. This combined with $|\nabla u|\in L^\infty(\mathbb{R}^N)$ implies that for all $x\in\mathbb{R}^N$ and $\varepsilon\in(0,1)$

$$|x\cdot\nabla u(x)\,(\rho_\varepsilon * g(\cdot,u))(x)| \leqslant C|x|e^{-c|x|}\in L^1(\mathbb{R}^N).$$

By invoking the dominated convergence theorem we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} (x\cdot\nabla u(x))(\rho_\varepsilon * g(\cdot,u))(x)\,dx = \int_{\mathbb{R}^N} (x\cdot\nabla u(x))g(x,u(x))\,dx. \tag{4.29}$$

Hence, using $u\in C^1_b(\mathbb{R}^N)$, integration by parts, $\rho_\varepsilon * g(x,u)$ has exponential decay, (4.28), (4.29), we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon &= \lim_{\varepsilon\to 0}\left[-N\int_{\mathbb{R}^N} u(x)(\rho_\varepsilon * g(\cdot,u))(x)\,dx - \int_{\mathbb{R}^N} x\cdot\nabla u(x)(\rho_\varepsilon * g(\cdot,u))\,(x)dx\right] \\ &= -N\int_{\mathbb{R}^N} u(x)g(x,u(x))\,dx - \int_{\mathbb{R}^N}(x\cdot\nabla u(x))g(x,u(x))\,dx \\ &= -N\int_{\mathbb{R}^N} u(x)(-\Delta+m^2)^s u(x)\,dx - \int_{\mathbb{R}^N}(x\cdot\nabla u(x))g(x,u(x))\,dx \\ &= -N\int_{\mathbb{R}^N} (|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi - \int_{\mathbb{R}^N}(x\cdot\nabla u(x))g(x,u(x))\,dx\end{aligned} \tag{4.30}$$

where in the first integral in the last identity we used the fact that u solves (1.7). Clearly, by (4.28), we obtain

$$\lim_{\varepsilon\to 0} II_\varepsilon = 2s\int_{\mathbb{R}^N} u(x)g(x,u(x))\,dx = 2s\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi, \tag{4.31}$$

Since $u_\varepsilon\to u$ in $L^2(\mathbb{R}^N)$ as $\varepsilon\to 0$ and $s\in(0,1)$, we see that

$$\begin{aligned}\lim_{\varepsilon\to 0} III_\varepsilon &= -2sm^2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\xi|^2+m^2)^{s-1}\mathcal{F}u(\xi)\mathcal{F}u_\varepsilon(\xi)\,d\xi \\ &= -2sm^2\int_{\mathbb{R}^N}(|\xi|^2+m^2)^{s-1}|\mathcal{F}u(\xi)|^2\,d\xi.\end{aligned} \tag{4.32}$$

Combining (4.27), (4.30), (4.31) and (4.32), we get

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(-\Delta+m^2)^s u(x)\,(x\cdot\nabla u_\varepsilon(x))\,dx=(2s-N)\int_{\mathbb{R}^N}(|\xi|^2+m^2)^s|\mathcal{F}u(\xi)|^2\,d\xi$$
$$-\int_{\mathbb{R}^N}(x\cdot\nabla u(x))g(x,u(x))\,dx-2sm^2\int_{\mathbb{R}^N}(|\xi|^2+m^2)^{s-1}|\mathcal{F}u(\xi)|^2\,d\xi. \tag{4.33}$$

On the other hand, from $u\in C^1_b(\mathbb{R}^N)$, $g(x,u)$ has exponential decay and the Young inequality, we have

$$\begin{aligned}|g(x,u(x))x\cdot\nabla u_\varepsilon(x)|&\leqslant\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}|g(x,u(x))||x|\\&\leqslant\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\|\nabla u\|_{L^\infty(\mathbb{R}^N)}|g(x,u(x))||x|\\&\leqslant C_3|x|e^{-C_2|x|}\in L^1(\mathbb{R}^N),\end{aligned}$$

and observing that $\nabla u_\varepsilon\to\nabla u$ uniformly on compact sets of $\mathbb{R}^N$ as $\varepsilon\to 0$, we can use the dominated convergence theorem to see that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}g(x,u(x))\,(x\cdot\nabla u_\varepsilon(x))\,dx=\int_{\mathbb{R}^N}g(x,u(x))\,(x\cdot\nabla u(x))\,dx.$$

Now, noting that

$$g(x,u(x))\,(x\cdot\nabla u(x))=(x\cdot\nabla_x)G(x,u)-(x\cdot\nabla_x G)(x,u),$$

and using the exponential decay of $G(x,u)$, an integration by parts yields

$$\int_{\mathbb{R}^N}g(x,u(x))\,(x\cdot\nabla u(x))\,dx=-N\int_{\mathbb{R}^N}G(x,u(x))\,dx-\int_{\mathbb{R}^N}(x\cdot\nabla_x G)(x,u)\,dx. \tag{4.34}$$

Consequently,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}g(x,u(x))\,(x\cdot\nabla u_\varepsilon(x))\,dx=-N\int_{\mathbb{R}^N}G(x,u(x))\,dx-\int_{\mathbb{R}^N}(x\cdot\nabla_x G)(x,u)\,dx. \tag{4.35}$$

Gathering (4.26), (4.33), (4.34) and (4.35), we get the thesis. □

4.4. **Radial symmetry.** In this subsection we use the method of moving planes to study the radial symmetry of positive solutions to

$$(-\Delta+m^2)^s u=g(u)\text{ in }\mathbb{R}^N. \tag{4.36}$$

In what follows, we take inspiration by the approaches in [17, 18]. Put

$$H_{\nu,m}(t):=\frac{K_\nu(mt)}{t^\nu}\quad(t>0)\,,\ \text{where }\nu:=\frac{N+2s}{2},$$

and observe that $H_{\nu,m}(t)$ is positive and decreasing for $t>0$ (since $K_\nu(t)>0$ and $(t^{-\nu}K_\nu(t))'=-t^{-\nu}K_{\nu+1}(t)<0$ for all $t>0$). In order to use the definition (1.3), we will work with regular functions belonging to the space $L^{\exp}_s$ defined as in Section 2. Firstly, we introduce some notations. Let

$$T_\lambda:=\{x\in\mathbb{R}^N:x_1=\lambda,\ \text{for some }\lambda\in\mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda:=\{x\in\mathbb{R}^N:x_1<\lambda\}$$

be the region to the left of the plane, and

$$x^\lambda:=(2\lambda-x_1,x_2,\dots,x_N)$$

be the reflection of x about the plane T_λ. We set $u_\lambda(x):=u(x^\lambda)$ and $w_\lambda(x):=u_\lambda(x)-u(x)$. Obviously, w_λ is an anti-symmetric function, namely $w_\lambda(x)=-w_\lambda(x^\lambda)$. Next we establish a maximum principle for anti-symmetric functions.

Theorem 4.3. *(Maximum principle for anti-symmetric functions) Let $\delta\leqslant m^{2s}$ and Ω be a bounded domain in Σ_λ. Assume that $w_\lambda\in C^{1,1}_{loc}(\mathbb{R}^N)\cap L^{\exp}_s$. If*

$$\begin{cases}(-\Delta+m^2)^s w_\lambda-\delta w_\lambda\geqslant 0 & \text{in }\Omega,\\ w_\lambda\geqslant 0 & \text{in }\Sigma_\lambda\setminus\Omega,\end{cases} \tag{4.37}$$

then $w_\lambda(x)\geqslant 0$ in Ω. Furthermore, if $w_\lambda=0$ at some point in Ω, then $w_\lambda=0$ a.e. in $\mathbb{R}^N$. These conclusions hold for unbounded region Ω if we further assume that

$$\liminf_{|x|\to\infty}w_\lambda(x)\geqslant 0. \tag{4.38}$$

Proof. We modify in a suitable way the proof of Theorem 2.2 in [17]. Suppose by contradiction that there exists a point $x^0 \in \Omega$ such that

$$w_\lambda(x^0) = \min_{\Omega} w_\lambda < 0. \tag{4.39}$$

Then, using (1.3) and that $w_\lambda(y^\lambda) = -w_\lambda(y)$ for $y \in \Sigma_\lambda$, we see that

$$\begin{aligned}
&(-\Delta + m^2)^s w_\lambda(x^0) - \delta w_\lambda(x^0) \\
&= (m^{2s} - \delta) w_\lambda(x^0) + C(N,s) m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} (w_\lambda(x^0) - w_\lambda(y)) H_{\nu,m}(|x^0 - y|)\, dy \\
&= (m^{2s} - \delta) w_\lambda(x^0) + C(N,s) m^{\frac{N+2s}{2}} P.V. \left\{ \int_{\Sigma_\lambda} \left[H_{\nu,m}(|x^0 - y|) - H_{\nu,m}(|x^0 - y^\lambda|) \right] (w_\lambda(x^0) - w_\lambda(y))\, dy \right. \\
&\quad \left. + 2 w_\lambda(x^0) \int_{\Sigma_\lambda} H_{\nu,m}(|x^0 - y^\lambda|)\, dy \right\} \\
&=: (m^{2s} - \delta) w_\lambda(x^0) + C(N,s) m^{\frac{N+2s}{2}} \{ I_1 + I_2 \}.
\end{aligned} \tag{4.40}$$

Note that $I_2 < 0$ because of $H_{\nu,m} > 0$ and (4.39). On the other hand, $I_1 < 0$ due to the facts that $|x - y^\lambda| > |x - y|$ for all $x, y \in \Sigma_\lambda$, $H_{\nu,m}(t)$ is decreasing for $t > 0$, and

$$w_\lambda(x^0) - w_\lambda(y) \leqslant 0 \text{ but } \not\equiv 0.$$

Consequently, $I_1 + I_2 < 0$. This fact combined with $\delta \leqslant m^{2s}$ and (4.39) gives

$$(-\Delta + m^2)^s w_\lambda(x^0) - \delta w_\lambda(x^0) < 0,$$

which contradicts (4.37). Hence, we must have $w_\lambda \geqslant 0$ in Ω. If there is some point $x^0 \in \Omega$ such that $w_\lambda(x^0) = 0$, then x^0 is a minimum point of w_λ in Ω, hence (4.40) holds with $I_2 = (m^{2s} - \delta) w_\lambda(x^0) = 0$. From (4.37), we deduce that $I_1 \geqslant 0$ and thus $0 \leqslant w_\lambda(x^0) - w_\lambda(y) = -w_\lambda(y)$ for almost every $y \in \Sigma_\lambda$. Therefore, we must have $w_\lambda = 0$ a.e. in Σ_λ and from the antisymmetry of w_λ we get $w_\lambda = 0$ a.e. in $\mathbb{R}^N$. When Ω is unbounded, under assumption (4.38), if it is not true that $w_\lambda \geqslant 0$ in Σ_λ, then a negative minimum of w_λ is achieved at some point $x^0 \in \Sigma_\lambda$. Repeating the above argument, we derive a contradiction. □

Finally, we present the main result of this subsection.

Theorem 4.4. *(Radial symmetry and monotonicity) Let $u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap L^{\exp}_s$ be a positive solution of (4.36) with $\lim_{|x|\to\infty} u(x) = 0$. Assume that $g \in C^1_{loc}(\mathbb{R})$ and $g'(t) \leqslant m^{2s}$ for $t > 0$ sufficiently small. Then u must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^N$.*

Proof. We follow some ideas found in the proof of Theorem 4.1 in [17] combined with Theorem 4.3.
Step 1 We start by showing that for λ sufficiently negative, it holds

$$w_\lambda(x) \geqslant 0 \quad \text{ for all } x \in \Sigma_\lambda. \tag{4.41}$$

In the light of (4.36), we deduce that

$$(-\Delta + m^2)^s w_\lambda(x) = g'(\xi_\lambda(x)) w_\lambda(x) \tag{4.42}$$

where $\xi_\lambda(x)$ is between $u_\lambda(x)$ and $u(x)$. Suppose by contradiction that (4.41) is false. Since $u(x) \to 0$ as $|x| \to \infty$, there exists $x^0 \in \Sigma_\lambda$ such that

$$w_\lambda(x^0) = \min_{\Sigma_\lambda} w_\lambda < 0. \tag{4.43}$$

Consequently,

$$u_\lambda(x^0) \leqslant \xi_\lambda(x^0) \leqslant u(x^0). \tag{4.44}$$

For sufficiently negative λ, $u(x^0)$ is small, hence $\xi_\lambda(x^0)$ is small, and using the assumption on g' we have $g'(\xi_\lambda(x^0)) \leqslant m^{2s}$. Thus, by (4.42) and (4.43), we get

$$(-\Delta + m^2)^s w_\lambda(x^0) \geqslant m^{2s} w_\lambda(x^0). \tag{4.45}$$

On the other hand, arguing as in the proof of Theorem 4.3, we obtain that

$$\begin{aligned}
&(-\Delta + m^2)^s w_\lambda(x^0) \\
&= m^{2s} w_\lambda(x^0) + C(N,s) m^{\frac{N+2s}{2}} P.V. \left\{ \int_{\Sigma_\lambda} \left[H_{\nu,m}(|x^0 - y|) - H_{\nu,m}(|x^0 - y^\lambda|) \right] (w_\lambda(x^0) - w_\lambda(y))\, dy \right.
\end{aligned}$$

$$+ 2w_\lambda(x^0)\int_{\Sigma_\lambda} H_{\nu,m}(|x^0-y^\lambda|)\,dy\Big\}$$
$$=: m^{2s}w_\lambda(x^0) + C(N,s)m^{\frac{N+2s}{2}}\{I_1+I_2\} < m^{2s}w_\lambda(x^0)$$

which is contrast with (4.45). Therefore, (4.41) is valid for λ sufficiently negative.

Step 2 Inequality (4.41) provides a starting point, from which we move the plane T_λ toward the right as long as (4.41) holds to its limiting position to show that u is symmetric about the limiting plane. More precisely, let

$$\lambda_0 := \sup\{\lambda : w_\mu(x) \geqslant 0,\ x\in\Sigma_\mu,\ \mu\leqslant\lambda\},$$

we show that u is symmetric about the limiting plane T_{λ_0}, or

$$w_{\lambda_0}(x)\equiv 0,\ x\in\Sigma_{\lambda_0}. \tag{4.46}$$

Assume by contradiction that (4.46) is not true. Then, the strong maximum principle (second part of Theorem 4.3) yields

$$w_{\lambda_0}(x) > 0 \quad \text{for all } x\in\Sigma_{\lambda_0}.$$

On the other hand, by the definition of λ_0, there exist (λ_k), with $\lambda_k \searrow \lambda_0$ as $k\to\infty$, and $(x^k)\subset\Sigma_{\lambda_k}$, such that

$$w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0, \text{ and } \nabla w_{\lambda_k}(x^k) = 0. \tag{4.47}$$

Now we prove that the assumption on g' guarantees that there exists a subsequence of (x^k) that converges to some point x^0. As before, we can write

$$(-\Delta+m^2)^s w_{\lambda_k}(x^k) = g'(\xi_{\lambda_k}(x^k))w_{\lambda_k}(x^k). \tag{4.48}$$

If $|x^k|$ is sufficiently large, $u(x^k)$ is small, and thus $\xi_{\lambda_k}(x^k)$ is small, which implies $g'(\xi_{\lambda_k}(x^k))\leqslant m^{2s}$. Hence, by (4.47) and (4.48), we infer that

$$(-\Delta+m^2)^s w_{\lambda_k}(x^k) \geqslant m^{2s}w_{\lambda_k}(x^k).$$

This contradicts the fact that x^k is a negative minimum of w_{λ_k} since, arguing as in Step 1, we should have

$$(-\Delta+m^2)^s w_{\lambda_k}(x^k) < m^{2s}w_{\lambda_k}(x^k).$$

Therefore, (x^k) is bounded, and there exists a subsequence of (x^k) (still denoted by (x^k)) such that $x^k\to x^0$ as $k\to\infty$. From (4.47) and the continuity of $w_\lambda(x)$ and its derivative with respect to both x and λ, we find

$$w_{\lambda_0}(x^0)\leqslant 0,\ \text{hence } x^0\in\partial\Sigma_{\lambda_0};\ \text{and } \nabla w_{\lambda_0}(x^0)=0.$$

Consequently,

$$\frac{w_{\lambda_k}(x^k)}{\delta_k}\to 0 \quad \text{as } k\to\infty, \tag{4.49}$$

where

$$\delta_k := \operatorname{dist}(x^k,\partial\Sigma_k) \equiv |\lambda^k - x_1^k|.$$

Now, we prove that

$$\limsup_{\delta_k\to 0}\frac{1}{\delta_k}[(-\Delta+m^2)^s w_{\lambda_k}(x^k)] < 0. \tag{4.50}$$

Arguing as in Step 1, we deduce that

$$\frac{1}{\delta_k}[(-\Delta+m^2)^s w_{\lambda_k}(x^k)]$$
$$= m^{2s}\frac{w_{\lambda_k}(x^k)}{\delta_k} + \frac{C(N,s)m^{\frac{N+2s}{2}}}{\delta_k}P.V.\Big\{\int_{\Sigma_{\lambda_k}}\left[H_{\nu,m}(|x^k-y|)-H_{\nu,m}(|x^k-y^{\lambda_k}|)\right](w_{\lambda_k}(x^k)-w_{\lambda_k}(y))\,dy$$
$$+ 2w_{\lambda_k}(x^k)\int_{\Sigma_{\lambda_k}} H_{\nu,m}(|x^k-y^{\lambda_k}|)\,dy\Big\}$$
$$=: m^{2s}\frac{w_{\lambda_k}(x^k)}{\delta_k} + C(N,s)m^{\frac{N+2s}{2}}\{I_{1k}+I_{2k}\}.$$

Clearly, by (4.47) and $H_{\nu,m} > 0$, we know that $I_{2k} < 0$. On the other hand, from the mean value theorem, $H'_{\nu,m}(t) = -t^{-\nu} K_{\nu+1}(mt) < 0$ for $t > 0$, $|x - y^{\lambda}| > |x - y|$ for all $x, y \in \Sigma_{\lambda}$, and observing that, as $k \to \infty$,

$$w_{\lambda_k}(x^k) - w_{\lambda_k}(y) \to w_{\lambda_0}(x^0) - w_{\lambda_0}(y) < 0 \quad \text{for all } y \in \Sigma_{\lambda_0},$$

we have

$$\limsup_{\delta_k \to 0} I_{1k} < 0.$$

Then, by using this fact, $I_{2k} < 0$ and (4.49), we obtain that (4.50) holds. Combining (4.48), $|g'(\xi_{\lambda_k}(x^k))| \leqslant C$ for all $k \in \mathbb{N}$, (4.49), and (4.50), we get a contradiction.
Since the x_1-direction can be chosen arbitrarily, we conclude that the solution u must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^N$. □

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