Multiplicity of positive solutions for a fractional p&q-Laplacian problem in RN

MULTIPLICITY OF POSITIVE SOLUTIONS FOR A FRACTIONAL $p\&q$-LAPLACIAN PROBLEM IN $\mathbb{R}^N$

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Abstract. In this paper we deal with the following fractional $p\&q$-Laplacian problem:

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon\, x) \left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u(x) > 0 & \text{for a.e. } x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, $\varepsilon > 0$ is a small parameter, $2 \leq p < q < \frac{N}{s}$, $(-\Delta)_t^s$, with $t \in \{p,q\}$, is the fractional (s,t)-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the global Rabinowitz condition, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth. Using suitable variational arguments and Ljusternik-Schnirelmann category theory, we prove that the above problem admits multiple solutions for $\varepsilon > 0$ small enough.

1. Introduction

In this paper we focus our attention on the existence and multiplicity of solutions for the following fractional $p\&q$-Laplacian type problem:

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon\, x) \left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u(x) > 0 & \text{for a.e. } x \in \mathbb{R}^N, \end{cases} \tag{P_ε}$$

where $s \in (0,1)$, $\varepsilon > 0$ is a small parameter, $2 \leq p < q < \frac{N}{s}$, $W^{s,t}(\mathbb{R}^N)$, with $t \in \{p,q\}$, is defined as the set of the functions $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^t(\mathbb{R}^N)$ such that

$$[u]^t_{W^{s,t}(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^t}{|x-y|^{N+st}} dxdy < \infty.$$

The leading operator $(-\Delta)_t^s$ is the fractional (s,t)-Laplacian operator which may be defined, up to a normalization constant, by setting

$$(-\Delta)_t^s u(x) = 2 \lim_{r \to 0} \int_{\mathbb{R}^N \setminus \mathcal{B}_r(x)} \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}} dy \quad (x \in \mathbb{R}^N)$$

for any $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$; we refer to [23, 38] for more details on nonlocal fractional operators.

In the local case, that is when $s = 1$, (P_ε) reduces to a $p\&q$-Laplacian equation of the type

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x,u) \quad \text{in } \mathbb{R}^N.$$

These class of problems arise from a general reaction-diffusion system

$$u_t = \operatorname{div}(D(u)\nabla u) + f(x,u) \quad x \in \mathbb{R}^N, t > 0,$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. As underlined in [18], this equation appears in a lot of applications such as biophysics, plasma physics and chemical reaction design. In these applications, u describes a concentration, $\operatorname{div}(D(u)\nabla u)$ corresponds to the diffusion with a diffusion coefficient $D(u)$, and the reaction term $f(x,u)$ relates to source and loss processes. Classical $p\&q$ Laplacian problems in

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bounded or unbounded domains have been studied by several authors; see for instance [2, 13, 14, 16, 20, 25, 27, 31–33, 39, 41] and references therein.

Coming back to the fractional setting, we point out that in these last years, the study of non-local problems driven by fractional operators has received a great interest from the mathematical community both for their interesting theoretical structure and in view of concrete applications, such as, obstacle problem, optimization, finance, phase transition, and so on. For more details we refer to [23, 38]. Indeed, when $p = q = 2$, equation (P_ε) appears in the study of standing wave solutions, i.e. solutions of the form $\psi(x,t) = u(x)e^{-\imath\omega t}$, to the following fractional Schrödinger equation

$$\imath\hbar\frac{\partial\psi}{\partial t} = \hbar^2(-\Delta)^s\psi + W(x)\psi - f(|\psi|) \quad \text{in } \mathbb{R}^N \times \mathbb{R},$$

where $\hbar$ is the Planck constant, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential and f is a suitable nonlinearity. The fractional Schrödinger equation has been introduced for the first time by Laskin [30] due to its fundamental importance in the study of particles on stochastic fields modeled by Lévy processes. After that, a remarkable attention has been devoted to the study of fractional Schrödinger equations, and several existence, multiplicity and qualitative results have been established; we refer the interested reader to [4, 7, 24, 28, 42] and references therein.

When $p = q \neq 2$, (P_ε) boils down to the following fractional p-Laplacian equation:

$$(-\Delta)^s_p u + V(\varepsilon x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

We stress that the fractional p-Laplacian has a great attractive since two phenomena are present in it: the nonlinearity of the operator and its nonlocal character. Indeed, standard tools used to investigate the linear case $p = 2$ seem not to be trivially adaptable in the case $p \neq 2$ due to the lack of Hilbertian structure of $W^{s,p}(\mathbb{R}^N)$ for $p \neq 2$. For these reasons, there has been a source of interest around nonlocal and fractional problems involving the fractional p-Laplacian operator; see for instance [5, 8, 9, 22, 26, 29, 34, 35, 37] and references therein.

On the other hand, in recent years, a great attention has been devoted to the study of fractional $p\&q$-Laplacian problems; we mention [1, 6, 10–12, 15, 17] for some existence and multiplicity results, and [21] (see also [1, 12]) for some regularity results. However, only few papers deal with fractional problems like (P_ε) and the main purpose of this paper is to give a further result in this direction.

In order to precisely state our result, we introduce the assumptions on the potential V and the nonlinearity f. Throughout the paper we will assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following condition introduced by Rabinowitz [40]:

$$V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0, \tag{V}$$

and we consider both cases $V_\infty < \infty$ and $V_\infty = \infty$.

Concerning the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ we suppose that

(f_1) $f \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$ and $f(t) = 0$ for all $t < 0$;

(f_2) $\displaystyle\lim_{|t|\to 0}\frac{|f(t)|}{|t|^{p-1}} = 0$;

(f_3) there exists $r \in (q, q^*_s)$, with $q^*_s = \frac{Nq}{N-sq}$, such that $\displaystyle\lim_{|t|\to\infty}\frac{|f(t)|}{|t|^{r-1}} = 0$;

(f_4) there exists $\vartheta \in (q, q^*_s)$ such that

$$0 < \vartheta F(t) = \vartheta\int_0^t f(\tau)\,d\tau \leq tf(t) \quad \text{for all } t > 0;$$

(f_5) the map $t \mapsto \dfrac{f(t)}{t^{q-1}}$ is increasing in $(0,\infty)$.

Since we deal with the multiplicity of solutions to (P_ε), we recall that if Y is a given closed set of a topological space X, we denote by $cat_Y(Y)$ the Ljusternik-Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [44] for more details.

Let us denote by

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\} \quad \text{and} \quad M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\}, \text{ for } \delta > 0.$$

Our first main result can be stated as follows:

Theorem 1.1. *Assume that* (V) *and* (f_1)*-*(f_5) *hold. Then, for any* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$*, problem* ($P_\varepsilon$) *has at least* $cat_{M_\delta}(M)$ *positive solutions.*

The proof of Theorem 1.1 is obtained by using variational techniques and category theory. We note that Theorem 1.1 can be seen as the fractional analogue of Theorem 1.1 in [2]. Anyway, we are able to improve this result because in [2] the authors assumed $f \in \mathcal{C}^1$ and that there exist $C > 0$ and $\nu \in (p, q_s^*)$ such that

$$f'(t)t^2 - (q-1)f(t)t \geq Ct^\nu \quad \text{for all } t \geq 0.$$

Therefore, since we require that f is merely continuous, the classical Nehari manifold arguments used in [2] do not work in our context, and in order to overcome the non-differentiability of the Nehari manifold, we borrow some ideas in [8] which combine the critical point theorems in [43] with Ljusternik-Schnirelmann category theory. Clearly, due to the fact that (P_ε) includes the sum of two nonlocal operators with different scaling properties, we have to face with some additional technical difficulties. We would like to point out that our arguments are rather flexible and can be applied in other situations to study fractional $p\&q$-Laplacian problems like (P_ε). Moreover, our proofs also work in the local case $s = 1$.

The paper is organized as follows: in Section 2 we collect some facts about the involved fractional Sobolev spaces and some useful lemmas. In Section 3 we provide some technical results which will be fundamental to prove our main theorem. In Section 4 we deal with the autonomous problems related to (P_ε). In Section 5 we obtain an existence result to (P_ε) for small ε. The last section is devoted to the multiplicity result for (P_ε).

2. Preliminaries

In this preliminary section we recall some facts about the fractional Sobolev spaces and we prove some technical lemmas which we will use later.

Let $p \in [1, \infty]$ and $A \subset \mathbb{R}^N$. We denote by $|u|_{L^p(A)}$ the $L^p(A)$-norm of a function $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^p(A)$. When $A = \mathbb{R}^N$ we simply write $|u|_p$ instead of $|u|_{L^p(\mathbb{R}^N)}$. For $s \in (0, 1)$ and $p \in (1, \infty)$, we define $\mathcal{D}^{s,p}(\mathbb{R}^N)$ as the closure of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy.$$

Let us indicate by $W^{s,p}(\mathbb{R}^N)$ the set of functions $u \in L^p(\mathbb{R}^N)$ such that $[u]_{s,p} < \infty$, endowed with the natural norm

$$\|u\|_{s,p}^p = [u]_{s,p}^p + |u|_p^p.$$

We begin by recalling the following embeddings of the fractional Sobolev spaces into Lebesgue spaces.

Theorem 2.1. *[23] Let* $s \in (0, 1)$ *and* $N > sp$*. Then there exists a constant* $S_* > 0$ *such that, for any* $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)$*,*

$$|u|_{p_s^*}^p \leq S_*^{-1} [u]_{s,p}^p.$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$ and compactly in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [1, p_s^*)$.

We recall the following compactness-Lions type result.

Lemma 2.1. *[8] Let $N > sp$ and $r \in [p, p_s^*)$. If $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathcal{B}_R(y)} |u_n|^r dx = 0, \tag{2.1}$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p_s^*)$.

The next result provides a way to manipulate smooth truncations for the fractional p-Laplacian.

Lemma 2.2. *[8] Let $u \in W^{s,p}(\mathbb{R}^N)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ in $\mathcal{B}_1(0)$ and $\phi = 0$ in $\mathcal{B}_2^c(0)$. Set $\phi_r(x) = \phi(\frac{x}{r})$. Then*

$$\lim_{r\to\infty} [u\phi_r - u]_{s,p} = 0 \quad \text{and} \quad \lim_{r\to\infty} |u\phi_r - u|_p = 0.$$

We also have the following useful lemma.

Lemma 2.3. *[8] Let $w \in \mathcal{D}^{s,p}(\mathbb{R}^N)$ and $\{z_n\} \subset \mathcal{D}^{s,p}(\mathbb{R}^N)$ be a sequence such that $z_n \to 0$ a.e. in $\mathbb{R}^N$ and $[z_n]_{s,p} \leq C$ for any $n \in \mathbb{N}$. Then we have*

$$\iint_{\mathbb{R}^{2N}} |\mathcal{A}(z_n + w) - \mathcal{A}(z_n) - \mathcal{A}(w)|^{p'} dx = o_n(1),$$

where $\mathcal{A}(u) = \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+sp}{p'}}}$ and $p' = \frac{p}{p-1}$.

Let us define the space

$$\mathbb{X}_\varepsilon = \left\{ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon\, x)\, (|u|^p + |u|^q)\, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon = \|u\|_{V,p} + \|u\|_{V,q},$$

where

$$\|u\|^t_{V,t} = [u]^t_{s,t} + \int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^t \, dx \quad \text{for all } t > 1.$$

Then we have the following embeddings:

Lemma 2.4. *[1] The space $\mathbb{X}_\varepsilon$ is continuously embedded into $W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$. Therefore, $\mathbb{X}_\varepsilon$ is continuously embedded into $L^t(\mathbb{R}^N)$ for any $t \in [p, q_s^*]$ and compactly embedded into $L^t(\mathcal{B}_R(0))$, for all $R > 0$ and for any $t \in [1, q_s^*)$.*

Lemma 2.5. *[1] If $V_\infty = \infty$, the embedding $\mathbb{X}_\varepsilon \subset L^m(\mathbb{R}^N)$ is compact for any $p \leq m < q_s^*$.*

Finally we recall the following splitting lemma which will be very useful in this work.

Lemma 2.6. *[1] Let $\{u_n\} \subset \mathbb{X}_\varepsilon$ be a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$. Set $v_n = u_n - u$. Then we have*

(i) $[v_n]^p_{s,p} + [v_n]^q_{s,q} = \left([u_n]^p_{s,p} + [u_n]^q_{s,q}\right) - \left([u]^p_{s,p} + [u]^q_{s,q}\right) + o_n(1)$,

(ii) $\int_{\mathbb{R}^N} V(\varepsilon\, x)\, (|v_n|^p + |v_n|^q)\, dx = \int_{\mathbb{R}^N} V(\varepsilon\, x)\, (|u_n|^p + |u_n|^q)\, dx - \int_{\mathbb{R}^N} V(\varepsilon\, x)\, (|u|^p + |u|^q)\, dx + o_n(1)$

(iii) $\int_{\mathbb{R}^N} (F(v_n) - F(u_n) + F(u))\, dx = o_n(1)$,

(*iv*) $\sup_{\|w\|_\varepsilon \le 1} \int_{\mathbb{R}^N} |\left(f(v_n) - f(u_n) + f(u)\right) w| \, dx = o_n(1)$.

3. Functional setting

In this section we consider the following problem

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u + V(\varepsilon\, x) \left(|u|^{p-2}u + |u|^{p-2}u\right) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases}$$

In order to study (P_ε), we look for critical points of the functional $\mathcal{I}_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(u) = \frac{1}{p}[u]^p_{s,p} + \frac{1}{q}[u]^q_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon\, x) \left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) dx - \int_{\mathbb{R}^N} F(u)\, dx.$$

It is easy to see that $\mathcal{I}_\varepsilon \in \mathcal{C}^1(\mathbb{X}_\varepsilon, \mathbb{R})$ and its differential is given by

$$\begin{aligned} \langle \mathcal{I}'_\varepsilon(u), \varphi\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} (\varphi(x) - \varphi(y))\, dxdy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x-y|^{N+sq}} (\varphi(x) - \varphi(y))\, dxdy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon\, x)(|u|^{p-2}u + |u|^{q-2}u)\, \varphi\, dx - \int_{\mathbb{R}^N} f(u)\varphi\, dx \end{aligned}$$

for any $u, \varphi \in \mathbb{X}_\varepsilon$. Now, let us introduce the Nehari manifold associated to $\mathcal{I}_\varepsilon$, that is

$$\mathcal{N}_\varepsilon = \left\{u \in \mathbb{X}_\varepsilon \setminus \{0\} : \langle \mathcal{I}'_\varepsilon(u), u\rangle = 0\right\},$$

and define

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u).$$

Let us note that $\mathcal{I}_\varepsilon$ possesses a mountain pass geometry [3].

Lemma 3.1. *The functional $\mathcal{I}_\varepsilon$ satisfies the following conditions:*
(*i*) *there exist $\alpha, \rho > 0$ such that $\mathcal{I}_\varepsilon(u) \ge \alpha$ with $\|u\|_\varepsilon = \rho$;*
(*ii*) *there exists $e \in \mathbb{X}_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $\mathcal{I}_\varepsilon(e) < 0$.*

Proof. (*i*) Using the growth assumptions on f, for any given $\xi > 0$ there exists $C_\xi > 0$ such that

$$|f(t)| \le \xi |t|^{p-1} + C_\xi |t|^{r-1} \quad \text{for any } t \in \mathbb{R}, \tag{3.1}$$

$$|F(t)| \le \frac{\xi}{p}|t|^p + \frac{C_\xi}{r}|t|^r \quad \text{for any } t \in \mathbb{R}. \tag{3.2}$$

Hence, taking $\xi \in (0, V_0)$, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(u) &\ge \frac{1}{p}\|u\|^p_{V,p} + \frac{1}{q}\|u\|^q_{V,q} - \frac{\xi}{p}|u|^p_p - \frac{C_\xi}{r}|u|^r_r \\ &\ge C_1\|u\|^p_{V,p} + \frac{1}{q}\|u\|^q_{V,q} - C'_\xi\|u\|^r_\varepsilon. \end{aligned}$$

Choosing $\|u\|_\varepsilon = \rho \in (0,1)$ and using $1 < p < q$, we have $\|u\|_{V,p} < 1$ and then $\|u\|^p_{V,p} \ge \|u\|^q_{V,p}$ which combined with $a^t + b^t \ge C_t(a+b)^t$ for any $a, b \ge 0$ and $t > 1$, yields

$$\mathcal{I}_\varepsilon(u) \ge C\|u\|^q_\varepsilon - C'_\xi\|u\|^r_\varepsilon.$$

Since $r > q$ we can find $\alpha > 0$ such that $\mathcal{I}_\varepsilon(u) \ge \alpha > 0$ for $\|u\|_\varepsilon = \rho$.
(*ii*) By (f_4) we can infer

$$F(t) \ge C_1|t|^\vartheta - C_2 \quad \text{for any } t \ge 0,$$

for some $C_1, C_2 > 0$. Taking $v \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $v \geq 0$, $v \not\equiv 0$, we have

$$\mathcal{I}_\varepsilon(tv) \leq \frac{t^p}{p}\|v\|_\varepsilon^p + \frac{t^q}{q}\|v\|_\varepsilon^q - t^\vartheta C_1 \int_{\operatorname{supp} v} v^\vartheta\, dx + C_2|\operatorname{supp} v| \to -\infty \text{ as } t \to \infty.$$

□

Now, in view of Lemma 3.1, we can use a version of mountain pass theorem without the Palais-Smale condition [44] to deduce the existence of a Palais-Smale sequence $\{u_n\}$ at level c'_ε, namely

$$\mathcal{I}_\varepsilon(u_n) \to c'_\varepsilon \quad \text{ and } \quad \mathcal{I}'_\varepsilon(u_n) \to 0,$$

where c'_ε is the mountain pass level of $\mathcal{I}_\varepsilon$ defined as

$$c'_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_\varepsilon(\gamma(t)),$$

and $\Gamma = \{\gamma \in \mathcal{C}^0([0,1], \mathbb{X}_\varepsilon) : \gamma(0) = 0, \mathcal{I}_\varepsilon(\gamma(1)) < 0\}$.

Lemma 3.2. *It holds*

$$c'_\varepsilon = c_\varepsilon = \inf_{u \in \mathbb{X}_\varepsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\varepsilon(tu).$$

Proof. For each $u \in \mathbb{X}_\varepsilon \setminus \{0\}$ and $t > 0$, let us introduce the function $h(t) = \mathcal{I}_\varepsilon(tu)$. Following the same arguments in the proof of Lemma 3.1 we deduce that $h(0) = 0$, $h(t) < 0$ for t large and $h(t) > 0$ for t small. Hence, $\max_{t\geq 0} h(t)$ is achieved at $t = t_u > 0$ satisfying $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

Note that, if $u \in \mathcal{N}_\varepsilon$ then $u^+ \not\equiv 0$. Indeed, from (f_1), we deduce that

$$\|u\|_{V,p}^p + \|u\|_{V,q}^q = \int_{\mathbb{R}^N} f(u)u\, dx = \int_{\mathbb{R}^N} f(u^+)u^+\, dx.$$

Now, if $u^+ \equiv 0$, then $\|u\|_{V,p}^p + \|u\|_{V,q}^q = 0$ that is $u \equiv 0$ and this is a contradiction in view of $u \in \mathcal{N}_\varepsilon$.

Next, we prove that t_u is the unique critical point of h. Assume by contradiction that there are t_1 and t_2 such that $t_1 u, t_2 u \in \mathcal{N}_\varepsilon$, that is

$$t_1^{p-q}[u]_{s,p}^p + [u]_{s,q}^q + t_1^{p-q}\int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^p\, dx + \int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^q\, dx = \int_{\{u>0\}} \frac{f(t_1 u)}{(t_1 u)^{q-1}} u^q\, dx$$

and

$$t_2^{p-q}[u]_{s,p}^p + [u]_{s,q}^q + t_2^{p-q}\int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^p\, dx + \int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^q\, dx = \int_{\{u>0\}} \frac{f(t_2 u)}{(t_2 u)^{q-1}} u^q\, dx.$$

Subtracting terms by terms the above equalities we get

$$(t_1^{p-q} - t_2^{p-q})[u]_{s,p}^p + (t_1^{p-q} - t_2^{p-q})\int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^p\, dx = \int_{\{u>0\}} \left[\frac{f(t_1 u)}{(t_1 u)^{q-1}} - \frac{f(t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx.$$

Now, if $t_1 < t_2$, exploiting (f_5) and recalling that $p < q$, we infer

$$0 < (t_1^{p-q} - t_2^{p-q})[u]_{s,p}^p + (t_1^{p-q} - t_2^{p-q})\int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^p\, dx = \int_{\{u>0\}} \left[\frac{f(t_1 u)}{(t_1 u)^{q-1}} - \frac{f(t_2 u)}{(t_2 u)^{q-1}}\right] u^q dx < 0,$$

which gives a contradiction. Then we can argue as in [44]. □

Next we prove the following useful result.

Lemma 3.3. *Let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\varepsilon$ at level c. Then*

(i) $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$.

(ii) $u_n^- \to 0$ in $\mathbb{X}_\varepsilon$ and we may assume that $u_n \geq 0$ for any $n \in \mathbb{N}$.

Proof. (i) From (f_4) we have

$$\begin{aligned}
C(1+\|u_n\|_\varepsilon) &\geq \mathcal{I}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{I}'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p} + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^q_{V,q} + \frac{1}{\vartheta}\int_{\mathbb{R}^N}(f(u_n)u_n - \vartheta F(u_n))\,dx \\
&\geq \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p} + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^q_{V,q} \\
&\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)(\|u_n\|^p_{V,p} + \|u_n\|^q_{V,q}).
\end{aligned}$$

Now, assume by contradiction that $\|u_n\|_\varepsilon \to \infty$. Then we distinguish the following cases:

(1) $\|u_n\|_{V,p} \to \infty$ and $\|u_n\|_{V,q} \to \infty$.
Since $p<q$, we have, for n sufficiently large, that $\|u_n\|^{q-p}_{V,q} \geq 1$, that is $\|u_n\|^q_{V,q} \geq \|u_n\|^p_{V,q}$, and thus

$$\begin{aligned}
C(1+\|u_n\|_\varepsilon) &\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\left(\|u_n\|^p_{V,p} + \|u_n\|^p_{V,q}\right) \\
&\geq C_1\left(\|u_n\|_{V,p} + \|u_n\|_{V,q}\right)^p = C_1\|u_n\|^p_\varepsilon,
\end{aligned}$$

which gives a contradiction.

(2) $\|u_n\|_{V,p} \to \infty$ and $\|u_n\|_{V,q}$ is bounded.
We can see that

$$C\left(1+\|u_n\|_{V,p} + \|u_n\|_{V,q}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p}$$

implies

$$C\left(\frac{1}{\|u_n\|^p_{V,p}} + \frac{1}{\|u_n\|^{p-1}_{V,p}} + \frac{\|u_n\|_{V,q}}{\|u_n\|^p_{V,p}}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right),$$

and letting $n\to\infty$, we get $0 \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right) > 0$, which yields a contradiction.

(3) $\|u_n\|_{V,p}$ is bounded and $\|u_n\|_{V,q} \to \infty$.
We can proceed similarly to the case (2).

Hence, $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$ and we may assume that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

(ii) Since $\langle \mathcal{I}'_\varepsilon(u_n), u_n\rangle = o_n(1)$, where $u_n^- = \min\{u_n, 0\}$, and $f(t)=0$ for $t\leq 0$, we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}}(u_n^-(x)-u_n^-(y))\,dxdy \\
&+ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+sq}}(u_n^-(x)-u_n^-(y))\,dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n)\,u_n^-\,dx = o_n(1).
\end{aligned}$$

Observing that for all $t\geq 2$ and $x,y\in\mathbb{R}^N$

$$|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y)) \geq |u_n^-(x)-u_n^-(y)|^t,$$

we get

$$\|u_n^-\|^p_{V,p} + \|u_n^-\|^q_{V,q} \leq o_n(1),$$

that is $u_n^- \to 0$ in $\mathbb{X}_\varepsilon$. Moreover, $\{u_n^+\}$ is bounded in $\mathbb{X}_\varepsilon$. Now, we prove that $\mathcal{I}_\varepsilon(u_n^+)\to c$ and $\mathcal{I}'_\varepsilon(u_n^+) = o_n(1)$. Clearly, $\|u_n\|_{V,t} = \|u_n^+\|_{V,t} + o_n(1)$ for $t \in \{p, q\}$. On the other hand, by (3.2), the mean value theorem, $u_n = u_n^+ + u_n^-$, we have

$$\left|\int_{\mathbb{R}^N} F(u_n)\,dx - \int_{\mathbb{R}^N} F(u_n^+)\,dx\right| \leq C\int_{\mathbb{R}^N}(|u_n|^{p-1}+|u_n|^{r-1})|u_n^-|\,dx$$
$$\leq C|u_n^-|_p + C|u_n^-|_r \leq C\|u_n^-\|_{V,p} + C\|u_n^-\|_{V,q} \leq C\|u_n^-\|_\varepsilon = o_n(1).$$

This shows that $\mathcal{I}_\varepsilon(u_n^+)\to c$. Next we claim that $\mathcal{I}'_\varepsilon(u_n^+) = o_n(1)$. Fix $\varphi \in \mathbb{X}_\varepsilon$ such that $\|\varphi\|_\varepsilon \leq 1$. Then we have

$$\begin{aligned}
&\left|\langle \mathcal{I}'_\varepsilon(u_n),\varphi\rangle - \langle \mathcal{I}'_\varepsilon(u_n^+),\varphi\rangle\right| \\
&= \Bigg|\iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{p-2}(u_n^+(x)-u_n^+(y))]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy \\
&+ \iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{q-2}(u_n^+(x)-u_n^+(y))]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+sq}}\,dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)[(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n) - (|u_n^+|^{p-2}u_n^+ + |u_n^+|^{q-2}u_n^+)]\,\varphi\,dx \\
&- \int_{\mathbb{R}^N}[f(u_n)-f(u_n^+)]\varphi\,dx\Bigg|.
\end{aligned}$$

Now, recalling that for all $\xi > 0$ there exists $C_\xi > 0$ such that

$$||a+b|^{t-2}(a+b) - |a|^{t-2}a| \leq \xi|a|^{t-1} + C_\xi|b|^{t-1} \quad \text{for all } a, b \in \mathbb{R}, \text{ and } t \geq 2,$$

and using it with

$$a = u_n^+(x) - u_n^+(y) \quad \text{and} \quad b = u_n^-(x) - u_n^-(y),$$

we see that for $t \in \{p, q\}$ it holds

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{t-2}(u_n^+(x)-u_n^+(y))]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+st}}\,dxdy\right| \\
&\leq \iint_{\mathbb{R}^{2N}}[\xi|u_n^+(x)-u_n^+(y)|^{t-1} + C_\xi|u_n^-(x)-u_n^-(y)|^{t-1}]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+st}}\,dxdy \\
&\leq \xi[u_n^+]_{s,t}^{t-1}[\varphi]_{s,t} + C_\xi[u_n^-]_{s,t}^{t-1}[\varphi]_{s,t} \\
&\leq \xi C + C'_\xi\|u_n^-\|_\varepsilon^{t-1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{t-2}(u_n^+(x)-u_n^+(y))]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+st}}\,dxdy\right| \\
&\leq \xi C
\end{aligned}$$

and by the arbitrariness of $\xi > 0$ we get

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}[|u_n(x)-u_n(y)|^{t-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{t-2}(u_n^+(x)-u_n^+(y))]\frac{(\varphi(x)-\varphi(y))}{|x-y|^{N+st}}\,dxdy = 0.$$

A similar argument shows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(\varepsilon x)[(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n) - (|u_n^+|^{p-2}u_n^+ + |u_n^+|^{q-2}u_n^+)]\,\varphi\,dx = 0.$$

Observing that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}[f(u_n)-f(u_n^+)]\varphi\,dx\right| &= \left|\int_{\mathbb{R}^N} f(u_n^-)\varphi\,dx\right| \\ &\leq C\int_{\mathbb{R}^N}(|u_n^-|^{p-1}+|u_n^-|^{r-1})|\varphi|\,dx \\ &\leq C(|u_n^-|_p^{p-1}|\varphi|_p+|u_n^-|_r^{r-1}|\varphi|_r) \\ &\leq C(\|u_n^-\|_\varepsilon^{p-1}+\|u_n^-\|_\varepsilon^{r-1})=o_n(1),\end{aligned}$$

we can deduce that $|\langle \mathcal{I}_\varepsilon'(u_n),\varphi\rangle-\langle \mathcal{I}_\varepsilon'(u_n^+),\varphi\rangle|=o_n(1)$. Since $\langle \mathcal{I}_\varepsilon'(u_n),\varphi\rangle=o_n(1)$, we conclude that $\mathcal{I}_\varepsilon'(u_n^+)=o_n(1)$. □

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$. We begin by proving some properties for the functional $\mathcal{I}_\varepsilon$.

Lemma 3.4. *Under assumptions (V) and (f_1)-(f_5), for any $\varepsilon>0$ we have:*
(i) $\mathcal{I}_\varepsilon'$ maps bounded sets of $\mathbb{X}_\varepsilon$ into bounded sets of $\mathbb{X}_\varepsilon$.
(ii) $\mathcal{I}_\varepsilon'$ is weakly sequentially continuous in $\mathbb{X}_\varepsilon$.
(iii) $\mathcal{I}_\varepsilon(t_nu_n)\to-\infty$ as $t_n\to\infty$, where $u_n\in K$ and $K\subset\mathbb{X}_\varepsilon\setminus\{0\}$ is a compact subset.

Proof. (i) Let $\{u_n\}$ be a bounded sequence in $\mathbb{X}_\varepsilon$ and $v\in\mathbb{X}_\varepsilon$. Then, from assumptions (f_2) and (f_3) we deduce that

$$\langle \mathcal{I}_\varepsilon'(u_n),v\rangle\leq C_1\|u_n\|_\varepsilon^{p-1}\|v\|_\varepsilon+C_2\|u_n\|_\varepsilon^{q-1}\|v\|_\varepsilon+C_3\|u_n\|_\varepsilon^{r-1}\|v\|_\varepsilon\leq C.$$

(ii) Let $u_n\rightharpoonup u$ in $\mathbb{X}_\varepsilon$. By Lemma 2.4, we have that $u_n\to u$ in $L^t_{loc}(\mathbb{R}^N)$ for all $t\in[1,q_s^*)$ and $u_n\to u$ a.e. in $\mathbb{R}^N$. Then, for all $v\in\mathcal{C}_c^\infty(\mathbb{R}^N)$, it follows from (3.1) and the dominated convergence theorem that

$$\langle \mathcal{I}_\varepsilon'(u_n),v\rangle\to\langle \mathcal{I}_\varepsilon'(u),v\rangle. \tag{3.3}$$

Since $\mathcal{C}_c^\infty(\mathbb{R}^N)$ is dense in $\mathbb{X}_\varepsilon$, we can take $\{v_j\}\subset\mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\|v_j-v\|_\varepsilon\to0$ as $j\to\infty$. Note that (3.1) and Lemma 2.4 yield

$$\begin{aligned}|\langle \mathcal{I}_\varepsilon'(u_n),v\rangle-\langle \mathcal{I}_\varepsilon'(u),v\rangle| &\leq |\langle \mathcal{I}_\varepsilon'(u_n)-\mathcal{I}_\varepsilon'(u),v_j\rangle|+|\langle \mathcal{I}_\varepsilon'(u_n)-\mathcal{I}_\varepsilon'(u),v-v_j\rangle| \\ &\leq |\langle \mathcal{I}_\varepsilon'(u_n)-\mathcal{I}_\varepsilon'(u),v_j\rangle|+C\int_{\mathbb{R}^N}(|u_n|^{p-1}+|u|^{p-1}+|u_n|^{r-1}+|u|^{r-1})|v-v_j|\,dx \\ &\leq |\langle \mathcal{I}_\varepsilon'(u_n)-\mathcal{I}_\varepsilon'(u),v_j\rangle|+C\|v_j-v\|_\varepsilon.\end{aligned}$$

For any $\zeta>0$, fix $j_0\in\mathbb{N}$ such that $\|v_{j_0}-v\|_\varepsilon<\frac{\zeta}{2C}$. By (3.3) there is $n_0\in\mathbb{N}$ such that

$$|\langle \mathcal{I}_\varepsilon'(u_n)-\mathcal{I}_\varepsilon'(u),v_{j_0}\rangle|<\frac{\zeta}{2}\quad\text{for all } n\geq n_0.$$

Thus

$$|\langle \mathcal{I}_\varepsilon'(u_n),v\rangle-\langle \mathcal{I}_\varepsilon'(u),v\rangle|<\zeta\quad\text{for all } n\geq n_0$$

and this shows that $\mathcal{I}_\varepsilon'$ is weakly sequentially continuous in $\mathbb{X}_\varepsilon$.
(iii) Without loss of generality, we may assume that $\|u\|_\varepsilon=1$ for each $u\in K$. For $u_n\in K$, after passing to a subsequence, we obtain that $u_n\to u\in\mathbb{S}_\varepsilon$. Then, using (f_4) and Fatou's lemma, we can see that

$$\begin{aligned}\mathcal{I}_\varepsilon(t_nu_n) &= \frac{t_n^p}{p}\|u_n\|_\varepsilon^p+\frac{t_n^q}{q}\|u_n\|_\varepsilon^q-\int_{\mathbb{R}^N}F(t_nu_n)\,dx \\ &\leq t_n^\vartheta\left(\frac{\|u_n\|_\varepsilon^p}{t_n^{\vartheta-p}}+\frac{\|u_n\|_\varepsilon^q}{t_n^{\vartheta-q}}-\int_{\mathbb{R}^N}\frac{F(t_nu_n)}{t_n^\vartheta}\,dx\right)\to-\infty\ \text{ as } n\to\infty.\end{aligned}$$

□

Lemma 3.5. *Under the assumptions of Lemma 3.4, for $\varepsilon > 0$ we have:*

(i) for all $u \in \mathbb{S}_\varepsilon$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon(u) = t_u u$ is the unique maximum of $\mathcal{I}_\varepsilon$ on $\mathbb{X}_\varepsilon$, where $\mathbb{S}_\varepsilon = \{u \in \mathbb{X}_\varepsilon : \|u\|_\varepsilon = 1\}$.

(ii) The set $\mathcal{N}_\varepsilon$ is bounded away from 0. Furthermore $\mathcal{N}_\varepsilon$ is closed in $\mathbb{X}_\varepsilon$.

(iii) There exists $\alpha > 0$ such that $t_u \geq \alpha$ for each $u \in \mathbb{S}_\varepsilon$ and, for each compact subset $W \subset \mathbb{S}_\varepsilon$, there exists $C_W > 0$ such that $t_u \leq C_W$ for all $u \in W$.

(iv) For each $u \in \mathcal{N}_\varepsilon$, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in \mathcal{N}_\varepsilon$. In particular, $\mathcal{N}_\varepsilon$ is a regular manifold diffeomorphic to the sphere in $\mathbb{X}_\varepsilon$.

(v) $c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon \geq \rho > 0$ and $\mathcal{I}_\varepsilon$ is bounded below on $\mathcal{N}_\varepsilon$, where ρ is independent of ε.

Proof. (i) The proof follows the same lines of the proof of Lemma 3.2.
(ii) Using (3.1) and Lemma 2.4, for any $u \in \mathcal{N}_\varepsilon$ we have

$$\|u\|_{V,p}^p + \|u\|_{V,q}^q = \int_{\mathbb{R}^N} f(u)u\, dx \leq \frac{\xi}{V_0}\|u\|_{V,p}^p + C_\xi \|u\|_\varepsilon^r.$$

Taking $\xi > 0$ sufficiently small we deduce that

$$C_1\|u\|_{V,p}^p + \|u\|_{V,q}^q \leq C\|u\|_\varepsilon^r.$$

Now, if $\|u\|_\varepsilon \geq 1$, we have done. If $\|u\|_\varepsilon < 1$, then $\|u\|_{V,p}^p \geq \|u\|_{V,p}^q$ so we get

$$C\|u\|_\varepsilon^r \geq C_1\|u\|_{V,p}^p + \|u\|_{V,q}^q \geq C_1\|u\|_{V,p}^q + \|u\|_{V,q}^q \geq C_2\|u\|_\varepsilon^q,$$

which implies that $\|u\|_\varepsilon \geq \kappa$ for some $\kappa > 0$.
Next we prove that $\mathcal{N}_\varepsilon$ is closed in $\mathbb{X}_\varepsilon$. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. From Lemma 3.4 we infer that $\mathcal{I}'_\varepsilon(u_n)$ is bounded, so

$$\langle \mathcal{I}'_\varepsilon(u_n), u_n\rangle - \langle \mathcal{I}'_\varepsilon(u), u\rangle = \langle \mathcal{I}'_\varepsilon(u_n) - \mathcal{I}'_\varepsilon(u), u\rangle + \langle \mathcal{I}'_\varepsilon(u_n), u_n - u\rangle \to 0,$$

that is $\langle \mathcal{I}'_\varepsilon(u), u\rangle = 0$, which combined with $\|u\|_\varepsilon \geq \kappa$ implies that

$$\|u\|_\varepsilon = \lim_{n\to\infty} \|u_n\|_\varepsilon \geq \kappa > 0,$$

hence $u \in \mathcal{N}_\varepsilon$.
(iii) For each $u \in \mathbb{S}_\varepsilon$ there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Then, using $\|u\|_\varepsilon \geq \kappa$, we also have $t_u = \|t_u u\|_\varepsilon \geq \kappa$. It remains we prove that $t_u \leq C_W$ for all $u \in W \subset \mathbb{S}_\varepsilon$. We argue by contradiction and we suppose that there exists a sequence $\{u_n\} \subset W \subset \mathbb{S}_\varepsilon$ such that $t_{u_n} \to \infty$. Since W is compact, we can find $u \in W$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

Now, using (f_4) we have

$$\begin{aligned}\mathcal{I}_\varepsilon(u) &= \mathcal{I}_\varepsilon(u) - \frac{1}{q}\langle \mathcal{I}'_\varepsilon(u), u\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)[u]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u|^p dx - \int_{\mathbb{R}^N}\left(F(u) - \frac{1}{q}f(u)u\right) dx \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{V,p}^p - \int_{\mathbb{R}^N}\left(F(u) - \frac{1}{q}f(u)u\right) dx \geq 0,\end{aligned}$$

and this is in contrast with Lemma 3.4-(iii) for which $\mathcal{I}_\varepsilon(t_{u_n}u_n) \to -\infty$ as $n \to \infty$.
(iv) Let us define the maps $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon \setminus \{0\} \to \mathcal{N}_\varepsilon$ and $m_\varepsilon : \mathbb{S}_\varepsilon \to \mathcal{N}_\varepsilon$ by setting

$$\hat{m}_\varepsilon(u) = t_u u \quad \text{and} \quad m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon}. \tag{3.4}$$

In view of (i)-(iii) and Proposition 3.1 in [43] we can deduce that m_ε is a homeomorphism between $\mathbb{S}_\varepsilon$ and $\mathcal{N}_\varepsilon$ and the inverse of m_ε is given by $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$. Therefore $\mathcal{N}_\varepsilon$ is a regular manifold diffeomorphic to $\mathbb{S}_\varepsilon$.

(v) For $\varepsilon>0$, $t>0$ and $u\in\mathbb{X}_\varepsilon\setminus\{0\}$, we can see that (3.2) yields

$$\begin{aligned}\mathcal{I}_\varepsilon(tu)&\geq\frac{t^p}{p}[u]^p_{s,p}+\frac{t^q}{q}[u]^q_{s,q}+\int_{\mathbb{R}^N}V(\varepsilon x)\left(\frac{t^p}{p}|u|^p+\frac{t^q}{q}|u|^q\right)dx-\frac{\xi t^p}{V_0}\int_{\mathbb{R}^N}V_0|u|^p\,dx-C_\xi t^r\int_{\mathbb{R}^N}|u|^r\,dx\\&\geq\frac{t^p}{p}\left(1-\frac{\xi}{V_0}\right)\|u\|^p_{V,p}+\frac{t^q}{q}\|u\|^q_{V,q}-C_\xi t^r\|u\|^r_\varepsilon\end{aligned}$$

so we can find $\rho>0$ such that $\mathcal{I}_\varepsilon(tu)\geq\rho>0$ for $t>0$ small enough. On the other hand, by using (i)-(iii), we know (see [43]) that

$$c_\varepsilon=\inf_{u\in\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon(u)=\inf_{u\in\mathbb{X}_\varepsilon\setminus\{0\}}\max_{t>0}\mathcal{I}_\varepsilon(tu)=\inf_{u\in\mathbb{S}_\varepsilon}\max_{t>0}\mathcal{I}_\varepsilon(tu)\tag{3.5}$$

which implies $c_\varepsilon\geq\rho$ and $\mathcal{I}_\varepsilon|_{\mathcal{N}_\varepsilon}\geq\rho$. □

Now we introduce the following functionals $\hat{\Psi}_\varepsilon:\mathbb{X}_\varepsilon\setminus\{0\}\to\mathbb{R}$ and $\Psi_\varepsilon:\mathbb{S}_\varepsilon\to\mathbb{R}$ defined by

$$\hat{\Psi}_\varepsilon=\mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))\quad\text{ and }\quad\Psi_\varepsilon=\hat{\Psi}_\varepsilon|_{\mathbb{S}_\varepsilon},$$

where $\hat{m}_\varepsilon(u)=t_u u$ is given in (3.4). As in [43], we have the following result:

Lemma 3.6. *Under the assumptions of Lemma 3.4, we have that for $\varepsilon>0$:*
(i) *$\Psi_\varepsilon\in\mathcal{C}^1(\mathbb{S}_\varepsilon,\mathbb{R})$, and*

$$\langle\Psi'_\varepsilon(w),v\rangle=\|m_\varepsilon(w)\|_\varepsilon\langle\mathcal{I}'_\varepsilon(m_\varepsilon(w)),v\rangle\quad\text{ for } v\in T_w(\mathbb{S}_\varepsilon).$$

(ii) *$\{w_n\}$ is a Palais-Smale sequence for Ψ_ε if and only if $\{m_\varepsilon(w_n)\}$ is a Palais-Smale sequence for $\mathcal{I}_\varepsilon$. If $\{u_n\}\subset\mathcal{N}_\varepsilon$ is a bounded Palais-Smale sequence for $\mathcal{I}_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}$ is a Palais-Smale sequence for Ψ_ε.*
(iii) *$u\in\mathbb{S}_\varepsilon$ is a critical point of Ψ_ε if and only if $m_\varepsilon(u)$ is a critical point of $\mathcal{I}_\varepsilon$. Moreover the corresponding critical values coincide and*

$$\inf_{\mathbb{S}_\varepsilon}\Psi_\varepsilon=\inf_{\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon=c_\varepsilon.$$

4. The autonomous problem

In this section we deal with the autonomous problem associated with (P_ε), that is

$$\begin{cases}(-\Delta)^s_p u+(-\Delta)^s_q u+\mu(|u|^{p-2}u+|u|^{q-2}u)=f(u) & \text{in } \mathbb{R}^N\\ u\in W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \mu>0.\end{cases}\tag{AP_μ}$$

The functional associated with (AP_μ) is given by

$$\mathcal{J}_\mu(u)=\frac{1}{p}[u]^p_{s,p}+\frac{1}{q}[u]^q_{s,q}+\mu\left[\frac{1}{p}|u|^p_p+\frac{1}{q}|u|^q_q\right]-\int_{\mathbb{R}^N}F(u)\,dx$$

which is well-defined on the space $\mathbb{Y}_\mu=W^{s,p}(\mathbb{R}^N)\cap W^{s,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\mu=\|u\|_{\mu,p}+\|u\|_{\mu,q},$$

where

$$\|u\|^t_{\mu,t}=[u]^t_{s,t}+\mu|u|^t_t\quad\text{ for all } t>1.$$

It is easy to check that $\mathcal{J}_\mu \in \mathcal{C}^1(\mathbb{Y}_\mu, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathcal{J}_\mu'(u), \varphi\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sq}}\,dxdy \\ &\quad + \mu\left[\int_{\mathbb{R}^N} |u|^{p-2}u\,\varphi\,dx + \int_{\mathbb{R}^N} |u|^{q-2}u\,\varphi\,dx\right] - \int_{\mathbb{R}^N} f(u)\varphi\,dx\end{aligned}$$

for any $u, \varphi \in \mathbb{Y}_\mu$. Let us define the Nehari manifold associated with $\mathcal{J}_\mu$

$$\mathcal{M}_\mu = \{u \in \mathbb{Y}_\mu \setminus \{0\} : \langle \mathcal{J}_\mu'(u), u\rangle = 0\}.$$

We note that (f_4) yields

$$\begin{aligned}\mathcal{J}_\mu(u) &= \mathcal{J}_\mu(u) - \frac{1}{q}\langle \mathcal{J}_\mu'(u), u\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mu,p}^p - \int_{\mathbb{R}^N}\left(F(u) - \frac{1}{q}f(u)u\right)dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{\mu,p}^p \quad \text{for all } u \in \mathcal{M}_\mu. \end{aligned} \tag{4.1}$$

Arguing as in the previous section and using (4.1), it is easy to prove the following lemma.

Lemma 4.1. *Under the assumptions of Lemma 3.4, for $\mu > 0$ we have:*

- (i) *for all $u \in \mathbb{S}_\mu$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{M}_\mu$. Moreover, $m_\mu(u) = t_u u$ is the unique maximum of $\mathcal{J}_\mu$ on $\mathbb{Y}_\mu$, where $\mathbb{S}_\mu = \{u \in \mathbb{Y}_\mu : \|u\|_\mu = 1\}$.*
- (ii) *The set $\mathcal{M}_\mu$ is bounded away from 0. Furthermore $\mathcal{M}_\mu$ is closed in $\mathbb{Y}_\mu$.*
- (iii) *There exists $\alpha > 0$ such that $t_u \geq \alpha$ for each $u \in \mathbb{S}_\mu$ and, for each compact subset $W \subset \mathbb{S}_\mu$, there exists $C_W > 0$ such that $t_u \leq C_W$ for all $u \in W$.*
- (iv) *$\mathcal{M}_\mu$ is a regular manifold diffeomorphic to the sphere in $\mathbb{Y}_\mu$.*
- (v) *$d_\mu = \inf_{\mathcal{M}_\mu} \mathcal{J}_\mu > 0$ and $\mathcal{J}_\mu$ is bounded below on $\mathcal{M}_\mu$ by some positive constant.*
- (vi) *$\mathcal{J}_\mu$ is coercive on $\mathcal{M}_\mu$.*

Now we define the following functionals $\hat{\Psi}_\mu : \mathbb{Y}_\mu \setminus \{0\} \to \mathbb{R}$ and $\Psi_\mu : \mathbb{S}_\mu \to \mathbb{R}$ by setting

$$\hat{\Psi}_\mu = \mathcal{J}_\mu(\hat{m}_\mu(u)) \quad \text{and} \quad \Psi_\mu = \hat{\Psi}_\mu|_{\mathbb{S}_\mu}.$$

Then we have the following result:

Lemma 4.2. *Under the assumptions of Lemma 3.4, we have that for $\mu > 0$:*

- (i) *$\Psi_\mu \in \mathcal{C}^1(\mathbb{S}_\mu, \mathbb{R})$, and*
$$\langle \Psi_\mu'(w), v\rangle = \|m_\mu(w)\|_\mu \langle \mathcal{J}_\mu'(m_\mu(w)), v\rangle \quad \text{for } v \in T_w(\mathbb{S}_\mu).$$
- (ii) *$\{w_n\}$ is a Palais-Smale sequence for Ψ_μ if and only if $\{m_\mu(w_n)\}$ is a Palais-Smale sequence for $\mathcal{J}_\mu$. If $\{u_n\} \subset \mathcal{M}_\mu$ is a bounded Palais-Smale sequence for $\mathcal{J}_\mu$, then $\{m_\mu^{-1}(u_n)\}$ is a Palais-Smale sequence for Ψ_μ.*
- (iii) *$u \in \mathbb{S}_\mu$ is a critical point of Ψ_μ if and only if $m_\mu(u)$ is a critical point of $\mathcal{J}_\mu$. Moreover the corresponding critical values coincide and*
$$\inf_{\mathbb{S}_\mu} \Psi_\mu = \inf_{\mathcal{M}_\mu} \mathcal{J}_\mu = d_\mu.$$

Remark 4.1. *As in (3.5), from (i)-(iii) of Lemma 4.1, we can see that d_μ admits the following minimax characterization*

$$d_\mu = \inf_{u \in \mathcal{M}_\mu} \mathcal{J}_\mu(u) = \inf_{u \in \mathbb{Y}_\mu \setminus \{0\}} \max_{t>0} \mathcal{J}_\mu(tu) = \inf_{u \in \mathbb{S}_\mu} \max_{t>0} \mathcal{J}_\mu(tu). \tag{4.2}$$

Lemma 4.3. *Let $\{u_n\} \subset \mathcal{M}_\mu$ be a minimizing sequence for $\mathcal{J}_\mu$. Then, $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$ and there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q dx \geq \beta > 0.$$

Proof. Arguing as in the proof of Lemma 3.3, we can see that $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$. Now, in order to prove the latter conclusion of this lemma, we argue by contradiction. Assume that for any $R > 0$ it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathcal{B}_R(y)} |u_n|^q dx = 0.$$

Since $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$, it follows from Lemma 2.1 that

$$u_n \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{ for any } t \in (q, q_s^*). \tag{4.3}$$

Fix $\xi \in (0, \mu)$. Then, taking into account that $\{u_n\} \subset \mathcal{M}_\mu$ and (3.1), we have

$$\begin{aligned} 0 &= \langle \mathcal{J}'_\mu(u_n), u_n \rangle \\ &\geq [u_n]^p_{s,p} + [u_n]^q_{s,q} + \mu \left[|u_n|^p_p + |u_n|^q_q \right] - \xi |u_n|^p_p - C_\xi |u_n|^r_r \\ &\geq C_1 \|u_n\|^p_{s,p} + C_2 \|u_n\|^q_{s,q} - C_3 |u_n|^r_r, \end{aligned}$$

and in view of (4.3) we have that $\|u_n\|_\mu \to 0$. □

Now, we prove the following useful compactness result for the autonomous problem.

Lemma 4.4. *The problem (AP_μ) has a positive ground state solution.*

Proof. From (v) of Lemma 4.1, we know that $d_\mu > 0$ for each $\mu > 0$. Moreover, if $u \in \mathcal{M}_\mu$ verifies $\mathcal{J}_\mu(u) = d_\mu$, then $m_\mu^{-1}(u)$ is a minimizer of Ψ_μ and it is a critical point of Ψ_μ. In view of Lemma 4.2, we can see that u is a critical point of $\mathcal{J}_\mu$. Now we show that there exists a minimizer of $\mathcal{J}_\mu|_{\mathcal{M}_\mu}$. By applying Ekeland's variational principle [44] there exists a sequence $\{\nu_n\} \subset \mathbb{S}_\mu$ such that $\Psi_\mu(\nu_n) \to d_\mu$ and $\Psi'_\mu(\nu_n) \to 0$ as $n \to \infty$. Let $u_n = m_\mu(\nu_n) \in \mathcal{M}_\mu$. Then, thanks to Lemma 4.2, $\mathcal{J}_\mu(u_n) \to d_\mu$ and $\mathcal{J}'_\mu(u_n) \to 0$ as $n \to \infty$. Therefore, arguing as in the proof of Lemma 3.3, $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$ which is a reflexive space, so we may assume that $u_n \rightharpoonup u$ in $\mathbb{Y}_\mu$ for some $u \in \mathbb{Y}_\mu$.

In what follows, we show that $\mathcal{J}'_\mu(u) = 0$. Consider the sequence

$$h_n(x,y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{\frac{N+sp}{p'}}},$$

and let

$$h(x,y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{\frac{N+sp}{p'}}},$$

where $p' = \frac{p}{p-1}$. It is easy to check that $\{h_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^{2N})$ with $h_n \to h$ a.e. in $\mathbb{R}^{2N}$. Since $L^{p'}(\mathbb{R}^{2N})$ is a reflexive space, there exists a subsequence, still denoted by $\{h_n\}$, such that $h_n \rightharpoonup h$ in $L^{p'}(\mathbb{R}^{2N})$, that is

$$\iint_{\mathbb{R}^{2N}} h_n(x,y) g(x,y) dxdy \to \iint_{\mathbb{R}^{2N}} h(x,y) g(x,y) dxdy \quad \forall g \in L^p(\mathbb{R}^{2N}).$$

Then, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, we know that

$$g(x,y) = \frac{(\phi(x) - \phi(y))}{|x-y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^{2N}),$$

and we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}dxdy \\ \to \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}dxdy.$$

In a similar way we can prove that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sq}}dxdy \\ \to \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sq}}dxdy.$$

Since it is clear that

$$\int_{\mathbb{R}^N} |u_n|^{t-2}u_n\phi\, dx \to \int_{\mathbb{R}^N} |u|^{t-2}u\phi\, dx, \qquad \text{for } t\in\{p,q\},$$
$$\int_{\mathbb{R}^N} f(u_n)\phi\, dx \to \int_{\mathbb{R}^N} f(u)\phi\, dx,$$

and using the fact that $\langle \mathcal{J}'_\mu(u_n),\phi\rangle = o_n(1)$, we can deduce that $\langle \mathcal{J}'_\mu(u),\phi\rangle = 0$ for all $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^N)$. By the density of $\mathcal{C}^\infty_c(\mathbb{R}^N)$ in $\mathbb{Y}_\mu$, we obtain that u is a critical point of $\mathcal{J}_\mu$.

Now, if $u\neq 0$, then u is a nontrivial solution to (AP_μ). Assume that $u=0$. Then $\|u_n\|_\mu \not\to 0$ in $\mathbb{Y}_\mu$. Hence, arguing as in the proof of Lemma 4.3 we can find a sequence $\{y_n\}\subset\mathbb{R}^N$ and constants $R,\beta>0$ such that

$$\liminf_{n\to\infty}\int_{\mathcal{B}_R(y_n)} |u_n|^q dx \geq \beta > 0. \tag{4.4}$$

Now, let us define

$$\tilde{v}_n(x) = u_n(x+y_n).$$

From the invariance by translations of $\mathbb{R}^N$, it is clear that $\|\tilde{v}_n\|_{\mu,t} = \|u_n\|_{\mu,t}$, with $t\in\{p,q\}$, so $\{\tilde{v}_n\}$ is bounded in $\mathbb{Y}_\mu$ and there exists $\tilde{v}$ such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{Y}_\mu$, $\tilde{v}_n\to\tilde{v}$ in $L^m_{loc}(\mathbb{R}^N)$ for any $m\in[1,q^*_s)$ and $\tilde{v}\neq 0$ in view of (4.4). Moreover, $\mathcal{J}_\mu(\tilde{v}_n) = \mathcal{J}_\mu(u_n)$ and $\mathcal{J}'_\mu(\tilde{v}_n) = o_n(1)$, and arguing as before it is easy to check that $\mathcal{J}'_\mu(\tilde{v}) = 0$.

Now let u be the solution obtained from the previous study, and we prove that u is a ground state solution. It is clear that $d_\mu \leq \mathcal{J}_\mu(u)$. On the other hand, from Fatou's lemma we can see that

$$\mathcal{J}_\mu(u) = \mathcal{J}_\mu(u) - \frac{1}{q}\langle \mathcal{J}'_\mu(u),u\rangle \leq \liminf_{n\to\infty}\left[\mathcal{J}_\mu(u_n) - \frac{1}{q}\langle \mathcal{J}'_\mu(u_n),u_n\rangle\right] = d_\mu,$$

which implies that $d_\mu = \mathcal{J}_\mu(u)$.

Finally we prove that the ground state obtained before is positive. Indeed, taking $u^- = \min\{u,0\}$ as test function in (AP_μ), and exploiting (f_1) and the following inequality

$$|x-y|^{t-2}(x-y)(x^- - y^-) \geq |x^- - y^-|^t \quad \forall t\geq 1,$$

we can see that

$$
\begin{aligned}
&\|u^-\|_{\mu,p}^p + \|u^-\|_{\mu,q}^q \\
&\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(u^-(x)-u^-(y))\,dxdy + \int_{\mathbb{R}^N} \mu |u|^{p-2}uu^-\,dx \\
&+ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{N+sq}}(u^-(x)-u^-(y))\,dxdy + \int_{\mathbb{R}^N} \mu |u|^{q-2}uu^-\,dx \\
&= \int_{\mathbb{R}^N} f(u)u^-\,dx = 0
\end{aligned}
$$

which implies that $u^- = 0$, that is $u \geq 0$ in $\mathbb{R}^N$. Arguing as in the proof of Lemma 6.1 in [1], we can prove that $u \in L^\infty(\mathbb{R}^N)$, and thus, by Corollary 2.1 in [12], we deduce that u is continuous in $\mathbb{R}^N$. In view of Lemmas 2.1 and 2.2 in [12], we can argue as in the proof of Theorem 1.1-(ii) in [29] to see that $u > 0$ in $\mathbb{R}^N$. This completes the proof of lemma.

□

5. A first existence result for (P_ε)

In this section we focus on the existence of a solution to (P_ε) provided that ε is sufficiently small. Let us start with the following useful lemma.

Lemma 5.1. *Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $\mathcal{I}_\varepsilon(u_n) \to c$ and $u_n \rightharpoonup 0$ in $\mathbb{X}_\varepsilon$. Then, one of the following alternatives occurs:*
(a) $u_n \to 0$ in $\mathbb{X}_\varepsilon$;
(b) there are a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q dx \geq \beta > 0.$$

Proof. Assume that (b) does not hold true. Then, for any $R > 0$, it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathcal{B}_R(y)} |u_n|^q dx = 0.$$

Since $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$, it follows from Lemma 2.1 that

$$u_n \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{ for any } t \in (q, q_s^*). \tag{5.1}$$

Now, we can argue as in the proof of Lemma 4.3 to deduce that $\|u_n\|_\varepsilon \to 0$ as $n \to \infty$. □

In order to get a compactness result for $\mathcal{I}_\varepsilon$, we need to prove the following auxiliary lemma.

Lemma 5.2. *Assume that $V_\infty < \infty$ and let $\{v_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $\mathcal{I}_\varepsilon(v_n) \to d$ with $v_n \rightharpoonup 0$ in $\mathbb{X}_\varepsilon$. If $v_n \not\to 0$ in $\mathbb{X}_\varepsilon$, then $d \geq d_{V_\infty}$, where d_{V_∞} is the infimum of $\mathcal{J}_{V_\infty}$ over $\mathcal{M}_{V_\infty}$.*

Proof. Let $\{t_n\} \subset (0, \infty)$ be such that $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$. Our aim is to show that $\limsup_{n\to\infty} t_n \leq 1$. Assume by contradiction that there exist $\delta > 0$ and a subsequence, denoted again by $\{t_n\}$, such that

$$t_n \geq 1 + \delta \quad \text{ for any } n \in \mathbb{N}. \tag{5.2}$$

Since $\{v_n\} \subset \mathbb{X}_\varepsilon$ is a bounded Palais-Smale sequence for $\mathcal{I}_\varepsilon$, we have that $\langle \mathcal{I}'_\varepsilon(v_n), v_n\rangle = o_n(1)$, or equivalently

$$[v_n]_{s,p}^p + [v_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon\, x)|v_n|^p dx + \int_{\mathbb{R}^N} V(\varepsilon\, x)|v_n|^q dx - \int_{\mathbb{R}^N} f(v_n)v_n\,dx = o_n(1). \tag{5.3}$$

Since $t_n v_n \in \mathcal{M}_{V_\infty}$, we also have that

$$t_n^{p-q}[v_n]_{s,p}^p + [v_n]_{s,q}^q + t_n^{p-q} V_\infty \int_{\mathbb{R}^N} |v_n|^p dx + V_\infty \int_{\mathbb{R}^N} |v_n|^q dx - \int_{\mathbb{R}^N} \frac{f(t_n v_n)}{(t_n v_n)^{q-1}} v_n^q \, dx = 0. \tag{5.4}$$

Putting together (5.3) and (5.4) we get

$$\int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{(v_n)^{q-1}} \right) v_n^q \, dx \leq \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x)) |v_n|^q dx. \tag{5.5}$$

Now, using assumption (V) we can see that, given $\zeta > 0$ there exists $R = R(\zeta) > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \zeta \quad \text{ for any } |x| \geq R. \tag{5.6}$$

From this, taking into account that $v_n \to 0$ in $L^q(\mathcal{B}_R(0))$ and the boundedness of $\{v_n\}$ in $\mathbb{X}_\varepsilon$, we can infer

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x)) |v_n|^q dx &= \int_{\mathcal{B}_R(0)} (V_\infty - V(\varepsilon x)) |v_n|^q dx + \int_{\mathbb{R}^N \setminus \mathcal{B}_R(0)} (V_\infty - V(\varepsilon x)) |v_n|^q dx \\ &\leq V_\infty \int_{\mathcal{B}_R(0)} |v_n|^q dx + \zeta \int_{\mathbb{R}^N \setminus \mathcal{B}_R(0)} |v_n|^q dx \\ &\leq o_n(1) + \zeta C. \end{aligned} \tag{5.7}$$

Combining (5.5) and (5.7) we have

$$\int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{(v_n)^{q-1}} \right) v_n^q \, dx \leq o_n(1) + \zeta C. \tag{5.8}$$

Since $v_n \not\to 0$ in $\mathbb{X}_\varepsilon$, we can apply Lemma 5.1 to deduce the existence of a sequence $\{y_n\} \subset \mathbb{R}^N$ and two positive numbers $\bar{R}, \beta$ such that

$$\int_{\mathcal{B}_{\bar{R}}(y_n)} |v_n|^q \, dx \geq \beta > 0. \tag{5.9}$$

Let us consider $\tilde{v}_n = v_n(x + y_n)$. Then we may assume that, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{X}_\varepsilon$. By (5.9) there exists $\Omega \subset \mathbb{R}^N$ with positive measure and such that $\tilde{v} > 0$ in Ω. From (5.2), (f_4) and (5.8), we can infer that

$$0 < \int_\Omega \left(\frac{f((1+\delta)\tilde{v}_n)}{((1+\delta)\tilde{v}_n)^{q-1}} - \frac{f(\tilde{v}_n)}{(\tilde{v}_n)^{q-1}} \right) \tilde{v}_n^q \, dx \leq o_n(1) + \zeta C.$$

Taking the limit as $n \to \infty$ and applying Fatou's lemma, we obtain

$$0 < \int_\Omega \left(\frac{f((1+\delta)\tilde{v})}{((1+\delta)\tilde{v})^{q-1}} - \frac{f(\tilde{v})}{(\tilde{v})^{q-1}} \right) \tilde{v}^q \, dx \leq \zeta C \quad \text{ for any } \zeta > 0,$$

and this is a contradiction.

Now, we consider the following cases:

CASE 1: Assume that $\limsup_{n\to\infty} t_n = 1$. Thus there exists $\{t_n\}$ such that $t_n \to 1$. Taking into account that $\mathcal{I}_\varepsilon(v_n) \to c$, we have

$$\begin{aligned} c + o_n(1) &= \mathcal{I}_\varepsilon(v_n) \\ &= \mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + \mathcal{J}_{V_\infty}(t_n v_n) \\ &\geq \mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + d_{V_\infty}. \end{aligned} \tag{5.10}$$

Now, let us point out that

$$\begin{aligned} &\mathcal{I}_\varepsilon(v_n)-\mathcal{J}_{V_\infty}(t_nv_n)\\ &=\frac{1-t_n^p}{p}[v_n]_{s,p}^p+\frac{1-t_n^q}{q}[v_n]_{s,q}^q+\frac{1}{p}\int_{\mathbb{R}^N}\left(V(\varepsilon x)-t_n^pV_\infty\right)|v_n|^p dx\\ &\quad+\frac{1}{q}\int_{\mathbb{R}^N}\left(V(\varepsilon x)-t_n^qV_\infty\right)|v_n|^q dx+\int_{\mathbb{R}^N}\left(F(t_nv_n)-F(v_n)\right)\,dx. \end{aligned} \tag{5.11}$$

Using condition (V), $v_n\to 0$ in $L^p(\mathcal{B}_R(0))$, $t_n\to 1$, (5.6), and the fact that

$$V(\varepsilon x)-t_n^pV_\infty=(V(\varepsilon x)-V_\infty)+(1-t_n^p)V_\infty\geq -\zeta+(1-t_n^p)V_\infty \text{ for any } |x|\geq R,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\left(V(\varepsilon x)-t_n^pV_\infty\right)|v_n|^p dx\\ &=\int_{\mathcal{B}_R(0)}\left(V(\varepsilon x)-t_n^pV_\infty\right)|v_n|^p dx+\int_{\mathbb{R}^N\setminus\mathcal{B}_R(0)}\left(V(\varepsilon x)-t_n^pV_\infty\right)|v_n|^p dx\\ &\geq (V_0-t_n^pV_\infty)\int_{\mathcal{B}_R(0)}|v_n|^p dx-\zeta\int_{\mathbb{R}^N\setminus\mathcal{B}_R(0)}|v_n|^p dx+V_\infty(1-t_n^p)\int_{\mathbb{R}^N\setminus\mathcal{B}_R(0)}|v_n|^p dx\\ &\geq o_n(1)-\zeta C. \end{aligned} \tag{5.12}$$

In a similar fashion we can prove that

$$\int_{\mathbb{R}^N}\left(V(\varepsilon x)-t_n^qV_\infty\right)|v_n|^q dx\geq o_n(1)-\zeta C. \tag{5.13}$$

Since $\{v_n\}$ is bounded in $\mathbb{X}_\varepsilon$, we can conclude that

$$\frac{(1-t_n^p)}{p}[v_n]_{s,p}^p=o_n(1) \quad \text{and} \quad \frac{(1-t_n^q)}{q}[v_n]_{s,q}^q=o_n(1). \tag{5.14}$$

Thus, putting together (5.11), (5.12), (5.13) and (5.14) we obtain

$$\mathcal{I}_\varepsilon(v_n)-\mathcal{J}_{V_\infty}(t_nv_n)\geq\int_{\mathbb{R}^N}\left(F(t_nv_n)-F(v_n)\right)\,dx+o_n(1)-\zeta C. \tag{5.15}$$

At this point, we aim to show that

$$\int_{\mathbb{R}^N}\left(F(t_nv_n)-F(v_n)\right)\,dx=o_n(1). \tag{5.16}$$

Applying the mean value theorem and (3.1), we deduce that

$$\int_{\mathbb{R}^N}|F(t_nv_n)-F(v_n)|\,dx\leq C|t_n-1|\int_{\mathbb{R}^N}|v_n|^p dx+C|t_n-1|\int_{\mathbb{R}^N}|v_n|^r dx.$$

Exploiting the boundedness of $\{v_n\}$ we get the thesis. Gathering (5.10), (5.15) and (5.16), we can infer that

$$c+o_n(1)\geq o_n(1)-\zeta C+d_{V_\infty},$$

and taking the limit as $\zeta\to 0$ we get $c\geq d_{V_\infty}$.

CASE 2: Assume that $\limsup_{n\to\infty}t_n=t_0<1$. Then, there is a subsequence, still denoted by $\{t_n\}$, such that $t_n\to t_0(<1)$ and $t_n<1$ for any $n\in\mathbb{N}$. Let us observe that

$$\begin{aligned} c+o_n(1)&=\mathcal{I}_\varepsilon(v_n)-\frac{1}{q}\langle\mathcal{I}_\varepsilon'(v_n),v_n\rangle\\ &=\left(\frac{1}{p}-\frac{1}{q}\right)\|v_n\|_{V,p}^p+\int_{\mathbb{R}^N}\left(\frac{1}{q}f(v_n)v_n-F(v_n)\right)\,dx. \end{aligned} \tag{5.17}$$

Recalling that $t_n v_n \in \mathcal{M}_{V_\infty}$, and using (f_5) and (5.17), we obtain

$$\begin{aligned} d_{V_\infty} &\leq \mathcal{J}_{V_\infty}(t_n v_n) \\ &= \mathcal{J}_{V_\infty}(t_n v_n) - \frac{1}{q}\langle \mathcal{J}'_{V_\infty}(t_n v_n), t_n v_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\|t_n v_n\|^p_{V,p} + \int_{\mathbb{R}^N}\left(\frac{1}{q}f(t_n v_n)t_n v_n - F(t_n v_n)\right)dx \\ &\leq \left(\frac{1}{p}-\frac{1}{q}\right)\|v_n\|^p_{V,p} + \int_{\mathbb{R}^N}\left(\frac{1}{q}f(v_n)v_n - F(v_n)\right)dx \\ &= c + o_n(1). \end{aligned}$$

Taking the limit as $n\to\infty$ we get $c \geq d_{V_\infty}$. □

At this point we are able to prove the following compactness result.

Proposition 5.1. *Let $\{u_n\}\subset \mathcal{N}_\varepsilon$ be such that $\mathcal{I}_\varepsilon(u_n)\to c$, where $c < d_{V_\infty}$ if $V_\infty<\infty$ and $c\in\mathbb{R}$ if $V_\infty=\infty$. Then $\{u_n\}$ has a convergent subsequence in $\mathbb{X}_\varepsilon$.*

Proof. It is easy to see that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$. Then, up to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } \mathbb{X}_\varepsilon, \\ &u_n \to u \text{ in } L^m_{loc}(\mathbb{R}^N) \quad \text{for any } m\in[1,q^*_s), \\ &u_n \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{5.18}$$

By using assumptions (f_2)-(f_3), (5.18) and the fact that $\mathcal{C}^\infty_c(\mathbb{R}^N)$ is dense in $\mathbb{X}_\varepsilon$, it is standard to check that $\mathcal{I}'_\varepsilon(u)=0$.
Now, let $v_n = u_n - u$. By Lemma 2.6, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(v_n) &= \mathcal{I}_\varepsilon(u_n) - \mathcal{I}_\varepsilon(u) + o_n(1) \\ &= c - \mathcal{I}_\varepsilon(u) + o_n(1) = d + o_n(1). \end{aligned} \tag{5.19}$$

Now, we prove that $\mathcal{I}'_\varepsilon(v_n)=o_n(1)$. For $t\in\{p,q\}$, by using Lemma 2.3 with $z_n=v_n$ and $w=u$, we get

$$\iint_{\mathbb{R}^{2N}} |\mathcal{A}(u_n)-\mathcal{A}(v_n)-\mathcal{A}(u)|^{t'}dx = o_n(1), \tag{5.20}$$

and arguing as in the proof of Lemma 3.3 in [36], we can see that

$$\int_{\mathbb{R}^N} V(\varepsilon\, x)||v_n|^{t-2}v_n - |u_n|^{t-2}u_n + |u|^{t-2}u|^{t'}dx = o_n(1). \tag{5.21}$$

Hence, by using Hölder inequality, for any $\varphi \in \mathbb{X}_\varepsilon$ such that $\|\varphi\|_\varepsilon \leq 1$, it holds

$$\begin{aligned}
&|\langle \mathcal{I}'_\varepsilon(v_n) - \mathcal{I}'_\varepsilon(u_n) + \mathcal{I}'_\varepsilon(u), \varphi\rangle| \\
&\leq \left(\iint_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{p'} dxdy\right)^{\frac{1}{p'}} [\varphi]_{s,p} \\
&+ \left(\iint_{\mathbb{R}^{2N}} |\mathcal{A}(u_n) - \mathcal{A}(v_n) - \mathcal{A}(u)|^{q'} dxdy\right)^{\frac{1}{q'}} [\varphi]_{s,q} \\
&+ \left(\int_{\mathbb{R}^N} V(\varepsilon x)||v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u|^{p'} dx\right)^{p'} \left(\int_{\mathbb{R}^N} V(\varepsilon x)|\varphi|^p dx\right)^{\frac{1}{p}} \\
&+ \left(\int_{\mathbb{R}^N} V(\varepsilon x)||v_n|^{q-2}v_n - |u_n|^{q-2}u_n + |u|^{q-2}u|^{q'} dx\right)^{q'} \left(\int_{\mathbb{R}^N} V(\varepsilon x)|\varphi|^q dx\right)^{\frac{1}{q}} \\
&+ \int_{\mathbb{R}^N} |(f(v_n) - f(u_n) + f(u))\varphi| dx,
\end{aligned}$$

and in view of (iv) of Lemma 2.6, (5.20), (5.21), $\mathcal{I}'_\varepsilon(u_n) = 0$ and $\mathcal{I}'_\varepsilon(u) = 0$ we obtain the thesis.

Now, we note that by using (f_4) we can see that

$$\mathcal{I}_\varepsilon(u) = \mathcal{I}_\varepsilon(u) - \frac{1}{q}\langle \mathcal{I}'_\varepsilon(u), u\rangle \geq 0. \tag{5.22}$$

Assume $V_\infty < \infty$. It follows from (5.19) and (5.22) that

$$d \leq c < d_{V_\infty}$$

which together Lemma 5.2 gives $v_n \to 0$ in $\mathbb{X}_\varepsilon$, that is $u_n \to u$ in $\mathbb{X}_\varepsilon$.
Let us consider the case $V_\infty = \infty$. Then, we can use Lemma 2.5 to deduce that $v_n \to 0$ in $L^m(\mathbb{R}^N)$ for all $m \in [p, q_s^*)$. This combined with assumptions (f_2) and (f_3) implies that

$$\int_{\mathbb{R}^N} f(v_n) v_n dx = o_n(1). \tag{5.23}$$

Since $\langle \mathcal{I}'_\varepsilon(v_n), v_n\rangle = o_n(1)$ and applying (5.23) we can infer that

$$\|v_n\|_\varepsilon^p = o_n(1),$$

which yields $u_n \to u$ in $\mathbb{X}_\varepsilon$. □

We end this section by giving the proof of the existence of a ground state solution to (P_ε) whenever $\varepsilon > 0$ is small enough.

Theorem 5.1. *Assume that (V) and (f_1)-(f_5) hold. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) admits a ground state solution.*

Proof. From (v) of Lemma 3.5, we know that $c_\varepsilon \geq \rho > 0$ for each $\varepsilon > 0$. Moreover, if $u_\varepsilon \in \mathcal{N}_\varepsilon$ verifies $\mathcal{I}_\varepsilon(u_\varepsilon) = c_\varepsilon$, then $m_\varepsilon^{-1}(u_\varepsilon)$ is a minimizer of Ψ_ε and it is a critical point of Ψ_ε. In view of Lemma 3.6, we can see that u_ε is a critical point of $\mathcal{I}_\varepsilon$. It remains to show that there exists a minimizer of $\mathcal{I}_\varepsilon|_{\mathcal{N}_\varepsilon}$. By applying Ekeland's variational principle [44], there exists a sequence $\{v_n\} \subset \mathbb{S}_\varepsilon$ such that $\Psi_\varepsilon(v_n) \to c_\varepsilon$ and $\Psi'_\varepsilon(v_n) \to 0$ as $n \to \infty$. Let $u_n = m_\varepsilon(v_n) \in \mathcal{N}_\varepsilon$. Then, from Lemma 3.6, we deduce that $\mathcal{I}_\varepsilon(u_n) \to c_\varepsilon$, $\langle \mathcal{I}'_\varepsilon(u_n), u_n\rangle = 0$ and $\mathcal{I}'_\varepsilon(u_n) \to 0$ as $n \to \infty$. Therefore, $\{u_n\}$ is a Palais-Smale sequence for $\mathcal{I}_\varepsilon$ at level c_ε. It is standard to check that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$ and we denote by u its weak limit. It is easy to verify that $\mathcal{I}'_\varepsilon(u) = 0$.

When $V_\infty = \infty$, by using Lemma 2.5, we have $\mathcal{I}_\varepsilon(u) = c_\varepsilon$ and $\mathcal{I}'_\varepsilon(u) = 0$.
Now, we deal with the case $V_\infty < \infty$. In view of Proposition 5.1 it is enough to show that $c_\varepsilon < d_{V_\infty}$

for small ε. Without loss of generality, we may suppose that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x).$$

Let $\mu \in \mathbb{R}$ be such that $\mu \in (V_0, V_\infty)$. Clearly, $d_{V_0} < d_\mu < d_{V_\infty}$. Let us prove that there exists a function $w \in \mathbb{Y}_\mu$ with compact support such that

$$\mathcal{J}_\mu(w) = \max_{t \geq 0} \mathcal{J}_\mu(tw) \quad \text{and} \quad \mathcal{J}_\mu(w) < d_{V_\infty}. \tag{5.24}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N, [0,1])$ be such that $\psi = 1$ in $\mathcal{B}_1(0)$ and $\psi = 2$ in $\mathbb{R}^N \setminus \mathcal{B}_2(0)$. For any $R > 0$, we set $\psi_R(x) = \psi(\frac{x}{R})$ and we consider the function $w_R(x) = \psi_R(x) w^\mu(x)$, where w^μ is a ground state solution to (AP_μ). By Lemma 2.2 we can see that

$$\lim_{R \to \infty} \|w_R - w^\mu\|_{s,p} + \|w_R - w^\mu\|_{s,q} = 0. \tag{5.25}$$

Let $t_R > 0$ be such that $\mathcal{J}_\mu(t_R w_R) = \max_{t \geq 0} \mathcal{J}_\mu(t w_R)$. Then, $t_R w_R \in \mathcal{M}_\mu$. Now we can see that there exists $\bar{r} > 0$ such that $\mathcal{J}_\mu(t_{\bar{r}} w_{\bar{r}}) < d_{V_\infty}$. Indeed, if $\mathcal{J}_\mu(t_R w_R) \geq d_{V_\infty}$ for any $R > 0$, using $t_R w_R \in \mathcal{M}_\mu$, (5.25) and w^μ is a ground state, we can deduce that $t_R \to 1$ and

$$d_{V_\infty} \leq \liminf_{R \to \infty} \mathcal{J}_\mu(t_R w_R) = \mathcal{J}_\mu(w^\mu) = d_\mu < d_{V_\infty},$$

which gives a contradiction. Then, taking $w = \psi_{\bar{r}} w^\mu$, we can conclude that (5.24) holds true.

Now, by (V), we obtain that for some $\bar{\varepsilon} > 0$

$$V(\varepsilon x) \leq \mu \quad \text{for all } x \in \operatorname{supp} w \text{ and } \varepsilon \in (0, \bar{\varepsilon}). \tag{5.26}$$

Then, in the light of (5.24) and (5.26), we have for all $\varepsilon \in (0, \bar{\varepsilon})$

$$\max_{t > 0} \mathcal{I}_\varepsilon(tw) \leq \max_{t > 0} \mathcal{J}_\mu(tw) = \mathcal{J}_\mu(w) < d_{V_\infty}.$$

It follows from (3.5) that $c_\varepsilon < d_{V_\infty}$ for all $\varepsilon \in (0, \bar{\varepsilon})$. □

6. Multiple solutions for (P_ε)

This section is devoted to the study of the multiplicity of solutions to (P_ε). We begin by proving the following result which will be needed to implement the barycenter machinery.

Proposition 6.1. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that the translated sequence*

$$v_n(x) = u_n(x + \tilde{y}_n)$$

has a subsequence which converges in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\} = \{\varepsilon_n \tilde{y}_n\}$ is such that $y_n \to y \in M$.

Proof. Since $\langle \mathcal{I}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$, we know that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$. From $d_{V_0} > 0$, we can infer that $\|u_n\|_{\varepsilon_n} \not\to 0$. Therefore, as in the proof of Lemma 5.1, we can find a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n \to \infty} \int_{\mathcal{B}_R(\tilde{y}_n)} |u_n|^q \, dx \geq \beta. \tag{6.1}$$

Let us define

$$v_n(x) = u_n(x + \tilde{y}_n).$$

In view of the boundedness of $\{u_n\}$ and (6.1), we may assume that $v_n \rightharpoonup v$ in $\mathbb{Y}_{V_0}$ for some $v \neq 0$. Let $\{t_n\} \subset (0, \infty)$ be such that $w_n = t_n v_n \in \mathcal{M}_{V_0}$, and we set $y_n = \varepsilon_n \tilde{y}_n$.

Thus, by using the change of variables $z \mapsto x + \tilde{y}_n$, $V(x) \geq V_0$ and the invariance by translation, we can see that

$$d_{V_0} \leq \mathcal{J}_{V_0}(w_n) \leq \mathcal{I}_{\varepsilon_n}(t_n v_n) \leq \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1).$$

Then we can infer $\mathcal{J}_{V_0}(w_n) \to d_{V_0}$. This fact and $\{w_n\} \subset \mathcal{M}_{V_0}$ imply that there exists $K > 0$ such that $\|w_n\|_{V_0} \leq K$ for all $n \in \mathbb{N}$. Moreover, we can prove that the sequence $\{t_n\}$ is bounded in $\mathbb{R}$. In fact, $v_n \not\to 0$ in $\mathbb{Y}_{V_0}$, so there exists $\alpha > 0$ such that $\|v_n\|_{V_0} \geq \alpha$. Consequently, for all $n \in \mathbb{N}$, we have

$$|t_n|\alpha \leq \|t_n v_n\|_{V_0} = \|w_n\|_{V_0} \leq K,$$

which yields $|t_n| \leq \frac{K}{\alpha}$ for all $n \in \mathbb{N}$. Therefore, up to a subsequence, we may suppose that $t_n \to t_0 \geq 0$. Let us show that $t_0 > 0$. Otherwise, if $t_0 = 0$, from the boundedness of $\{v_n\}$, we get $w_n = t_n v_n \to 0$ in $\mathbb{Y}_{V_0}$, that is $\mathcal{J}_{V_0}(w_n) \to 0$ in contrast with the fact $d_{V_0} > 0$. Thus $t_0 > 0$ and, up to a subsequence, we may assume that $w_n \rightharpoonup w = t_0 v \neq 0$ in $\mathbb{Y}_{V_0}$.
Hence, it holds

$$\mathcal{J}_{V_0}(w_n) \to d_{V_0} \quad \text{and} \quad w_n \rightharpoonup w \neq 0 \text{ in } \mathbb{Y}_{V_0}.$$

From Lemma 4.4, we deduce that $w_n \to w$ in $\mathbb{Y}_{V_0}$, that is $v_n \to v$ in $\mathbb{Y}_{V_0}$.
Now, we show that $\{y_n\}$ has a subsequence satisfying $y_n \to y \in M$. Firstly, we prove that $\{y_n\}$ is bounded in $\mathbb{R}^N$. Assume by contradiction that $\{y_n\}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \to \infty$.
Firstly, we deal with the case $V_\infty = \infty$. By using $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ and a change of variable, we can see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)dx \\
&\leq [v_n]^p_{s,p} + [v_n]^q_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)dx \\
&= \int_{\mathbb{R}^N} f(u_n)u_n \, dx = \int_{\mathbb{R}^N} f(v_n)v_n \, dx.
\end{aligned}$$

By applying Fatou's lemma and $v_n \to v$ in $\mathbb{Y}_{V_0}$, we deduce that

$$\infty = \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} f(v_n)v_n dx = \int_{\mathbb{R}^N} f(v)v \, dx < \infty,$$

which gives a contradiction.
Let us consider the case $V_\infty < \infty$. Taking into account $w_n \to w$ strongly in $\mathbb{Y}_{V_0}$, condition (V) and using the change of variable $z = x + \tilde{y}_n$, we have

$$\begin{aligned}
d_{V_0} &= \mathcal{J}_{V_0}(w) < \mathcal{J}_{V_\infty}(w) \\
&\leq \liminf_{n\to\infty} \left[\frac{1}{p}[w_n]^p_{s,p} + \frac{1}{q}[w_n]^q_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|w_n|^p + \frac{1}{q}|w_n|^q\right) dx - \int_{\mathbb{R}^N} F(w_n)\, dx\right] \\
&= \liminf_{n\to\infty} \left[\frac{t_n^p}{p}[u_n]^p_{s,p} + \frac{t_n^q}{q}[u_n]^q_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon_n z)\left(\frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q\right) dz - \int_{\mathbb{R}^N} F(t_n u_n)\, dz\right] \\
&= \liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0}
\end{aligned} \tag{6.2}$$

which is an absurd. Thus $\{y_n\}$ is bounded and, up to a subsequence, we may assume that $y_n \to y$. If $y \notin M$, then $V_0 < V(y)$ and we can argue as in (6.2) to get a contradiction. Therefore, we can conclude that $y \in M$. □

Let $\delta>0$ be fixed and let $\psi\in\mathcal{C}^\infty([0,+\infty),[0,1])$ be a nonincreasing function such that $\psi=1$ in $[0,\frac{\delta}{2}]$, $\psi=0$ in $[\delta,\infty)$ and $|\psi'|\leq C$ for some $C>0$. For any $y\in M$, we define

$$\Upsilon_{\varepsilon,y}(x)=\psi(|\,\varepsilon\, x-y|)\omega\left(\frac{\varepsilon\, x-y}{\varepsilon}\right),$$

where $\omega\in\mathbb{X}_{V_0}$ is a ground state solution to (AP_{V_0}) which there exists thanks to Lemma 4.4.

Let $t_\varepsilon>0$ be the unique positive number such that

$$\mathcal{I}_\varepsilon(t_\varepsilon\Upsilon_{\varepsilon,y})=\max_{t\geq 0}\mathcal{I}_\varepsilon(t\Upsilon_{\varepsilon,y})$$

and define the map $\Phi_\varepsilon:M\to\mathcal{N}_\varepsilon$ by setting $\Phi_\varepsilon(y):=t_\varepsilon\Upsilon_{\varepsilon,y}$. Then we can prove that

Lemma 6.1. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon(\Phi_\varepsilon(y))=d_{V_0}\ \textit{uniformly in } y\in M. \tag{6.3}$$

Proof. Assume by contradiction that there exist $\delta_0>0$, $\{y_n\}\subset M$ and $\varepsilon_n\to 0$ such that

$$|\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-d_{V_0}|\geq\delta_0. \tag{6.4}$$

Let us observe that Lemma 2.2 and the dominated convergence theorem imply

$$[\Upsilon_{\varepsilon_n,y_n}]^p_{s,p}+\int_{\mathbb{R}^N}V(\varepsilon_n\, x)|\Upsilon_{\varepsilon_n,y_n}|^p\,dx\to[\omega]^p_{s,p}+\int_{\mathbb{R}^N}V_0|\omega|^p\,dx \tag{6.5}$$

and

$$[\Upsilon_{\varepsilon_n,y_n}]^q_{s,q}+\int_{\mathbb{R}^N}V(\varepsilon_n\, x)|\Upsilon_{\varepsilon_n,y_n}|^q\,dx\to[\omega]^q_{s,q}+\int_{\mathbb{R}^N}V_0|\omega|^q\,dx. \tag{6.6}$$

Since $\langle\mathcal{I}'_{\varepsilon_n}(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}),t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}\rangle=0$, we can use the change of variable $z=\frac{\varepsilon_n\, x-y_n}{\varepsilon_n}$ to see that

$$\begin{aligned}
&[t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}]^p_{s,p}+[t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}]^q_{s,q}+\int_{\mathbb{R}^N}V(\varepsilon_n\, x)\left(|t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}|^p+|t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}|^q\right)\,dx\\
&=\int_{\mathbb{R}^N}f(t_{\varepsilon_n}\Upsilon_{\varepsilon_n})t_{\varepsilon_n}\Upsilon_{\varepsilon_n}dx\\
&=\int_{\mathbb{R}^N}f(t_{\varepsilon_n}\psi(|\,\varepsilon_n\, z|)\omega(z))t_{\varepsilon_n}\psi(|\,\varepsilon_n\, z|)\omega(z)\,dz.
\end{aligned} \tag{6.7}$$

Now, we prove that $t_{\varepsilon_n}\to 1$. Firstly we show that $t_{\varepsilon_n}\to t_0<\infty$. Assume by contradiction that $|t_{\varepsilon_n}|\to\infty$. Then, using the fact that $\psi(|x|)=1$ for $x\in\mathcal{B}_{\frac{\delta}{2}}(0)$ and that $\mathcal{B}_{\frac{\delta}{2}}(0)\subset\mathcal{B}_{\frac{\delta}{2\varepsilon_n}}(0)$ for n sufficiently large, we can see that (6.7) and (f_5) give

$$\begin{aligned}
&t^{p-q}_{\varepsilon_n}[\Upsilon_{\varepsilon_n,y_n}]^p_{s,p}+[\Upsilon_{\varepsilon_n,y_n}]^q_{s,q}+\int_{\mathbb{R}^N}V(\varepsilon_n\, x)\left(t^{p-q}_{\varepsilon_n}|\Upsilon_{\varepsilon_n,y_n}|^p+|\Upsilon_{\varepsilon_n,y_n}|^q\right)\,dx\\
&\quad\geq\int_{\mathcal{B}_{\frac{\delta}{2}}(0)}\frac{f(t_{\varepsilon_n}\omega(z))}{(t_{\varepsilon_n}\omega(z))^{q-1}}(\omega(z))^q dz\geq\frac{f(t_{\varepsilon_n}\omega(\bar z))}{(t_{\varepsilon_n}\omega(\bar z))^{q-1}}\int_{\mathcal{B}_{\frac{\delta}{2}}(0)}(\omega(z))^q dz
\end{aligned} \tag{6.8}$$

where $\bar z\in\mathbb{R}^N$ is such that $\omega(\bar z)=\min\{\omega(z):|z|\leq\frac{\delta}{2}\}>0$ (remark that $\omega\in C(\mathbb{R}^N)$ and $\omega>0$ in $\mathbb{R}^N$). Putting together (f_4), $p<q$, $t_{\varepsilon_n}\to\infty$, (6.5) and (6.6), we can see that (6.8) implies that $\|\Upsilon_{\varepsilon_n,y_n}\|^q_{V,q}\to\infty$, which gives a contradiction. Therefore, up to a subsequence, we may assume that $t_{\varepsilon_n}\to t_0\geq 0$. If $t_0=0$, we can use (6.5), (6.6), (6.7), $p<q$ and (f_2), to get

$$\|\Upsilon_{\varepsilon_n,y_n}\|^p_{V,p}\to 0,$$

that is a contradiction. Hence, $t_0 > 0$. Now, we show that $t_0 = 1$. Letting $n \to \infty$ in (6.7), we can see that

$$t_0^{p-q}[\omega]_{s,p}^p + [\omega]_{s,q}^q + \int_{\mathbb{R}^N} V_0(t_0^{p-q}|\omega|^p dx + |\omega|^q)\, dx = \int_{\mathbb{R}^N} \frac{f(t_0\omega)}{(t_0\omega)^{q-1}}\omega^q\, dx. \tag{6.9}$$

Since $\omega \in \mathcal{M}_{V_0}$ we have

$$[\omega]_{s,p}^p + [\omega]_{s,q}^q + \int_{\mathbb{R}^N} V_0(|\omega|^p dx + |\omega|^q)\, dx = \int_{\mathbb{R}^N} f(\omega)\omega\, dx. \tag{6.10}$$

Putting together (6.11) and (6.10) we find

$$(t_0^{p-q} - 1)[\omega]_{s,p}^p + (t_0^{p-q} - 1)\int_{\mathbb{R}^N} V_0|\omega|^p\, dx = \int_{\mathbb{R}^N} \left(\frac{f(t_0\omega)}{(t_0\omega)^{q-1}} - \frac{f(\omega)}{\omega^{q-1}}\right)\omega^q\, dx. \tag{6.11}$$

By (f_5), we can deduce that $t_0 = 1$. This fact and the dominated convergence theorem yield

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n})\, dx = \int_{\mathbb{R}^N} F(\omega)\, dx. \tag{6.12}$$

Hence, taking the limit as $n \to \infty$ in

$$\begin{aligned}\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) =& \frac{t_{\varepsilon_n}^p}{p}[\Upsilon_{\varepsilon_n,y_n}]_{s,p}^p + \frac{t_{\varepsilon_n}^q}{q}[\Upsilon_{\varepsilon_n,y_n}]_{s,q}^q \\ &+ \int_{\mathbb{R}^N} V(\varepsilon_n x)\left(\frac{t_{\varepsilon_n}^p}{p}|\Upsilon_{\varepsilon_n,y_n}|^p + \frac{t_{\varepsilon_n}^q}{q}|\Upsilon_{\varepsilon_n,y_n}|^q\right) dx \\ &- \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n})\, dx\end{aligned}$$

and exploiting (6.5), (6.6) and (6.12), we can deduce that

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \mathcal{J}_{V_0}(\omega) = d_{V_0}$$

which is impossible in view of (6.4). □

Now, we are in the position to introduce the barycenter map. We take $\rho > 0$ such that $M_\delta \subset \mathcal{B}_\rho(0)$, and we set $\chi : \mathbb{R}^N \to \mathbb{R}^N$ as

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ as follows

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)\left(|u|^p + |u|^q\right)\, dx}{\int_{\mathbb{R}^N}\left(|u|^p + |u|^q\right)\, dx}.$$

Lemma 6.2. *The functional Φ_ε verifies the following limit*

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M. \tag{6.13}$$

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{6.14}$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n}, ψ and the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\chi(\varepsilon_n z + y_n) - y_n](|\psi(|\varepsilon_n z|)\omega(z)|^p + |\psi(|\varepsilon_n z|)\omega(z)|^q)\, dz}{\int_{\mathbb{R}^N}(|\psi(|\varepsilon_n z|)\omega(z)|^p + |\psi(|\varepsilon_n z|)\omega(z)|^q)\, dz}.$$

Taking into account $\{y_n\} \subset M \subset \mathcal{B}_\rho(0)$ and the dominated convergence theorem, we can infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which contradicts (6.14). □

At this point, we introduce a subset $\widetilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by taking a function $h:\mathbb{R}_+\to\mathbb{R}_+$ such that $h(\varepsilon)\to 0$ as $\varepsilon\to 0$, and setting

$$\widetilde{\mathcal{N}}_\varepsilon=\{u\in\mathcal{N}_\varepsilon:\mathcal{I}_\varepsilon(u)\leq d_{V_0}+h(\varepsilon)\},$$

where $h(\varepsilon)=\sup_{y\in M}|\mathcal{I}_\varepsilon(\Phi_\varepsilon(y))-d_{V_0}|$. By Lemma 6.1, we know that $h(\varepsilon)\to 0$ as $\varepsilon\to 0$. By definition of $h(\varepsilon)$, we deduce that for all $y\in M$ and $\varepsilon>0$, $\Phi_\varepsilon(y)\in\widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon\neq\emptyset$. Moreover, we have the following lemma.

Lemma 6.3. *For any $\delta>0$, there holds that*

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u),M_\delta)=0.$$

Proof. Let $\varepsilon_n\to 0$ as $n\to\infty$. For any $n\in\mathbb{N}$, there exists $\{u_n\}\subset\widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}}\inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u)-y|=\inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u_n)-y|+o_n(1).$$

Therefore, it suffices to prove that there exists $\{y_n\}\subset M_\delta$ such that

$$\lim_{n\to\infty}|\beta_{\varepsilon_n}(u_n)-y_n|=0. \tag{6.15}$$

Thus, recalling that $\{u_n\}\subset\widetilde{\mathcal{N}}_{\varepsilon_n}\subset\mathcal{N}_{\varepsilon_n}$, we deduce that

$$d_{V_0}\leq c_{\varepsilon_n}\leq\mathcal{I}_{\varepsilon_n}(u_n)\leq d_{V_0}+h(\varepsilon_n)$$

which implies that $\mathcal{I}_{\varepsilon_n}(u_n)\to d_{V_0}$. By Proposition 6.1, there exists $\{\tilde{y}_n\}\subset\mathbb{R}^N$ such that $y_n=\varepsilon_n\,\tilde{y}_n\in M_\delta$ for n sufficiently large. Thus

$$\beta_{\varepsilon_n}(u_n)=y_n+\frac{\int_{\mathbb{R}^N}[\chi(\varepsilon_n\,z+y_n)-y_n](|u_n(z+\tilde{y}_n)|^p+|u_n(z+\tilde{y}_n)|^q)\,dz}{\int_{\mathbb{R}^N}(|u_n(z+\tilde{y}_n)|^p+|u_n(z+\tilde{y}_n)|^q)\,dz}.$$

Since $u_n(\cdot+\tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n\,z+y_n\to y\in M$, we deduce that $\beta_{\varepsilon_n}(u_n)=y_n+o_n(1)$, that is (6.15) holds true. □

Now we show that (P_ε) admits at least $cat_{M_\delta}(M)$ solutions. In order to achieve our aim, we recall the following result for critical points involving Ljusternik-Schnirelmann category. For more details one can see [19].

Theorem 6.1. *Let U be a $\mathcal{C}^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space). Assume that $h\in\mathcal{C}^1(U,\mathbb{R})$ is bounded from below and satisfies $-\infty<\inf_U h<d<k<\infty$. Moreover, suppose that h satisfies the Palais-Smale condition on the sublevel $\{u\in U: h(u)\leq k\}$ and that d is not a critical level for h. Then*

$$card\{u\in h^d:\nabla h(u)=0\}\geq cat_{h^d}(h^d),$$

where $h^d=\{u\in U\,:\,h(u)\leq d\}$.

With a view to apply Theorem 6.1, the following abstract lemma provides a very useful tool in that it relates the topology of some sublevel of a functional to the topology of some subset of the space $\mathbb{R}^N$; see [19].

Lemma 6.4. *Let Ω,Ω_1 and Ω_2 be closed sets with $\Omega_1\subset\Omega_2$ and let $\pi:\Omega\to\Omega_2$, $\psi:\Omega_1\to\Omega$ be continuous maps such that $\pi\circ\psi$ is homotopically equivalent to the embedding $j:\Omega_1\to\Omega_2$. Then $cat_\Omega(\Omega)\geq cat_{\Omega_2}(\Omega_1)$.*

Since $\mathcal{N}_\varepsilon$ is not a $\mathcal{C}^1$ submanifold of $\mathbb{X}_\varepsilon$, we cannot directly apply Theorem 6.1. Fortunately, from Lemma 3.5, we know that the mapping m_ε is a homeomorphism between $\mathcal{N}_\varepsilon$ and $\mathbb{S}_\varepsilon$, and $\mathbb{S}_\varepsilon$ is a $\mathcal{C}^1$ submanifold of $\mathbb{X}_\varepsilon$. So we can apply Theorem 6.1 to $\Psi_\varepsilon(u) = \mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))|_{\mathbb{S}_\varepsilon} = \mathcal{I}_\varepsilon(m_\varepsilon(u))$, where Ψ_ε is given in Lemma 3.6. In the light of the above observations, we are ready to give the proof of the main result of this work.

Proof of Theorem 1.1. For any $\varepsilon > 0$, we define $\alpha_\varepsilon : M \to \mathbb{S}_\varepsilon$ by setting $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. By using Lemma 6.1 and the definition of Ψ_ε, we can see that

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \quad \text{uniformly in } y \in M.$$

Set $\tilde{\mathbb{S}}_\varepsilon := \{w \in \mathbb{S}_\varepsilon : \Psi_\varepsilon(w) \leq d_{V_0} + h(\varepsilon)\}$, where $h(\varepsilon) = \sup_{y\in M} |\Psi_\varepsilon(\alpha_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$. Thus, $\alpha_\varepsilon(y) \in \tilde{\mathbb{S}}_\varepsilon$ for all $y \in M$, and this yields $\tilde{\mathbb{S}}_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.

Taking into account Lemma 6.1, Lemma 3.5, Lemma 3.6 and Lemma 6.3, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram

$$M \overset{\Phi_\varepsilon}{\to} \widetilde{\mathcal{N}}_\varepsilon \overset{m_\varepsilon^{-1}}{\to} \tilde{\mathbb{S}}_\varepsilon \overset{m_\varepsilon}{\to} \widetilde{\mathcal{N}}_\varepsilon \overset{\beta_\varepsilon}{\to} M_\delta$$

is well defined for any $\varepsilon \in (0, \bar{\varepsilon})$. By using Lemma 6.2, there exists a function $\theta(\varepsilon, y)$ with $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$, for all $\varepsilon \in (0, \bar{\varepsilon})$, such that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$. Then we can see that $H(t, y) = y + (1-t)\theta(\varepsilon, y)$, with $(t, y) \in [0, 1] \times M$, is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \alpha_\varepsilon$ and the inclusion map $id : M \to M_\delta$. This fact and Lemma 6.4 imply that $cat_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon) \geq cat_{M_\delta}(M)$. On the other hand, let us choose a function $h(\varepsilon) > 0$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $d_{V_0} + h(\varepsilon)$ is not a critical level for $\mathcal{I}_\varepsilon$. For $\varepsilon > 0$ small enough, we deduce from Proposition 5.1 that $\mathcal{I}_\varepsilon$ satisfies the Palais-Smale condition in $\widetilde{\mathcal{N}}_\varepsilon$. So, by (ii) of Lemma 3.6, we infer that Ψ_ε satisfies the Palais-Smale condition in $\tilde{\mathbb{S}}_\varepsilon$. Hence, by using Theorem 6.1, we obtain that Ψ_ε has at least $cat_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon)$ critical points on $\tilde{\mathbb{S}}_\varepsilon$. Then, in view of (iii) of Lemma 3.6, we can infer that $\mathcal{I}_\varepsilon$ admits at least $cat_{M_\delta}(M)$ critical points. □

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