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An estimate concerning the difference between minimizer and boundary value in some polyconvex problems

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Abstract

This paper is concerned with regularity of minimizers of integral functionals with polyconvex potentials. In particular we obtain bounds on the difference $|u - u_*|_\infty$ for minimizers $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of problem

$$\min \left\{ \int_{\Omega} f(x, Dv(x)) dx, \quad v \in u_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\}.$$

Keywords: Regularity, minimizer, polyconvex, functional, non standard, growth

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1. Introduction

Let us consider mappings $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$ and the variational integral

$$\mathcal{F}(u) = \int_{\Omega} f(x, Du) dx, \quad (1.1)$$

where $f : \Omega \times \mathbb{R}^{N \times n} \rightarrow [0, +\infty)$ with $n \geq 2$ and $N \geq 1$. After fixing a suitable boundary value u_* , we deal with the problem of minimizing \mathcal{F} among mappings that agree with u_* on the boundary of Ω :

$$\min\{\mathcal{F}(u) : u = u_* \text{ on } \partial\Omega\}. \quad (1.2)$$

When we have p -coercivity

$$\nu|z|^p - a(x) \leq f(x, z), \quad (1.3)$$

for some exponent $p > 1$, then minimizing sequences for (1.2) are bounded in the Sobolev space $W^{1,p}$ and compact with respect to the weak convergence in $W^{1,p}$. In order to use direct methods, we need lower semicontinuity with respect to such a weak convergence. In the scalar case $N = 1$, \mathcal{F} is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if $z \rightarrow f(x, z)$ is convex, see Theorem 1.3 in [15]. In the vectorial case $N \geq 2$, when we also have p -growth from above

$$f(x, z) \leq M|z|^p + b(x), \quad (1.4)$$

then \mathcal{F} is sequentially weakly lower semicontinuous in $W^{1,p}$ if and only if $z \rightarrow f(x, z)$ is quasiconvex in the sense of Morrey, see Theorem 1.13 in [15]. Quasiconvexity is weaker than convexity in the vectorial case. Quasiconvexity is not a local condition, see [34]. A more friendly assumption is polyconvexity; this means that $f(x, z)$ can be written as a convex combination of minors taken from the $N \times n$ matrix z . Polyconvexity is weaker than convexity but stronger than quasiconvexity. When $N = n = 3$, polyconvexity can be stated as follows

$$f(x, z) = g(x, z, \text{adj}_2 z, \det z), \quad (1.5)$$

with

$$(z, \xi, t) \rightarrow g(x, z, \xi, t) \quad \text{convex}. \quad (1.6)$$

In nonlinear elasticity, $f(x, z) = g(x, z, \text{adj}_2 z, \det z)$ is the stored-energy function and $z, \text{adj}_2 z, \det z$ govern the deformations of line, surface, volume

elements respectively, see [1]. Existence results for minimization problem (1.2), when f is polyconvex, can be found in section 8.4.2 of [15].

This paper is concerned with regularity of minimizers of integral functional (1.1) when f is polyconvex. In this framework we recall partial regularity contained in [27], [25], [24], [45], [21], [32], [23], [9]. As far as everywhere regularity is concerned, only few contributions are recorded and they all consider the two dimensional case $n = 2$: [28], [17], [26], [5]. When L^∞ estimates are concerned, we mention global bounds in [35], [3], [4], [19], [38], [39], [37], [36], [6]. On the other hand, local L^∞ regularity is obtained in [12], [7], [13]. When the polyconvex term is of lower order, see [18], [14] and [30].

We focus our attention to the case where $N = n = 3$ and we are interested in studying the properties of solutions of the following minimization problem:

$$\min \left\{ \int_{\Omega} f(x, Dv(x)) dx, \quad v \in u_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\}. \quad (1.7)$$

Our aim is to obtain bounds of the quantity $|u - u_*|$ for minimizers u of problem (1.7). Here Ω is a bounded open set in \mathbb{R}^3 and $u, u_* : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Our main result is stated in Theorem 2.2 and gives an explicit estimate of the L^∞ norm of $u - u_*$ under structural assumptions on f that are formalized in the next section.

We underline the fact that in the present paper a “good” boundary datum allows minimizers to be “regular” under quite general assumptions. At the end of section 2 we will discuss in detail the fact that our assumptions allow non standard growth of the type

$$\tilde{\nu}|z|^p - 3a(x) \leq f(x, z) \leq \tilde{M}[b(x) + 1 + |z|^{\hat{p}}], \quad (1.8)$$

where $\tilde{\nu}, \tilde{M}$ are positive constants, $a, b : \Omega \rightarrow \mathbb{R}$ are measurable functions with suitable summability and $p < \hat{p}$. In the case of the example discussed in Remark 2.3 we get $p = 5/2 < 4 = \hat{p}$. In the framework of non standard growth, regularity for minimizers is usually obtained when p and \hat{p} are not too far apart, see [31], [40], [41], [29], [44], [33], [20], [22], [10], [11], [2], [16], sections 5 and 6 in [42] and sections 3,4 in [43]. In our case too a similar control on p and \hat{p} appears. See Remark 2.3 for a deeper discussion. In section 5 we apply our main Theorem to the special case where the boundary datum is Lipschitz.

Let us end this introduction by mentioning that, in the scalar case $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$, estimates of the difference between minimizers u and boundary values u_* are given in [8].

2. Notation and statement of the main result

2.1. Notation

In this paper we are concerned with $N = n = 3$, so $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$. We will use the following notation for u and for its Jacobian Matrix:

$$u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \quad Du = \begin{pmatrix} D_1u^1 & D_2u^1 & D_3u^1 \\ D_1u^2 & D_2u^2 & D_3u^2 \\ D_1u^3 & D_2u^3 & D_3u^3 \end{pmatrix}.$$

For simplicity, we will also denote the Jacobian matrix using the gradients of functions u^α :

$$Du = \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix}.$$

Moreover, we consider the adjugate matrix of the Jacobian:

$$adj_2(Du) = \begin{pmatrix} adj_2(Du)_1^1 & adj_2(Du)_2^1 & adj_2(Du)_3^1 \\ adj_2(Du)_1^2 & adj_2(Du)_2^2 & adj_2(Du)_3^2 \\ adj_2(Du)_1^3 & adj_2(Du)_2^3 & adj_2(Du)_3^3 \end{pmatrix},$$

where

$$adj_2(Du)_i^\alpha = (-1)^{\alpha+i} \det \begin{pmatrix} \frac{\partial u^\beta}{\partial x^r} & \frac{\partial u^\beta}{\partial x^s} \\ \frac{\partial u^\gamma}{\partial x^r} & \frac{\partial u^\gamma}{\partial x^s} \end{pmatrix},$$

with

$$\beta < \gamma, \quad \beta, \gamma \in \{1, 2, 3\} \setminus \{\alpha\}, \quad r < s, \quad r, s \in \{1, 2, 3\} \setminus \{i\}.$$

We write $adj_2(Du)^\alpha$ for the α -th row of the adjugate matrix.

Moreover we use also the following common notations: $|\Omega|$ is the Lebesgue measure of Ω ; $\|g\|_s$ is the $L^s(\Omega)$ norm of the function g and p^* is the critical Sobolev exponent associated to p . We remark that in our special case $p^* = \frac{3p}{3-p}$.

2.2. *Structure assumptions on f .*

Let p, q, r three strictly positive real numbers satisfying

$$0 < r < q < p < 3, \quad p \geq 1, \quad p > \max \left\{ \sqrt{3q}, \frac{3r}{q} \right\}. \quad (2.1)$$

Remark 2.1. *Note that $\sqrt{3q} < 3$ since $q < 3$ and $3r/q < 3$ since $r < q$; then, assumption (2.1) makes sense.*

We consider a polyconvex potential of the following form

$$\begin{aligned} f(x, Du) = & F_1(x, Du^1) + F_2(x, Du^2) + F_3(x, Du^3) \\ & + G_1(x, \text{adj}_2(Du)^1) + G_2(x, \text{adj}_2(Du)^2) + G_3(x, \text{adj}_2(Du)^3) \\ & + H(x, \det(Du)), \end{aligned}$$

where we suppose that $F_\alpha, G_\alpha : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $\alpha = 1, 2, 3$, and $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Caratheodory functions. Moreover we will assume that there exist two constants

$$0 < \nu, M < +\infty. \quad (2.2)$$

and two functions

$$0 \leq a(x), b(x) < +\infty, \quad a, b \in L^\sigma(\Omega), \quad \sigma > \frac{3}{p}, \quad (2.3)$$

such that

$$\nu|\xi|^p - a(x) \leq F_\alpha(x, \xi) \leq M|\xi|^p + b(x), \quad F_\alpha(x, \xi) \geq 0, \quad (2.4)$$

$$\nu|\xi|^q - a(x) \leq G_\alpha(x, \xi) \leq M|\xi|^q + b(x), \quad G_\alpha(x, \xi) \geq 0, \quad (2.5)$$

$$0 \leq H(x, t) \leq M|t|^r + b(x). \quad (2.6)$$

2.3. *Statement of the main Theorem*

We start defining some constants that will be used in Theorem 2.2:

- we denote by \tilde{p} a fixed real number such that

$$\tilde{p} > \max \left\{ 3, \frac{3pq}{p^2 - 3q}, \frac{3qr}{qp - 3r} \right\};$$

- the real number γ is defined by

$$\gamma := \min \left\{ \underbrace{1 - \frac{1}{\sigma}}_{\delta_1}, \underbrace{1 - \frac{p}{\tilde{p}}}_{\delta_2}, \underbrace{1 - \frac{q}{p} - \frac{q}{\tilde{p}}}_{\delta_3}, \underbrace{1 - \frac{r}{q} - \frac{r}{\tilde{p}}}_{\delta_4} \right\},$$

and we underline that, thanks to previous choices, $\gamma > \frac{p}{p^*}$.

We consider a minimization problem with fixed boundary datum u_* and we assume that $u_* \in W^{1, \tilde{p}}(\Omega, \mathbb{R}^3)$ such that

$$x \rightarrow f(x, Du_*(x)) \in L^1(\Omega). \quad (2.7)$$

The last constant that we will need is

$$\begin{aligned} K := & \frac{4}{\nu} \left(\frac{2p}{3-p} \right)^{\frac{3p}{3-p}} 2^{\frac{3(p-1)}{3-p}} \\ & \cdot \left(\max \left\{ 4\|b\|_\sigma + \|a\|_\sigma, M\|Du_*\|_{\tilde{p}}^p, \right. \right. \\ & \quad \left. \left. 2M3^q \|Du_*\|_{\tilde{p}}^q \left(\frac{1}{\nu} \int_\Omega f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}}, \right. \right. \\ & \quad \left. \left. M3^{\frac{r}{2}} \|Du_*\|_{\tilde{p}}^r \left(\frac{1}{\nu} \int_\Omega f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}} \right\} \right. \\ & \left. \cdot \max \left\{ |\Omega|^{\delta_1 - \gamma}, |\Omega|^{\delta_2 - \gamma}, |\Omega|^{\delta_3 - \gamma}, |\Omega|^{\delta_4 - \gamma} \right\} + \|Du_*\|_{\tilde{p}}^p |\Omega|^{\delta_2 - \gamma} \right)^{\frac{3}{3-p}}. \end{aligned} \quad (2.8)$$

We are ready now to state our main Theorem that will be proved in the next section.

Theorem 2.2. *Let $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a minimizer of problem (1.7) such that hypotheses (2.2)-(2.7) are fulfilled. In addition, we suppose that boundary datum u_* has the following high degree of integrability*

$$u_* \in W^{1, \tilde{p}}(\Omega), \quad (2.9)$$

Then,

$$|u^\alpha - u_*^\alpha| \leq L^*, \quad \text{in } \Omega, \quad \forall \alpha \in \{1, 2, 3\},$$

with

$$L^* = 2^{\frac{p^* \gamma}{p^* \gamma - p}} K^{\frac{1}{p^*}} |\Omega|^{\frac{\gamma}{p} - \frac{1}{p^*}}. \quad (2.10)$$

Remark 2.3. *We now give an application of the previous Theorem. We consider*

$$f(x, Du) = \sum_{\alpha=1}^3 (|Du^\alpha|^p + |\text{adj}_2(Du)^\alpha|^q) + |\det Du|^r. \quad (2.11)$$

Assuming $p, q, r > 1$ we have that the functional is both polyconvex and coercive and, so, the corresponding minimum problem (1.7) has a solution see Remark 8.32 (iii) in [15] and Theorem 3.1 in [12]. We choose $r = 4/3$, $q = 2$ and $p = 5/2$ and we observe that (2.1) is satisfied. For this choice we obtain $3pq/(p^2 - 3q) = 60$ and $3qr/(qp - 3r) = 8$; then our Theorem requires that the degree of integrability of the boundary datum u_* has to verify $\tilde{p} > 60$.

Let note that (2.11) has been considered in [12]: there, the conditions on p are more restrictive than the ones in the present paper, see section 4. We can guess it by recalling that [12] deals with local regularity and no boundary value u_* is fixed; on the contrary, in the present paper a “good” boundary value u_* allows minimizers to be “regular” under less restrictive assumptions.

Let us come back to our general structure described in section 2.2; then, our growth assumptions (2.4)-(2.6) imply

$$\tilde{\nu}|z|^p - 3a(x) \leq f(x, z) \leq \tilde{M}[b(x) + 1 + |z|^{\hat{p}}], \quad (2.12)$$

where $\tilde{\nu}, \tilde{M}$ are positive constants and

$$\hat{p} := \max\{p; 2q; 3r\}. \quad (2.13)$$

So, we are in the framework of non standard growth, that is, when the exponent p of the growth from below is less than the exponent \hat{p} of the growth from above: this happens in our previous example when $r = 4/3$, $q = 2$ and $p = 5/2$; in such a case $p = 5/2 < 4 = \hat{p}$. In the framework of non standard growth, regularity for minimizers is usually obtained when p and \hat{p} are not too far apart, see [31], [40], [41], [29], [44], [33], [20], [22], [10], [11], [2], [16], sections 5 and 6 in [42] and sections 3,4 in [43]. So, it is not surprising that also in our case a similar control on p and \hat{p} appears. Indeed, let us read the last inequality of (2.1) in a different way. Condition $p > \sqrt{3q}$ is equivalent to $2q < 2p^2/3$ and restriction $p > 3r/q$ can be seen as $3r < pq$; taking into account the previous condition on q , we get $3r < p^3/3$. This means that

$$\hat{p} = \max\{p; 2q; 3r\} \leq \max\left\{p; \frac{2}{3}p^2; \frac{p^3}{3}\right\}. \quad (2.14)$$

3. Proof of Theorem 2.2

We will present some preliminary estimates that are resumed in the following technical lemma and that will be useful later.

Lemma 3.1. *Let u be a minimizer of (1.7) satisfying (2.2)-(2.7); then:*

$$x \rightarrow f(x, Du(x)) \in L^1(\Omega), \quad (3.1)$$

$$\|Du\|_p^q \leq 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}}, \quad (3.2)$$

and

$$\|adj_2(Du)\|_q^r \leq 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}}. \quad (3.3)$$

Proof. For the first claim we observe that:

$$0 \leq \int_{\Omega} f(x, Du(x)) dx \leq \int_{\Omega} f(x, Du_*(x)) dx < +\infty.$$

Moreover,

$$|Du|^p = \left(\sum_{\alpha=1}^3 \sum_{i=1}^3 |D_i u^\alpha|^2 \right)^{\frac{p}{2}} \leq \left(\sum_{\alpha=1}^3 \sum_{i=1}^3 |D_i u^{\tilde{\alpha}}|^2 \right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \left(\sum_{i=1}^3 |D_i u^{\tilde{\alpha}}|^2 \right)^{\frac{p}{2}},$$

where the index $\tilde{\alpha}$ corresponds to the maximum of the quantity $\sum_{i=1}^3 |D_i u^\alpha|^2$ with $\alpha = 1, 2, 3$. From the previous inequality we have

$$|Du|^p \leq 3^{\frac{p}{2}} \sum_{\alpha=1}^3 \left(\sum_{i=1}^3 |D_i u^\alpha|^2 \right)^{\frac{p}{2}} = 3^{\frac{p}{2}} \sum_{\alpha=1}^3 (|Du^\alpha|^2)^{\frac{p}{2}} = 3^{\frac{p}{2}} \sum_{\alpha=1}^3 |Du^\alpha|^p. \quad (3.4)$$

Then, using (3.4), (2.2)-(2.7) we can write:

$$\begin{aligned}
\|Du\|_p^q &= \left(\int_{\Omega} |Du|^p \right)^{\frac{q}{p}} \\
&\leq \left(\int_{\Omega} 3^{\frac{p}{2}} \sum_{\alpha=1}^3 |Du^\alpha|^p \right)^{\frac{q}{p}} = 3^{\frac{q}{2}} \left(\sum_{\alpha=1}^3 \int_{\Omega} |Du^\alpha|^p \right)^{\frac{q}{p}} \\
&\leq 3^{\frac{q}{2}} \left(\sum_{\alpha=1}^3 \frac{1}{\nu} \int_{\Omega} [F_\alpha(x, Du^\alpha) + a(x)] dx \right)^{\frac{q}{p}} \\
&\leq 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}} \\
&\leq 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}},
\end{aligned}$$

where in the last inequality we have used the minimality of u . Now we pass to the second estimate. From a similar argument used for estimating $|Du|^p$ we obtain that

$$|adj_2(Du)|^q \leq 3^{\frac{q}{2}} \sum_{\alpha=1}^3 |adj_2(Du)^\alpha|^q, \quad (3.5)$$

and then, using (3.5), (2.2)-(2.7) and the minimality of u :

$$\begin{aligned}
\|adj_2(Du)\|_q^r &= \left(\int_{\Omega} |adj_2(Du)|^q dx \right)^{\frac{r}{q}} \\
&\leq \left(\int_{\Omega} 3^{\frac{q}{2}} \sum_{\alpha=1}^3 |adj_2(Du)^\alpha|^q dx \right)^{\frac{r}{q}} \\
&= 3^{\frac{r}{2}} \left(\sum_{\alpha=1}^3 \int_{\Omega} |adj_2(Du)^\alpha|^q dx \right)^{\frac{r}{q}} \\
&\leq 3^{\frac{r}{2}} \left(\sum_{\alpha=1}^3 \int_{\Omega} \frac{1}{\nu} [G_\alpha(x, adj_2(Du)^\alpha) + a(x)] dx \right)^{\frac{r}{q}} \\
&\leq 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}} \\
&\leq 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}}.
\end{aligned}$$

□

Remark 3.2. We observe that

$$\frac{3pq}{p^2 - 3q} > \frac{pq}{p - q}, \quad (3.6)$$

and

$$\frac{3qr}{qp - 3r} > \frac{qr}{q - r}. \quad (3.7)$$

The previous inequalities will be useful for the proof of Theorem 2.2. Moreover, (2.9) implies $\gamma > \frac{p}{p^*}$, see (3.14).

Now we are ready to prove Theorem 2.2

Proof. (Theorem 2.2) For $L > 0$ we introduce the following test function

$$v = \begin{pmatrix} v^1 \\ u^2 \\ u^3 \end{pmatrix},$$

where

$$v^1 = \begin{cases} u^1, & \text{on } \Omega \setminus A_L^1, \\ u_*^1 + L, & \text{on } A_L^1, \end{cases}$$

where

$$A_L^1 = \{x \in \Omega : u^1(x) > u_*^1(x) + L\}.$$

We first show that v is an admissible function for the minimum problem. Note that on $\partial\Omega$ we have

$$u^1(x) = u_*^1(x) < u_*^1(x) + L,$$

then

$$v^1(x) = u^1(x) \quad \text{on } \partial\Omega.$$

We observe that

$$Dv^1 = \begin{cases} Du^1, & \text{on } \Omega \setminus A_L^1, \\ Du_*^1, & \text{on } A_L^1, \end{cases}$$

and

$$Dv^2 = Du^2, \quad Dv^3 = Du^3.$$

Regarding the functions F_α we have:

$$F_1(x, Dv^1) = \begin{cases} F_1(x, Du^1), & \text{on } \Omega \setminus A_L^1, \\ F_1(x, Du_*^1), & \text{on } A_L^1, \end{cases},$$

$$F_2(x, Dv^2) = F_2(x, Du^2), \quad F_3(x, Dv^3) = F_3(x, Du^3).$$

On $\Omega \setminus A_L^1$ we have $Dv = Du$ and as a consequence $f(x, Dv) = f(x, Du)$.
On the set A_L^1 we have

$$Dv = \begin{pmatrix} Du_*^1 \\ Du^2 \\ Du^3 \end{pmatrix},$$

and for the first row of the adjugate matrix we have

$$adj_2(Dv)^1 = adj_2(Du)^1,$$

while for the other rows we have

$$adj_2(Dv)_i^2 = (-1)^{2+i} \det \begin{pmatrix} \frac{\partial u_*^1}{\partial x^r} & \frac{\partial u_*^1}{\partial x^s} \\ \frac{\partial u^3}{\partial x^r} & \frac{\partial u^3}{\partial x^s} \end{pmatrix},$$

with

$$r < s, \quad r, s \in \{1, 2, 3\} \setminus \{i\}.$$

The previous quantity can be estimated in the following way:

$$\begin{aligned} |adj_2(Dv)_i^2| &= \left| \frac{\partial u_*^1}{\partial x^r} \frac{\partial u^3}{\partial x^s} - \frac{\partial u_*^1}{\partial x^s} \frac{\partial u^3}{\partial x^r} \right| \leq \left| \frac{\partial u_*^1}{\partial x^r} \right| \left| \frac{\partial u^3}{\partial x^s} \right| + \left| \frac{\partial u_*^1}{\partial x^s} \right| \left| \frac{\partial u^3}{\partial x^r} \right| \\ &\leq \left(\left| \frac{\partial u_*^1}{\partial x^r} \right|^2 + \left| \frac{\partial u_*^1}{\partial x^s} \right|^2 \right)^{\frac{1}{2}} \left(\left| \frac{\partial u^3}{\partial x^r} \right|^2 + \left| \frac{\partial u^3}{\partial x^s} \right|^2 \right)^{\frac{1}{2}} \leq |Du_*^1| |Du^3|, \end{aligned} \quad (3.8)$$

where we have used Cauchy-Schwartz inequality. For the third row we have:

$$adj_2(Dv)_i^3 = (-1)^{3+i} \det \begin{pmatrix} \frac{\partial u_*^1}{\partial x^r} & \frac{\partial u_*^1}{\partial x^s} \\ \frac{\partial u^2}{\partial x^r} & \frac{\partial u^2}{\partial x^s} \end{pmatrix},$$

with

$$r < s, \quad r, s \in \{1, 2, 3\} \setminus \{i\},$$

for which we obtain in a similar way the following bound:

$$|\text{adj}_2(Dv)_i^3| \leq |Du_*^1| |Du^2|. \quad (3.9)$$

Then we have

$$\begin{aligned} G_1(\text{adj}_2(Dv)^1) &= G_1(\text{adj}_2(Du)^1), \\ G_2(\text{adj}_2(Dv)^2) &\leq M |\text{adj}_2(Dv)^2|^q + b(x) \leq M |Du_*^1|^q |Du^3|^q + b(x). \end{aligned}$$

Then in order to have $G_2(\text{adj}_2(Dv)^2) \in L^1(\Omega)$ it is sufficient to note that

$$\begin{aligned} \int_{\Omega} G_2(\text{adj}_2(Dv)^2) dx &\leq M \int_{\Omega} |Du_*^1|^q |Du^3|^q dx + \|b\|_1 \\ &\leq M \|Du^3\|_{L^p(\Omega)}^q \left(\int_{\Omega} |Du_*^1|^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{p}} + \|b\|_1 \\ &\leq c \|Du^3\|_p^q + \|b\|_1, \end{aligned}$$

where we have used (2.9) and (3.6) which ensure that

$$u_*^1 \in W^{1, \tilde{p}}, \quad \tilde{p} \geq \frac{pq}{p-q}.$$

The same reasoning can be used for G_3 .

We now consider the last term of the functional; we observe that the determinant of the Jacobian matrix can be computed using the first row:

$$\det(Dv) = D_1 u_*^1 \cdot \text{adj}_2(Du)_1^1 + D_2 u_*^1 \cdot \text{adj}_2(Du)_2^1 + D_3 u_*^1 \cdot \text{adj}_2(Du)_3^1,$$

then by Cauchy-Schwartz inequality we have

$$|\det(Dv)| \leq |Du_*^1| |\text{adj}_2(Du)^1|, \quad (3.10)$$

and as a consequence

$$0 \leq H(\det(Dv)) \leq M |\det(Dv)|^r + b(x) \leq M |Du_*^1|^r |\text{adj}_2(Du)^1|^r + b(x).$$

We observe that $\text{adj}_2(Du)^1 \in L^q(\Omega)$: indeed, from (2.5) we have

$$\nu |\text{adj}_2(Du)^1|^q \leq a(x) + G_1(x, \text{adj}_2(Du)^1) \leq a(x) + f(x, Du).$$

Then, using Hölder's inequality we obtain:

$$\begin{aligned} \int_{\Omega} H(\det(Dv)) dx &\leq M \int_{\Omega} |Du_*^1|^r |\text{adj}_2(Du)^1|^r dx + \|b\|_1 \\ &\leq M \left(\int_{\Omega} |Du_*^1|^{\frac{qr}{q-r}} dx \right)^{\frac{q-r}{q}} \left(\int_{\Omega} |\text{adj}_2(Du)^1|^q dx \right)^{\frac{r}{q}} + \|b\|_1, \end{aligned}$$

then we obtain $H(\det(Dv)) \in L^1(\Omega)$ by using hypothesis (2.9) and (3.7) for which we have

$$u_*^1 \in W^{1,\tilde{p}}, \quad \tilde{p} \geq \frac{rq}{q-r}.$$

Then we conclude that v is an admissible function. Now we pass to the bound of u . By minimality of u we can write

$$\int_{\Omega} f(x, Du(x)) dx \leq \int_{\Omega} f(x, Dv(x)) dx,$$

so that

$$\int_{\Omega \setminus A_L^1} f(x, Du) dx + \int_{A_L^1} f(x, Du) dx \leq \int_{\Omega \setminus A_L^1} f(x, Dv) dx + \int_{A_L^1} f(x, Dv) dx.$$

Since $u = v$ in $\Omega \setminus A_L^1$, we drop $\int_{\Omega \setminus A_L^1} f(x, Du) dx = \int_{\Omega \setminus A_L^1} f(x, Dv) dx$ from both sides and we obtain

$$\int_{A_L^1} f(x, Du(x)) dx \leq \int_{A_L^1} f(x, Dv(x)) dx.$$

We rewrite the previous inequality in details emphasising the terms that are equal (keep in mind that $v^2 = u^2$, $v^3 = u^3$ and $\text{adj}_2(Dv)^1 = \text{adj}_2(Du)^1$):

$$\begin{aligned} & \int_{A_L^1} F_1(x, Du^1) dx + \underbrace{\int_{A_L^1} F_2(x, Du^2) dx}_{(I)} + \underbrace{\int_{A_L^1} F_3(x, Du^3) dx}_{(II)} \\ & + \underbrace{\int_{A_L^1} G_1(x, \text{adj}_2(Du)^1) dx}_{(III)} + \int_{A_L^1} G_2(x, \text{adj}_2(Du)^2) dx \\ & + \int_{A_L^1} G_3(x, \text{adj}_2(Du)^3) dx + \int_{A_L^1} H(x, \det(Du)) dx \\ & \leq \int_{A_L^1} F_1(x, Dv^1) dx + \underbrace{\int_{A_L^1} F_2(x, Dv^2) dx}_{(I)} + \underbrace{\int_{A_L^1} F_3(x, Dv^3) dx}_{(II)} \\ & + \underbrace{\int_{A_L^1} G_1(x, \text{adj}_2(Dv)^1) dx}_{(III)} + \int_{A_L^1} G_2(x, \text{adj}_2(Dv)^2) dx \\ & + \int_{A_L^1} G_3(x, \text{adj}_2(Dv)^3) dx + \int_{A_L^1} H(x, \det(Dv)) dx; \end{aligned}$$

then, dropping the equal terms and using $Dv^1 = Du_*^1$ in A_L^1 , we have

$$\begin{aligned}
& \int_{A_L^1} F_1(x, Du^1) dx + \int_{A_L^1} G_2(x, \text{adj}_2(Du)^2) dx + \int_{A_L^1} G_3(x, \text{adj}_2(Du)^3) dx \\
& + \int_{A_L^1} H(x, \det(Du)) dx \\
& \leq \int_{A_L^1} F_1(x, Du_*^1) dx + \int_{A_L^1} G_2(x, \text{adj}_2(Dv)^2) dx + \int_{A_L^1} G_3(x, \text{adj}_2(Dv)^3) dx \\
& + \int_{A_L^1} H(x, \det(Dv)) dx,
\end{aligned}$$

and, using growth assumption (2.4), (2.5), (2.6) we obtain that the right hand side of the estimate above is less or equal to

$$\begin{aligned}
& M \int_{A_L^1} |Du_*^1|^p dx + \int_{A_L^1} b(x) dx + M \int_{A_L^1} |\text{adj}_2(Dv)^2|^q dx + \int_{A_L^1} b(x) dx \\
& + M \int_{A_L^1} |\text{adj}_2(Dv)^3|^q dx + \int_{A_L^1} b(x) dx + M \int_{A_L^1} |\det(Dv)|^r dx + \int_{A_L^1} b(x) dx.
\end{aligned}$$

Finally, by Hölder's inequality, (3.8), (3.9) and (3.10), the previous quantity can be be majorized by

$$\begin{aligned}
& 4 \left(\int_{\Omega} b(x)^\sigma dx \right)^{\frac{1}{\sigma}} |A_L^1|^{1-\frac{1}{\sigma}} + M \left(\int_{\Omega} |Du_*^1|^{\tilde{p}} dx \right)^{\frac{p}{\tilde{p}}} |A_L^1|^{1-\frac{p}{\tilde{p}}} \\
& + 2M3^{\frac{q}{2}} \int_{A_L^1} |Du_*^1|^q |Du|^q dx + M \int_{A_L^1} |Du_*^1|^r |\text{adj}_2(Du)|^r dx.
\end{aligned}$$

We estimate the last two terms of the previous expression separately. We first observe that the coefficient $3^{\frac{q}{2}}$ appears since (3.8) implies

$$|\text{adj}_2(Dv)^2| \leq \sqrt{3} |Du_*^1| |Du^3| \leq \sqrt{3} |Du_*^1| |Du|,$$

moreover (3.9) implies

$$|\text{adj}_2(Dv)^3| \leq \sqrt{3} |Du_*^1| |Du^2| \leq \sqrt{3} |Du_*^1| |Du|.$$

Note that (2.9) implies $\tilde{p} > \frac{pq}{p-q}$, see (3.6); then, for the first term we have

$$\begin{aligned}
\int_{A_L^1} |Du_*|^q |Du|^q dx &\leq \left(\int_{A_L^1} |Du_*|^{\frac{qp}{p-q}} dx \right)^{\frac{p-q}{p}} \left(\int_{\Omega} |Du|^p dx \right)^{\frac{q}{p}} \\
&\leq \left[\left(\int_{A_L^1} |Du_*|^{\tilde{p}} dx \right)^{\frac{qp}{(p-q)\tilde{p}}} |A_L^1|^{1-\frac{qp}{(p-q)\tilde{p}}} \right]^{\frac{p-q}{p}} \left(\int_{\Omega} |Du|^p dx \right)^{\frac{q}{p}} \\
&\leq \left(\int_{\Omega} |Du_*|^{\tilde{p}} dx \right)^{\frac{q}{\tilde{p}}} \left(\int_{\Omega} |Du|^p dx \right)^{\frac{q}{p}} \left[|A_L^1|^{1-\frac{qp}{(p-q)\tilde{p}}} \right]^{\frac{p-q}{p}} \\
&= \|Du_*\|_{\tilde{p}}^q \|Du\|_p^q |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}}.
\end{aligned}$$

Lemma 3.1 implies that the last term is less than

$$\|Du_*\|_{\tilde{p}}^q 3^{\frac{q}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}} |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} := c_1 |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}}.$$

Note that (2.9) implies $\tilde{p} > \frac{qr}{q-r}$, see (3.7), then, thanks again to Lemma 3.1 we can estimate the second term as follows

$$\begin{aligned}
\int_{A_L^1} |Du_*|^r |adj_2(Du)|^r dx &\leq \left(\int_{\Omega} |adj_2(Du)|^q dx \right)^{\frac{r}{q}} \left(\int_{A_L^1} |Du_*|^{\frac{rq}{q-r}} dx \right)^{\frac{q-r}{q}} \\
&\leq \left(\int_{\Omega} |adj_2(Du)|^q dx \right)^{\frac{r}{q}} \left[\left(\int_{\Omega} |Du_*|^{\tilde{p}} dx \right)^{\frac{rq}{(q-r)\tilde{p}}} |A_L^1|^{1-\frac{rq}{(q-r)\tilde{p}}} \right]^{\frac{q-r}{q}} \\
&\leq \left(\int_{\Omega} |adj_2(Du)|^q dx \right)^{\frac{r}{q}} \left(\int_{\Omega} |Du_*|^{\tilde{p}} dx \right)^{\frac{r}{\tilde{p}}} \left[|A_L^1|^{1-\frac{rq}{(q-r)\tilde{p}}} \right]^{\frac{q-r}{q}} \\
&= \|adj_2(Du)\|_q^r \|Du_*\|_{\tilde{p}}^r |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}} \\
&\leq \|Du_*\|_{\tilde{p}}^r 3^{\frac{r}{2}} \left(\frac{1}{\nu} \int_{\Omega} f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}} |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}} := c_2 |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{A_L^1} F_1(x, Du^1) dx + \int_{A_L^1} G_2(x, adj_2(Du)^2) dx + \int_{A_L^1} G_3(x, adj_2(Du)^3) dx \\
&+ \int_{A_L^1} H(x, \det(Du)) dx \\
&\leq 4\|b\|_{\sigma} |A_L^1|^{1-\frac{1}{\sigma}} + M \|Du_*\|_{\tilde{p}}^2 |A_L^1|^{1-\frac{2}{\tilde{p}}} + 2M 3^{\frac{q}{2}} c_1 |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} + M c_2 |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}.
\end{aligned}$$

By using the positiveness of G_α , H and the left hand side of (2.4) we can write:

$$\begin{aligned} \nu \int_{A_L^1} |Du^1|^p dx - \int_{A_L^1} a(x) dx &\leq \int_{A_L^1} F_1(x, Du^1) dx \\ &\leq k_1 |A_L^1|^{1-\frac{1}{\sigma}} + k_2 |A_L^1|^{1-\frac{p}{\tilde{p}}} + k_3 |A_L^1|^{1-\frac{q}{p}-\frac{q}{\tilde{p}}} + k_4 |A_L^1|^{1-\frac{r}{q}-\frac{r}{\tilde{p}}}, \end{aligned}$$

where

$$k_1 = 4\|b\|_\sigma, \quad k_2 = M\|Du_*\|_{\tilde{p}}^p, \quad k_3 = 2M3^{\frac{q}{2}}c_1, \quad k_4 = Mc_2.$$

We can estimate the term depending on the function $a(\cdot)$ in the following way:

$$\int_{A_L^1} a(x) dx \leq \left(\int_{\Omega} a(x)^\sigma dx \right)^{\frac{1}{\sigma}} |A_L^1|^{1-\frac{1}{\sigma}}.$$

Finally, we obtain

$$\int_{A_L^1} |Du^1|^p \leq k |A_L^1|^\gamma,$$

where

$$k = \frac{4}{\nu} \max\{k_1 + \|a\|_\sigma, k_2, k_3, k_4\} \max\{|\Omega|^{\delta_1-\gamma}, |\Omega|^{\delta_2-\gamma}, |\Omega|^{\delta_3-\gamma}, |\Omega|^{\delta_4-\gamma}\},$$

and

$$\gamma := \min \left\{ \underbrace{1 - \frac{1}{\sigma}}_{\delta_1}, \underbrace{1 - \frac{p}{\tilde{p}}}_{\delta_2}, \underbrace{1 - \frac{q}{p} - \frac{q}{\tilde{p}}}_{\delta_3}, \underbrace{1 - \frac{r}{q} - \frac{r}{\tilde{p}}}_{\delta_4} \right\}.$$

The previous estimate follows from the following reasoning,

$$|A_L^1|^\delta = |A_L^1|^\gamma |A_L^1|^{\delta-\gamma} \leq |A_L^1|^\gamma |\Omega|^{\delta-\gamma},$$

where $\delta \geq \gamma$. From Hölder's inequality we have

$$\int_{A_L^1} |Du_*^1|^p dx \leq \left(\int_{\Omega} |Du_*^1|^{\tilde{p}} \right)^{\frac{p}{\tilde{p}}} |A_L^1|^{1-\frac{p}{\tilde{p}}},$$

and as a consequence

$$\int_{A_L^1} |Du_*^1|^p dx \leq \|Du_*\|_{\tilde{p}}^p |A_L^1|^{1-\frac{p}{\tilde{p}}} \leq \tilde{k} |A_L^1|^\gamma,$$

where γ is as above and

$$\tilde{k} := \|Du_*\|_p^p |\Omega|^{\delta_2 - \gamma}.$$

Finally, we can write

$$\begin{aligned} \int_{A_L^1} |D[u^1(x) - u_*^1(x)]|^p dx &\leq 2^{p-1} \int_{A_L^1} |Du^1(x)|^p dx + 2^{p-1} \int_{A_L^1} |Du_*^1(x)|^p dx \\ &\leq 2^{p-1} (k + \tilde{k}) |A_L^1|^\gamma := \hat{k} |A_L^1|^\gamma. \end{aligned}$$

We use Sobolev inequality for $1 \leq p < 3$:

$$\begin{aligned} \int_{A_L^1} |(u^1 - u_*^1) - L|^{p^*} dx &= \int_{\Omega} |[(u^1 - u_*^1) - L] \vee 0|^{p^*} dx \\ &\leq C_{S,3} \left(\int_{\Omega} |D\{[(u^1 - u_*^1) - L] \vee 0\}|^p dx \right)^{\frac{p^*}{p}} \\ &= C_{S,3} \left(\int_{A_L^1} |D(u^1 - u_*^1)|^p dx \right)^{\frac{p^*}{p}}, \end{aligned} \quad (3.11)$$

where $C_{S,3} = \left(\frac{2p}{3-p}\right)^{\frac{3p}{3-p}}$ is the Sobolev constant in dimension 3, while p^* is the Sobolev exponent in dimension 3: $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. From the previous computation we obtain

$$\int_{A_L^1} |[u^1(x) - u_*^1(x)] - L|^{p^*} dx \leq K |A_L^1|^{\frac{p^*}{p} \gamma},$$

where $K = C_{S,3} (\hat{k})^{\frac{p^*}{p}}$. Now we take $L_1 > L$; from $A_{L_1}^1 \subset A_L^1$ we have

$$|A_{L_1}^1| [L_1 - L]^{p^*} \leq \int_{A_{L_1}^1} |[u^1(x) - u_*^1(x)] - L|^{p^*} dx \leq \int_{A_L^1} |[u^1(x) - u_*^1(x)] - L|^{p^*} dx.$$

Finally, we have obtained the following inequality:

$$|A_{L_1}^1| \leq \frac{K}{[L_1 - L]^{p^*}} |A_L^1|^{\frac{p^*}{p} \gamma}.$$

The previous inequality falls within the hypotheses of Stampacchia's Lemma that we write for the convenience of the reader.

Lemma 3.3. (see page 93 in[46]) Let $\varphi(t)$ a non negative and non increasing function defined for $t \geq k_0$ such that for $h > k \geq k_0$ we have

$$\varphi(h) \leq \frac{c}{(h-k)^\alpha} [\varphi(k)]^\beta,$$

with c, α, β positive constants. If $\beta > 1$ then we have

$$\varphi(k_0 + d) = 0,$$

where

$$d^\alpha = c[\varphi(k_0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}.$$

We use such a Lemma with

$$\alpha = p^* > 0, \quad \beta = \frac{p^*}{p} \gamma > 1,$$

where the second inequality is proved below.

Then, we have $|A_{\tilde{L}}^1| = 0$ for

$$\tilde{L} = L + 2^{\frac{\beta}{\beta-1}} \left\{ K |A_L^1|^{\beta-1} \right\}^{\frac{1}{\alpha}} \leq L + 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}}.$$

Since L was free in $(0, +\infty)$, taking $L = \frac{1}{N}$, with $N \in \mathbb{N}$, we have

$$|A_{L^*}^1| = 0, \quad \text{with } L^* = 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}}.$$

Then we conclude that

$$u^1 - u_*^1 \leq L^* = 2^{\frac{\beta}{\beta-1}} \left\{ K |\Omega|^{\beta-1} \right\}^{\frac{1}{\alpha}} = 2^{\frac{p^* \gamma}{p^* - p}} K^{\frac{1}{p^*}} |\Omega|^{\frac{\gamma}{p} - \frac{1}{p^*}} \quad (3.12)$$

where

$$\begin{aligned} K := & \frac{4}{\nu} \left(\frac{2p}{3-p} \right)^{\frac{3p}{3-p}} 2^{\frac{3(p-1)}{3-p}} \\ & \cdot \left(\max \left\{ 4 \|b\|_\sigma + \|a\|_\sigma, M \|Du_*\|_{\tilde{p}}^p, \right. \right. \\ & \quad \left. \left. 2M3^q \|Du_*\|_{\tilde{p}}^q \left(\frac{1}{\nu} \int_\Omega f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{q}{p}}, \right. \right. \\ & \quad \left. \left. M3^{\frac{r}{2}} \|Du_*\|_{\tilde{p}}^r \left(\frac{1}{\nu} \int_\Omega f(x, Du_*) dx + \frac{3}{\nu} \|a\|_1 \right)^{\frac{r}{q}} \right\} \right. \\ & \cdot \max \left\{ |\Omega|^{\delta_1 - \gamma}, |\Omega|^{\delta_2 - \gamma}, |\Omega|^{\delta_3 - \gamma}, |\Omega|^{\delta_4 - \gamma} \right\} + \|Du_*\|_{\tilde{p}}^p |\Omega|^{\delta_2 - \gamma} \Big)^{\frac{3}{3-p}}. \end{aligned} \quad (3.13)$$

It remains to check the inequality

$$\beta = \frac{p^*}{p}\gamma = \frac{3}{3-p}\gamma > 1; \quad (3.14)$$

we recall that γ may take four different values then we check the inequality for each value:

- (1) $\frac{3}{3-p} \left(1 - \frac{1}{\sigma}\right) > 1$, for $\sigma > \frac{3}{p}$,
- (2) $\frac{3}{3-p} \left(1 - \frac{p}{\tilde{p}}\right) > 1$, for $\tilde{p} > 3$,
- (3) $\frac{3}{3-p} \left(1 - \frac{q}{p} - \frac{q}{\tilde{p}}\right) > 1$, for $\tilde{p} > \frac{3pq}{p^2 - 3q}$, with $p > \sqrt{3q}$,
- (4) $\frac{3}{3-p} \left(1 - \frac{r}{q} - \frac{r}{\tilde{p}}\right) > 1$, for $\tilde{p} > \frac{3qr}{qp - 3r}$, with $p > \frac{3r}{q}$.

They are all verified thanks to the hypotheses (2.3),(2.9) and (2.1).

In order to provide a bound from below for the quantity $u^1(x) - u_*^1(x)$ we observe that $\tilde{u} = -u$ solves the following minimum problem:

$$\min \left\{ \int_{\Omega} \tilde{f}(x, Dh(x)) dx, \quad h \in \tilde{u}_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\},$$

where $\tilde{f}(x, z) = f(x, -z)$ and $\tilde{u}_*(x) = -u_*(x)$. We note that $D\tilde{u} = D(-u) = -Du$, $\text{adj}_2(D\tilde{u})_i^\alpha = \text{adj}_2(-Du)_i^\alpha = \text{adj}_2(Du)_i^\alpha$ and $\det(D\tilde{u}) = \det(-Du) = -\det(Du)$. Then

$$\tilde{f}(x, z) = \sum_{\alpha} F_{\alpha}(x, -z^{\alpha}) + \sum_{\alpha} G_{\alpha}(x, \text{adj}_2(z)^{\alpha}) + H(x, -\det(z)).$$

Since $F_{\alpha}(x, -\xi)$, $H(x, -t)$ satisfy again (2.4) and (2.6) we can repeat the same computations made for u without any changes. Then we obtain the following upper bound:

$$-u^1(x) + u_*^1(x) = \tilde{u}^1(x) - \tilde{u}_*^1(x) \leq L^*,$$

that is

$$u^1(x) - u_*^1(x) \geq -L^*. \quad (3.15)$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with \tilde{u}_* instead of u_* ; since $\tilde{u}_* = -u_*$ then $\|D\tilde{u}_*\|_s = \|Du_*\|_s$; since $\tilde{f}(x, z) = f(x, -z)$, then

$$\int_{\Omega} f(x, D\tilde{u}_*) dx = \int_{\Omega} f(x, Du_*) dx.$$

This means that L^* in (3.15) and L^* in (3.12) are the same. From the previous inequality and from (3.12) we conclude:

$$|u^1(x) - u_*^1(x)| \leq L^*. \quad (3.16)$$

Now we pass to analyse the second component of u . We introduce the following function by changing the order of the components of u :

$$\hat{u} = \begin{pmatrix} u^2 \\ u^1 \\ u^3 \end{pmatrix}, \quad \text{for which} \quad D\hat{u} = \begin{pmatrix} Du^2 \\ Du^1 \\ Du^3 \end{pmatrix}.$$

We denote by $C_{1,2}(z)$ the matrix obtained from z by inverting line 1 and line 2. Then

$$D\hat{u} = C_{1,2}(Du).$$

Since $\text{adj}_2(D\hat{u})^1 = -\text{adj}_2(Du)^2$, $\text{adj}_2(D\hat{u})^2 = -\text{adj}_2(Du)^1$ and $\text{adj}_2(D\hat{u})^3 = -\text{adj}_2(Du)^3$ we have

$$\text{adj}_2(D\hat{u}) = - \begin{pmatrix} \text{adj}_2(Du)^2 \\ \text{adj}_2(Du)^1 \\ \text{adj}_2(Du)^3 \end{pmatrix} = -C_{1,2}(\text{adj}_2(Du)),$$

moreover $\det(D\hat{u}) = -\det(Du)$. Then we have that \hat{u} solves

$$\min \left\{ \int_{\Omega} \hat{f}(x, Dh(x)) dx, \quad h \in \hat{u}_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\},$$

where $\hat{u}_*(x) = \begin{pmatrix} u_*^2 \\ u_*^1 \\ u_*^3 \end{pmatrix}$ and

$$\begin{aligned} \hat{f}(x, z) &= f(x, C_{1,2}(z)) \\ &= F_1(x, z^2) + F_2(x, z^1) + F_3(x, z^3) + G_1(x, -\text{adj}_2(z)^2) \\ &\quad + G_2(x, -\text{adj}_2(z)^1) + G_3(x, -\text{adj}_2(z)^3) + H(x, -\det(z)). \end{aligned}$$

Again, $\hat{f}(x, y)$ satisfies (2.4), (2.5), (2.6), and all the computations made for u can be repeated without any changes and we obtain:

$$|u^2(x) - u_*^2(x)| = |\hat{u}^1(x) - \hat{u}_*^1(x)| \leq L^*. \quad (3.17)$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with \hat{u}_* instead of u_* ; since $\hat{u}_* = C_{1,2}(u_*)$, then $\|D\hat{u}_*\|_s = \|Du_*\|_s$; since $\hat{f}(x, z) = f(x, C_{1,2}(z))$, then

$$\int_{\Omega} \hat{f}(x, D\hat{u}_*) dx = \int_{\Omega} f(x, Du_*) dx.$$

This means that L^* in (3.17) and L^* in (3.12) are the same. For the third component we introduce the function

$$\hat{u} = \begin{pmatrix} u^3 \\ u^2 \\ u^1 \end{pmatrix}.$$

In this case we denote by $C_{1,3}(z)$ the matrix obtained from z by inverting line 1 and line 3. Then

$$D\hat{u} = C_{1,3}(Du), \quad \text{adj}_2(D\hat{u}) = -C_{1,3}(\text{adj}_2(Du)), \quad \det(D\hat{u}) = -\det(Du).$$

Then we have that \hat{u} solves

$$\min \left\{ \int_{\Omega} \hat{f}(x, Dh(x)) dx, \quad h \in \hat{u}_* + W_0^{1,p}(\Omega, \mathbb{R}^3) \right\},$$

where $\hat{u}_*(x) = \begin{pmatrix} u_*^3 \\ u_*^2 \\ u_*^1 \end{pmatrix}$ and

$$\begin{aligned} \hat{f}(x, z) &= f(x, C_{1,3}(z)) \\ &= F_1(x, z^3) + F_2(x, z^2) + F_3(x, z^1) + G_1(x, -\text{adj}_2(z)^3) \\ &\quad + G_2(x, -\text{adj}_2(z)^2) + G_3(x, -\text{adj}_2(z)^1) + H(x, -\det(z)). \end{aligned}$$

Then we obtain as above that:

$$|u^3(x) - u_*^3(x)| = |\hat{u}^1(x) - \hat{u}_*^1(x)| \leq L^*. \quad (3.18)$$

Note that L^* is given by the right hand side of (3.12) and (3.13) with \hat{u}_* instead of u_* ; since $\hat{u}_* = C_{1,3}(u_*)$, then $\|D\hat{u}_*\|_s = \|Du_*\|_s$; since $\hat{f}(x, z) = f(x, C_{1,3}(z))$, then

$$\int_{\Omega} \hat{f}(x, D\hat{u}_*) dx = \int_{\Omega} f(x, Du_*) dx.$$

This means that L^* in (3.18) and L^* in (3.12) are the same. This concludes the proof. \square

4. Remarks on growth assumptions

We now consider model density (2.11) and we compare the restrictions on the growth assumptions considered in the present paper with the ones contained in [12]. Both requires r to be less than q which, in turn, must be less than p . Moreover, we are in the polyconvex case, so $1 \leq r$. The present paper requires $p < 3$, then $p^* = 3p/(3-p)$; in [12] also the case $p = 3$ is dealt with. Then, we confine ourselves to

$$1 \leq r < q < p < 3.$$

We recall that [12] asks for two conditions:

$$\frac{p}{p^*} < 1 - \frac{qp^*}{p(p^* - q)};$$

and

$$\frac{p}{p^*} < 1 - \frac{rp^*}{q(p^* - r)};$$

they can be written as follows

$$\frac{p}{3} > \frac{qp^*}{p(p^* - q)}; \tag{4.1}$$

and

$$\frac{p}{3} > \frac{rp^*}{q(p^* - r)}. \tag{4.2}$$

Let us study condition (4.1): it can be written as

$$\frac{p}{3} > \frac{q3p}{p(3p - q(3 - p))}; \tag{4.3}$$

this amounts to say

$$p(3p - q(3 - p)) > 3q3; \tag{4.4}$$

exploiting calculations we get

$$(3 + q)p^2 - 3qp - 9q > 0; \tag{4.5}$$

the corresponding second order equation has two roots: $p_1 := \frac{3q-3\sqrt{5q^2+12q}}{2(3+q)} < 0$ and $p_2 := \frac{3q+3\sqrt{5q^2+12q}}{2(3+q)}$; then, inequality (4.5) is equivalent, in our case, to $p > p_2$. We claim that

$$p_2 > \sqrt{3q}; \tag{4.6}$$

indeed, squaring both sides gives

$$\frac{9q^2 + 6q\sqrt{5q^2 + 12q} + 45q^2 + 108q}{4(3 + q)^2} > 3q; \quad (4.7)$$

q can be cut from both sides; eliminating the denominator and rearranging elements result in

$$3\sqrt{5q^2 + 12q} > 2q^2 + 3q; \quad (4.8)$$

squaring again both sides, rearranging elements and cutting q give

$$0 > q^3 + 3q^2 - 9q - 27 := g(q). \quad (4.9)$$

We study the function g : we have $g(3) = 0$ and $g'(q) = 3q^2 + 6q - 9$; equation $3q^2 + 6q - 9 = 0$ has two roots $q_1 = -3$ and $q_2 = 1$; then $g'(q) > 0$ for every $q > 1$ and g strictly increases in $[1, +\infty)$; in particular, $g(q) < g(3) = 0$ for $1 \leq q < 3$ and (4.9) holds true: this proves our claim (4.6). Now we turn our attention to (4.2): it can be written as

$$\frac{p}{3} > \frac{r3p}{q(3p - r(3 - p))}; \quad (4.10)$$

this amounts to say

$$q(3p - r(3 - p)) > 3r3; \quad (4.11)$$

dividing by q and exploiting calculations give

$$p > 3 \frac{r(q + 3)}{q(r + 3)}. \quad (4.12)$$

Note that

$$3 \frac{r(q + 3)}{q(r + 3)} > 3 \frac{r}{q}, \quad (4.13)$$

since $q > r$. Let us summarize: in [12] conditions on p are

$$\begin{cases} p > p_2, \\ p > 3 \frac{r(q+3)}{q(r+3)}; \end{cases} \quad (4.14)$$

if we keep in mind (4.6) and (4.13) we see that (4.14) implies

$$\begin{cases} p > \sqrt{3q}, \\ p > 3 \frac{r}{q}, \end{cases} \quad (4.15)$$

and these are our conditions on p . This shows that our conditions are weaker than the ones in [12] when dealing with (2.11) and $1 \leq r < q < p < 3$.

5. Lipschitz boundary datum

In the case where the boundary datum is a Lipschitz function it is possible to obtain a sharper bound for the minimizers. In this section we discuss in details this situation. We start fixing some notation.

Definition 5.1. Let $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ and let $u_* : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be such that u_*^i is Lipschitz with Lipschitz constant less than or equal to \mathcal{K}_i , $i = 1, 2, 3$. We will say that such a u_* is a \mathcal{K} -Lipschitz mapping. Let u_* be a \mathcal{K} -Lipschitz mapping, we will denote with $\text{Lip}_{\mathcal{K}_i, u_*^i}$ the set of Lipschitz functions $g : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ with Lipschitz constant less than or equal to \mathcal{K}_i that coincide with u_*^i on $\partial\Omega$. We will denote by $\text{Lip}_{\mathcal{K}_{u_*}}$ the set of functions $u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $u(x) = (u^1(x), u^2(x), u^3(x))$ with $u^i \in \text{Lip}_{\mathcal{K}_i, u_*^i}$ for every $i = 1, 2, 3$.

It is straightforward that, since we assume u_* is \mathcal{K} -Lipschitz, the set $\text{Lip}_{\mathcal{K}_i, u_*^i}$ is non empty. We also remark that it is compact with respect to the uniform convergence.

We fix $i \in \{1, 2, 3\}$ and we consider the functional

$$\text{Lip}_{\mathcal{K}_i, u_*^i} \ni g \rightarrow \int_{\Omega} g(x) dx. \quad (5.1)$$

The functional (5.1) is continuous with respect to the uniform convergence and then it admits both a minimum and a maximum on $\text{Lip}_{\mathcal{K}_i, u_*^i}$.

We denote the minimizer by u_-^i and the maximizer by u_+^i and we consider the functions $u_-, u_+ : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $u_- = (u_-^1, u_-^2, u_-^3)$ and $u_+ = (u_+^1, u_+^2, u_+^3)$.

Proposition 5.2. Let $u \in \text{Lip}_{\mathcal{K}_{u_*}}$. Then

$$u_-^i(x) \leq u^i(x) \leq u_+^i(x) \quad \text{for all } x \in \Omega, \quad \text{for all } i \in \{1, 2, 3\}.$$

Proof. It is sufficient to prove that $u_-^1(x) \leq u^1(x)$ for all $x \in \Omega$ since the other inequalities follow similarly. We argue by contradiction assuming that there exist $u \in \text{Lip}_{\mathcal{K}_1, u_*^1}$ and $x \in \Omega$ such that $u^1(x) < u_-^1(x)$. The continuity of both u^1 and u_-^1 implies that the set $A = \{x \in \Omega : u^1(x) < u_-^1(x)\}$ has positive measure. Let us consider the function

$$w = \min\{u^1; u_-^1\} = \begin{cases} u^1, & \text{if } x \in A, \\ u_-^1, & \text{otherwise.} \end{cases}$$

We obtain that $w \in \text{Lip}_{\mathcal{K}_1, u_*^1}$ and

$$\begin{aligned}
& \int_{\Omega} w(x) dx \\
&= \int_{\Omega \setminus A} u_-^1(x) dx + \int_A u^1(x) dx \\
&< \int_{\Omega \setminus A} u_-^1(x) dx + \int_A u_-^1(x) dx \\
&= \int_{\Omega} u_-^1(x) dx.
\end{aligned} \tag{5.2}$$

This is in contrast with the minimality of u_-^1 for the functional (5.1). \square

Remark 5.3. *The previous proposition shows also that the minimizer u_-^i is unique and, of course, the same is true for the maximizer u_+^i , for $i = 1, 2, 3$. There is only one case in which u_-^i and u_+^i coincide in Ω and it happens when there exists just one function with Lipschitz constant \mathcal{K}_i assuming the boundary datum u_*^i .*

Corollary 5.4. *Let u_* be a Lipschitz mapping with Lipschitz constant \mathcal{K} . Let u be a solution of the minimization problem 1.7. Then for almost every $x \in \Omega$ we have*

$$u_+^i(x) - L^+ \leq u^i(x) \leq u_-^i(x) + L^-, \tag{5.3}$$

for every $i \in \{1, 2, 3\}$, where L^- is the constant L^* that appears in (2.10) and (2.8) with u_- instead of u_* and, analogously, L^+ is the constant L^* with u_+ instead of u_* .

Proof. Since $u_- = u_+ = u_*$ on $\partial\Omega$, we have $u_* + W_0^{1,p}(\Omega) = u_- + W_0^{1,p}(\Omega) = u_+ + W_0^{1,p}(\Omega)$ and we can apply Theorem 2.2 both with u_- and u_+ instead of u_* . Then using just one side of the inequality (2.2) we get the estimate. \square

Remark 5.5. *Let us give an estimate for the constants L^- and L^+ . Looking at (2.10) and (2.8), we have to control the following items*

$$\|Du_-\|_{\bar{p}} \quad \text{and} \quad \int_{\Omega} f(x, Du_-) dx.$$

Let us set

$$\mathcal{K}_* := \max\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}.$$

We have

$$\begin{aligned} |Du_-|^{\tilde{p}} &= \left(\sum_{\alpha=1}^3 \sum_{i=1}^3 |D_i u_-^\alpha|^2 \right)^{\tilde{p}/2} \leq \left(\sum_{\alpha=1}^3 \sum_{i=1}^3 |\mathcal{K}_\alpha|^2 \right)^{\tilde{p}/2} \\ &\leq \left(\sum_{\alpha=1}^3 \sum_{i=1}^3 |\mathcal{K}_*|^2 \right)^{\tilde{p}/2} = (3\mathcal{K}_*)^{\tilde{p}}, \end{aligned}$$

then

$$\|Du_-\|_{\tilde{p}} \leq |\Omega|^{\frac{1}{\tilde{p}}} 3\mathcal{K}_*.$$

Moreover, using (2.4)-(2.2), we can write

$$\begin{aligned} \int_{\Omega} f(x, Du_-) dx &\leq 7\|b\|_1 + M \left\{ \sum_{\alpha=1}^3 \int_{\Omega} |Du_-^\alpha|^p dx + \sum_{\alpha=1}^3 \int_{\Omega} |adj_2(Du_-)^\alpha|^q dx + \right. \\ &\quad \left. \int_{\Omega} |det(Du_-)|^r dx \right\} \leq 7\|b\|_1 + M|\Omega| \left\{ 3^{1+\frac{p}{2}} \mathcal{K}_*^p + 3 \cdot 12^{\frac{q}{2}} \mathcal{K}_*^{2q} + 6^r \mathcal{K}_*^{3r} \right\}, \end{aligned}$$

where we have used the estimates

$$|adj_2(Du_-)^\alpha|^q = \left[\sum_{k=1}^3 |adj_2(Du_*)^\alpha_k|^2 \right]^{\frac{q}{2}} \leq \left[\sum_{k=1}^3 (2\mathcal{K}_*^2)^2 \right]^{\frac{q}{2}} = 12^{\frac{q}{2}} \mathcal{K}_*^{2q},$$

and

$$|det(Du_-)|^r \leq (3!\mathcal{K}_*^3)^r = 6^r \mathcal{K}_*^{3r}.$$

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