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(Article begins on next page)

An approximated computational method for fast stress reconstruction in large strain plasticity

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The paper describes a numerical method to reconstruct the stress field starting from strain data in elasto-plasticity. Usually, this reconstruction is performed using the radial return algorithm, commonly implemented also in finite element codes. However, that method requires iterations to converge and can bring to errors if applied to experimental strain data affected by noise. A different solution is proposed here, where an approximated numerical method is used to derive the stress from the strain data with no iterations. The method is general and can be applied to any plasticity model with a convex surface of the yield locus in non-proportional loading. The theoretical basis of the method is described and then it is implemented on two constitutive models of anisotropic plasticity, namely Hill48 and Yld2000-2D. The accuracy of the proposed method and the advantage in terms of computational time with respect to the classical radial-return algorithm are discussed. The possibility of using such method to reconstruct the stress field in case of few temporal data and noisy strain fields is also investigated.

KEYWORDS

Solids, Plasticity, Integration, Large deformation

1 | INTRODUCTION

The computation of stress from strain is a well-known problem in elasto-plasticity. According to the adopted constitutive model, the stress depends on the strain history in a non-linear way and, in general, a closed form solution to obtain directly the stress from the strain does not exist. Since the integration of such elasto-plastic constitutive equations is required, for instance, to implement plasticity models in finite element analysis (FEA), many numerical methods have been proposed during the years to solve efficaciously the problem.

Most of the integration algorithms were developed during the '80s although the first approach was already described by Wilkins [1] in 1964 who introduced the *radial return* algorithm. This method was further developed by other authors [2–4] and then extensively used in finite element applications [5, 6]. The procedure is also referred to as the closest point projection method (CPPM), because it finds the closest point projection of the elastic predictor or trial stress to the yield surface. Several variations of the CPPM were developed [7, 8] and extended to finite plasticity [9].

Another numerical method often used to integrate the stress is known as *cutting-plane* algorithm [10]. It does not require the evaluation of the second derivative of the plastic potential and is convenient when such derivation becomes complicated [11]. An in-depth description of various integration methods can be found in several books [12–14]. The above-mentioned algorithms are nowadays implemented in most commercial FE software.

In recent years, thanks to the advance in full-field measurements, a large interest is increasing on the use of inverse methods to identify the properties of materials [15]. Mainly, two approaches can be used: either a finite element model of the test is used to identify the material properties by subsequent iterations [16–18] – this is commonly known as finite element model updating (FEMU) – or the measured strain field is directly employed in the identification procedure exploiting equilibrium equations [19–21]. Among the methods that follow the second approach, the virtual fields method (VFM) [22] is probably the most used and widespread. Recently, integrated methods that combine the two approaches have also been developed, see for instance the iDIC [23, 24].

In plasticity, when the second approach is used, the stress state has to be derived from the measured strain field, using appropriate integration methods. However, in this case, the standard integration algorithms described above may not be optimal, especially if large step size or highly non-linear constitutive models are used [25, 26]. Moreover, since such algorithms require Newton's iterative solution, the computational time may increase, reducing one of the main advantage of those methods if compared with FEMU.

A different implementation was provided by Rossi and Pierron [27], who used a direct derivation of the stress with no iterations. The method can be seen as a generalization of the classical Levy-Mises equations to complex constitutive models, large strain and non-proportional loading. The same approach was successfully applied by Rossi et al. [28] on advanced constitutive models for anisotropic plasticity [29], however most of the other authors working in the field still prefer to use the classical radial return scheme [19, 30–33]. One of the reason is that a thorough review and error assessment of the direct stress reconstruction method was never performed.

In this paper, the direct method is studied in detail and compared with the classical integration algorithm, using validated FE models built up with commercial codes. The accuracy and the advantages in terms of computational time and data handling are investigated as well as the main limitations and the introduced approximations. Although the algorithm was initially designed to be used in conjunction with an inverse method, it could be also useful in other applications where it is necessary to reconstruct the stress field from strain data.

In Section 2 the theoretical background of the stress reconstruction algorithm is presented, in Section 3 the numerical implementation of the method is described in detail and applied to two popular plasticity models, *i.e.* Hill48 and Yld2000-2D. In Section 4 the performance and potentiality of the algorithm are verified and discussed using FE models as reference.

2 | THEORETICAL BACKGROUND

A general plasticity model is characterized by a *yield criterion*, which defines the limit at which the material becomes plastic, a *flow rule*, which describes the relationship between the stress and strain rate during the plastic deformation and a *hardening rule*, which describes how the yield stress evolves because of plastic deformation.

The yield criterion is usually described by a scalar function of the Cauchy stress tensor $\mathbf{T}$:

$$\Phi_p(\mathbf{T}, \xi_i^Y) = 0 \quad \text{with } i = 1, \dots, n_Y \quad (1)$$

where ξ_i^Y is a set of n_Y constitutive parameters that depends on the material. If $\Phi_p < 0$ the material is in the elastic regime, while if $\Phi_p = 0$ plastic deformation occurs. The yield criterion often assumes the following form:

$$\Phi_p(\mathbf{T}, \xi_i^Y) = \sigma_T(\mathbf{T}, \xi_i^Y) - \sigma_Y = 0 \quad (2)$$

where $\sigma_T(\mathbf{T}, \xi_i^Y)$ is the equivalent stress, a function that assigns to each stress tensor a non-negative scalar value, and σ_Y is the yield stress, the critical value of equivalent stress where plasticity occurs.

The flow rule defines the direction of the plastic flow that is the direction of the plastic strain rate during plastic deformation. The total strain rate is described by the rate of deformation tensor $\mathbf{D}$, defined as:

$$\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T) \quad (3)$$

where $\mathbf{L}$ is the spatial velocity gradient:

$$\mathbf{L} = \mathbf{F}\mathbf{F}^{-1} \quad (4)$$

and $\mathbf{F}$ is the deformation gradient, which can be retrieved from the gradient of the displacement vector $\mathbf{u}$:

$$\mathbf{F} = \text{Grad } \mathbf{u} + \mathbf{I} \quad (5)$$

Adopting the additive decomposition of the strain, the rate of deformation tensor $\mathbf{D}$ is divided in the elastic and plastic part:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \quad (6)$$

Accordingly, the direction of the plastic flow can be defined as the direction of the plastic part $\mathbf{D}^p$ of the rate of deformation tensor :

$$\hat{\mathbf{N}}_p = \frac{\mathbf{D}^p}{|\mathbf{D}^p|} \quad (7)$$

where the norm of a tensor $\mathbf{A}$ is defined as $|\mathbf{A}| := (\mathbf{A} \cdot \mathbf{A})^{\frac{1}{2}}$ with $\cdot$ scalar product [34].

As mentioned before, the direction of the plastic flow is governed by the flow rule, which, usually is derived from a plastic potential π so that:

$$\mathbf{D}^p = \lambda \frac{\partial \pi(\mathbf{T}, \xi_i^\pi)}{\partial \mathbf{T}} \quad \text{with } i = 1, \dots, n_\pi \quad (8)$$

where ξ_i^π are a set of n_π constitutive parameters and λ is a scalar multiplier. Eq. 8 can be rewritten in terms of the normalized tensor $\hat{\mathbf{N}}_p$ according to Eq. 7:

$$\hat{\mathbf{N}}_p = \frac{\partial \pi(\mathbf{T}, \xi_i^\pi)}{\partial \mathbf{T}} \bigg/ \left| \frac{\partial \pi(\mathbf{T}, \xi_i^\pi)}{\partial \mathbf{T}} \right| \quad (9)$$

the plastic flow direction does not depend on the scalar multiplier λ . Similarly to the yield criterion, the equation $\pi = 0$ represents a surface in the stress space. The plastic flow direction can be accordingly considered as the normal to this surface, a geometrical representation is illustrated Figure 1.

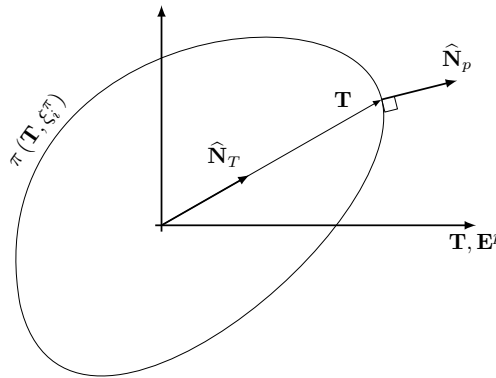


FIGURE 1 Geometrical representation of the plastic flow direction and the corresponding Cauchy stress tensor.

If the surface is convex, every normal corresponds to a unique stress state, therefore, if the constitutive param-

ters are known, the stress state in a material point can be directly derived from the direction of the plastic flow. This consideration is the basis for the proposed stress computation algorithm. The approach is general and can be applied to every plasticity model that involves a convex surface for the plastic potential. Here, we limit the study to the associated flow rule, which means that the plastic potential and the yield criterion coincide, i.e. $\pi = \Phi_p$ and $\xi_i^\pi = \xi_i^Y$. The procedure described below can be readily adapted to non-associated flow rules using the corresponding plastic potential in place of the yield criterion. Under the associated flow rule assumption, using the yield criterion described in Eq. 2, Eq. 9 becomes:

$$\hat{\mathbf{N}}_p = \frac{\partial \sigma_T(\mathbf{T}, \xi_i^Y)}{\partial \mathbf{T}} \bigg/ \left| \frac{\partial \sigma_T(\mathbf{T}, \xi_i^Y)}{\partial \mathbf{T}} \right| \quad (10)$$

If the equivalent stress function used in the yield criterion has a multiplicative scaling behaviour with respect to a scalar parameter α , i.e.:

$$\sigma_T(\alpha \mathbf{T}, \xi_i^Y) = \alpha \sigma_T(\mathbf{T}, \xi_i^Y) \quad (11)$$

the Cauchy stress tensor can be expressed in terms of its normalized tensor $\hat{\mathbf{N}}_T$, named here stress direction:

$$\mathbf{T} = |\mathbf{T}| \hat{\mathbf{N}}_T \quad (12)$$

where $|\mathbf{T}|$ is the absolute value of the tensor $\mathbf{T}$. A normalized equivalent stress $\hat{\sigma}_T$ is then defined as follows:

$$\hat{\sigma}_T = \sigma_T(\hat{\mathbf{N}}_T, \xi_i^Y) \quad (13)$$

so that, exploiting the property of Eq. 11:

$$\sigma_T(\mathbf{T}, \xi_i^Y) = |\mathbf{T}| \hat{\sigma}_T \quad (14)$$

Using Eq. 10, 11 and 12 we can derive a function $\mathfrak{F}$ that associates the stress direction to the plastic flow direction:

$$\mathfrak{F}(\hat{\mathbf{N}}_T, \xi_i^Y) = \frac{\partial \sigma_T(\hat{\mathbf{N}}_T, \xi_i^Y)}{\partial \mathbf{T}} \bigg/ \left| \frac{\partial \sigma_T(\hat{\mathbf{N}}_T, \xi_i^Y)}{\partial \mathbf{T}} \right| = \hat{\mathbf{N}}_p \quad (15)$$

If $\mathfrak{F}$ is invertible, the stress direction can be derived from the plastic flow direction:

$$\widehat{\mathbf{N}}_T = \widehat{\mathfrak{F}}^{-1} \left(\widehat{\mathbf{N}}_p, \xi_i^Y \right) \quad (16)$$

The norm of stress tensor can now be retrieved using Eq. 2 and Eq. 14, viz:

$$|\mathbf{T}| = \frac{\sigma_Y}{\widehat{\sigma}_T} \quad (17)$$

finally, using Eq. 12, 16 and 17, the Cauchy stress tensor can be derived directly from the direction of the plastic flow as:

$$\mathbf{T} = \frac{\sigma_Y}{\widehat{\sigma}_T} \widehat{\mathfrak{F}}^{-1} \left(\widehat{\mathbf{N}}_p, \xi_i^Y \right) \quad (18)$$

The equation gives a direct evaluation of the stress from the plastic flow direction. The equation is valid at any point of the material subjected to plastic deformation. It is worth noting that, usually, in plasticity, the stress tensor is obtained by multiplying the elastic stiffness matrix to the elastic part of the strain tensor, however Eq. 18 shows that the same result can be obtained only looking at the plastic deformation, if the shape of the yield surface is entirely known.

During the deformation process, the shape of the yield surface can change because of the hardening or because of modifications of the anisotropic behaviour. If isotropic hardening is considered, σ_Y is the only quantity that has to be updated in Eq. 18, since the shape of the yield surface does not change with deformation. In this case, σ_Y can be derived from the hardening law that usually has this form:

$$\sigma_Y = \sigma_Y \left(p, \xi_i^H \right) \quad \text{with } i = 1, \dots, n_H \quad (19)$$

where ξ_i^H are material parameters and p is a scalar value that represents the equivalent cumulated plastic strain. Kinematic or differential hardening could also be introduced but it is beyond the scope of this paper and thier implementation is postponed to future studies.

From Eq. 18 the stress tensor can be derived directly from the plastic flow and can be used effectively in all those problems where the deformation is known *a priori*. The main drawbacks are:

- in order to evaluate the plastic flow direction it is necessary to separate the plastic strain rate from the total one;
- the inverse function $\widehat{\mathfrak{F}}^{-1}$ is not always available in closed-form.

Both problems can be solved using an approximated numerical solution, as discussed in details in the following.

3 | APPROXIMATED NUMERICAL SOLUTION

In this section, a numerical method to derive the stress from the strain using Eq. 18 is presented and implemented to two well-known plasticity models. Some approximations are introduced and their influence on the accuracy of the stress reconstruction is thoroughly addressed in the paper.

The implementation follows the framework of incremental deformation theory based on the minimum plastic work path, as discussed by Yoon et. al in [35]. Using this approach, the logarithmic strain increment $\Delta \mathbf{E}$ is used instead of the deformation rate tensor $\mathbf{D}$ to describe the plastic flow. The logarithmic strain, or material Hencky strain, is defined as $\mathbf{E} = \ln \mathbf{U}$, where $\mathbf{U}$ is the right stretch tensor obtained by the polar decomposition of the deformation gradient, i.e. $\mathbf{F} = \mathbf{R}\mathbf{U}$. A detailed explanation of the incremental deformation theory is given in the Appendix.

In this section, the stress and strain tensors are expressed in terms of vectors, using the Voigt notations, which is usually adopted to implement stress computation algorithms into common scientific programming languages (e.g. Matlab, Python, Fortran). Accordingly

$$\begin{aligned} \mathbf{T} &\longrightarrow \boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}] \\ \mathbf{E} &\longrightarrow \boldsymbol{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12}] \\ \widehat{\mathbf{N}}_T &\longrightarrow \widehat{\mathbf{n}}_\sigma = [\widehat{n}_{11}^\sigma, \widehat{n}_{22}^\sigma, \widehat{n}_{33}^\sigma, \widehat{n}_{23}^\sigma, \widehat{n}_{13}^\sigma, \widehat{n}_{12}^\sigma] \\ \widehat{\mathbf{N}}_p &\longrightarrow \widehat{\mathbf{n}}_p = [\widehat{n}_{11}^p, \widehat{n}_{22}^p, \widehat{n}_{33}^p, \widehat{n}_{23}^p, \widehat{n}_{13}^p, \widehat{n}_{12}^p] \end{aligned} \quad (20)$$

where $\gamma_{ij} = 2\epsilon_{ij}$. The versors $\widehat{\mathbf{n}}_\sigma$ and $\widehat{\mathbf{n}}_p$ are derived by normalizing the corresponding stress or strain vector. In the numerical examples discussed below, we make the further assumption of plane stress, i.e. $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$. In this case, all vectors have only three components (e.g. $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]$)¹.

Assuming that strain field is known and the stress field at load step k is also known, the objective of the algorithm is reconstructing the stress at step $k+1$. The known quantities at load step k are ϵ^k , p^k and σ^k while, at load step $k+1$, only ϵ^{k+1} is known and p^{k+1} and σ^{k+1} have to be retrieved. First, a trial stress is computed making the hypothesis that the increment is in the elastic regime:

$$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}^k + \mathbf{C}_e \Delta \boldsymbol{\epsilon} \quad (21)$$

where $\Delta \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{k+1} - \boldsymbol{\epsilon}^k$ and $\mathbf{C}_e$ is the elastic stiffness matrix. Then the yielding condition is checked:

$$\sigma_T(\boldsymbol{\sigma}^{\text{trial}}, \xi_i^Y) > \sigma_Y(p^k) \quad (22)$$

If Eq. 22 is not satisfied, the step is in the elastic regime and the stress and the equivalent plastic strain at step $k+1$ are

¹The application of the proposed direct method to three-dimensional strain fields is possible but, in that case, usually only the stress deviator can be obtained. An example can be found in [27]. The complete 3D stress reconstruction can be instead obtained iteratively using the standard return algorithms

$$\begin{cases} \rho^{k+1} = \rho^k \\ \sigma^{k+1} = \sigma^{\text{trial}} \end{cases} \quad (23)$$

If the step is plastic, the strain increment has to be divided in the elastic and plastic part, analogously to Eq.6:

$$\Delta \epsilon = \Delta \epsilon^e + \Delta \epsilon^p \quad (24)$$

however, if plastic flow is fully developed within the increment, the elastic part is much smaller than the plastic one and can be neglected (the impact of such approximation will be evaluated later on in the paper). In fact, since the plastic deformation is isochoric, here the plastic strain increment is approximated to the deviatoric part of the total strain increment. In case of plane stress, only the planar components of the strain tensor are considered, accordingly $\Delta \epsilon^p \approx \Delta \epsilon$ and the out of plane component is obtained using the volume constancy as $\epsilon_{33} = -(\epsilon_{11} + \epsilon_{22})$. Assuming that the proportional loading condition is maintained within the increment, the plastic flow direction is obtained as:

$$\hat{\mathbf{n}}_p^{k+1} = \frac{\Delta \epsilon}{|\Delta \epsilon|} \quad (25)$$

where the strain increment $\Delta \epsilon$ is used instead of tensor $\mathbf{D}$ in Eq. 7. The normalized stress is derived from Eq. 16:

$$\hat{\mathbf{n}}_\sigma^{k+1} = \tilde{\gamma}^{-1} \left(\hat{\mathbf{n}}_p^{k+1}, \xi_i^Y \right) \quad (26)$$

The normalized stress is then used to evaluate the increment of equivalent plastic strain, exploiting the work-equivalence principle that, for incremental deformation, is written as:

$$\sigma_Y \Delta p = \sigma : \Delta \epsilon \quad (27)$$

Using the stress normalization of Eq. 12 and Eq. 14 in Eq. 27, it follows:

$$|\sigma| \hat{\sigma}_T \Delta p = |\sigma| \hat{\mathbf{n}}_\sigma : \Delta \epsilon \quad (28)$$

This equation shows that the increment of equivalent plastic strain only depends on the stress direction. Therefore, the equivalent plastic strain increment at step $k + 1$ can be directly derived as:

$$\Delta p = \frac{\hat{\mathbf{n}}_\sigma^{k+1} : \Delta \epsilon}{\hat{\sigma}_T \left(\hat{\mathbf{n}}_\sigma^{k+1}, \xi_i^Y \right)} \quad (29)$$

Finally, the stress and equivalent plastic strain at step $k + 1$ are readily obtained using Eq. 18:

$$\begin{cases} p^{k+1} = p^k + \Delta p \\ \sigma^{k+1} = \frac{\sigma_Y(p^{k+1}, \xi_i^H)}{\bar{\sigma}_T(\hat{n}_\sigma^{k+1}, \xi_i^Y)} \bar{\gamma}^{-1}(\hat{n}_\sigma^{k+1}, \xi_i^Y) \end{cases} \quad (30)$$

It is worth noting that no evolution law is required to integrate the hardening law. The reconstruction algorithm is summarized in the flow diagram of Figure 2. The integration scheme is direct and no iterations are required to compute the stress at step $k + 1$. It is important to underline that this method is different from the classical explicit stress reconstruction method, where the stress is derived from the elastic strain using the strain decomposition. The classical explicit stress reconstruction method do not require iterations but has severe convergency problems if the strain increment is large, for this reason it is usually avoided in FE applications.

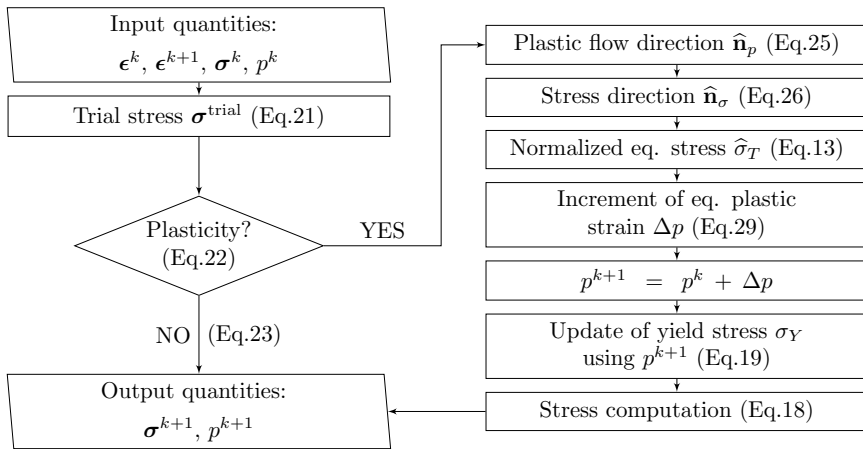


FIGURE 2 Flow diagram of the proposed direct method of stress reconstruction

The crucial part of the present method is the evaluation of the stress direction from the flow stress direction (Eq. 26). The derivation of this inverse function is discussed below for two plasticity models, Hill48, where a closed-form solution is found for $\bar{\gamma}^{-1}$ and Yld2000-2D, where the inverse function is obtained numerically using an interpolation scheme.

3.1 | Implementation of Hill48

In plane stress condition, the Hill48 criterion can be written as [36]:

$$\sigma_T(\sigma) = \left[(g + h) \sigma_{11}^2 + (f + h) \sigma_{22}^2 - 2h \sigma_{11} \sigma_{22} + 2n \sigma_{12}^2 \right]^{\frac{1}{2}} \quad (31)$$

where $\xi^Y = \{g, f, h, n\}$ are four anisotropic constants that can be derived from the Lankford parameter R , measured at least in three different orientations [37, 38]. R is the ratio of the transverse to the through thickness strain in a

uniaxial tensile test. If the orientations used are 0° , 45° and 90° with respect to the rolling direction, the anisotropic parameters can be obtained as follows:

$$f = \frac{R_0}{(1+R_0)R_{90}}; g = \frac{1}{1+R_0}; h = \frac{R_0}{1+R_0}; n = \left(\frac{1}{2} + R_{45}\right) \left(1 + \frac{R_0}{R_{90}}\right) g; \quad (32)$$

with R evaluated at 0° , 45° and 90° with respect to the first anisotropic direction, which is usually the rolling one in sheet metals. For this model, the normalized equivalent stress of Eq. 13 is:

$$\hat{\sigma}_T = \left[(g+h) \hat{n}_{11}^{\sigma^2} + (f+h) \hat{n}_{22}^{\sigma^2} - 2h \hat{n}_{11}^{\sigma} \hat{n}_{22}^{\sigma} + n \hat{n}_{12}^{\sigma^2} \right]^{\frac{1}{2}} \quad (33)$$

The direction of the plastic flow is achieved making the derivative of Eq. 33:

$$\frac{\partial \sigma_T(\hat{\mathbf{n}}_\sigma, \xi_i)}{\partial \sigma} = \hat{\sigma}_T^{-1} \begin{bmatrix} (h+g) & -h & 0 \\ -h & (f+h) & 0 \\ 0 & 0 & 2n \end{bmatrix} \begin{Bmatrix} \hat{n}_{11}^{\sigma} \\ \hat{n}_{22}^{\sigma} \\ \hat{n}_{12}^{\sigma} \end{Bmatrix} \quad (34)$$

In this case, the plastic flow direction is a linear transformation of the stress direction $\hat{\mathbf{n}}_\sigma$. Thus, the inverse function $\hat{\gamma}^{-1}$ can be easily obtained as the inverse of the transformation matrix of Eq. 34:

$$\hat{\mathbf{n}}_\sigma = \frac{\mathbf{A} \hat{\mathbf{n}}_p}{|\mathbf{A} \hat{\mathbf{n}}_p|} \text{ with } \mathbf{A} = \begin{bmatrix} f+h & h & 0 \\ h & h+g & 0 \\ 0 & 0 & \frac{gf+gh+hf}{2n} \end{bmatrix} \quad (35)$$

and the stress can be directly derived with no iteration using Eq. 18. Therefore, for Hill48, it stands:

$$\sigma = \frac{\sigma_Y}{\hat{\sigma}_T} \frac{\mathbf{A} \hat{\mathbf{n}}_p}{|\mathbf{A} \hat{\mathbf{n}}_p|} \quad (36)$$

3.2 | Implementation of Yld2000-2D

The equivalent stress for the Yld2000-2D criterion is:

$$\sigma_T(\sigma) = \left[\frac{1}{2} \left(|X'_1 - X'_2|^a + |2X''_2 + X''_1|^a + |2X''_1 + X''_2|^a \right) \right]^{\frac{1}{a}} \quad (37)$$

where X'_1 , X'_2 and X''_1 , X''_2 are the principal values of the $\mathbf{X}'$ and $\mathbf{X}''$ tensors obtained from the Cauchy stress σ using two linear transformations, according to $\mathbf{L}'$ and $\mathbf{L}''$, respectively. It follows:

$$\begin{cases} \mathbf{X}' = \mathbf{L}'\boldsymbol{\sigma} \\ \mathbf{X}'' = \mathbf{L}''\boldsymbol{\sigma} \end{cases} \quad (38)$$

with:

$$\mathbf{L}' = \begin{bmatrix} \frac{2\alpha_1}{3} & -\frac{2\alpha_1}{3} & 0 \\ -\frac{2\alpha_2}{3} & \frac{2\alpha_2}{3} & 0 \\ 0 & 0 & \alpha_7 \end{bmatrix} \quad \mathbf{L}'' = \begin{bmatrix} \frac{8\alpha_5-2\alpha_3-2\alpha_6+2\alpha_4}{9} & -\frac{4\alpha_6-4\alpha_4-4\alpha_5+\alpha_3}{9} & 0 \\ -\frac{4\alpha_3-4\alpha_5-4\alpha_4+\alpha_6}{9} & \frac{8\alpha_4-2\alpha_6-2\alpha_3+2\alpha_5}{9} & 0 \\ 0 & 0 & \alpha_8 \end{bmatrix} \quad (39)$$

where $[\alpha_1, \dots, \alpha_8]$ are eight parameters that describe the material anisotropy. The yield function is non-quadratic and the exponent a depends on the type of material.

The derivative of Eq. 37, required to obtain the direction of the plastic flow, is not straightforward. It can be obtained using the chain rule on the different stress transformations. The details are given in the Appendix of [35] and will not be repeated here. Conversely to Hill48, for Yld2000-2D, to the best authors' knowledge, a direct formula for the inverse function $\tilde{\gamma}^{-1}$ is not available. Therefore, a numerical algorithm is employed using scattered-data interpolants.

A scattered-data interpolant is an interpolation function $\Im$ that is created starting from an initial set of scattered points $\mathbf{x}_0$ and the corresponding $\mathbf{y}_0 = f(\mathbf{x}_0)$. It can be used to evaluate the value $\mathbf{y}$ corresponding to a generic $\mathbf{x}$ as $\mathbf{y} = \Im(\mathbf{x})$ by interpolation of the initial values. Once the interpolant is generated, its use is computationally very effective as it will be demonstrated later on. Routines to generate scattered-data interpolant are in-built in Matlab^{®2}. Similar routines can be developed also in other programming languages.

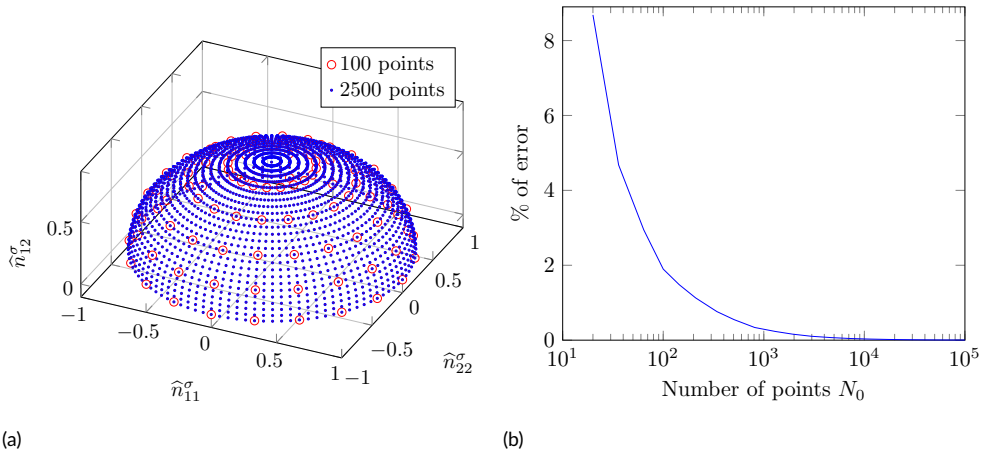


FIGURE 3 Initial values used in the interpolation function: a) initial points used to generate the interpolant $\Im$, in the graph two sets of points are shown with $N_0 = 100$ and $N_0 = 2500$; b) approximation error, as function of the number of points N_0 used to create the interpolant $\Im$.

²In Matlab R2018a, i.e. the version used in this work, the interpolation routine is named "scatteredInterpolant".

In the present application, N_0 samples of stress normals $\hat{\mathbf{n}}_\sigma$ are selected from the normalised stress domain, as illustrated in Figure 3a, and the corresponding flow normals are computed using the direct function (known):

$$\hat{\mathbf{n}}_p^{(i)} = \frac{\partial \sigma_T(\hat{\mathbf{n}}_\sigma^{(i)}, \xi_i)}{\partial \sigma} / \left| \frac{\partial \sigma_T(\hat{\mathbf{n}}_\sigma^{(i)}, \xi_i)}{\partial \sigma} \right| \quad \text{for } i = 1, \dots, N_0 \quad (40)$$

Then an interpolant is created using $x_0^{(i)} = \hat{\mathbf{n}}_p^{(i)}$ and $y_0^{(i)} = \hat{\mathbf{n}}_\sigma^{(i)}$. Such interpolant can be used to evaluate the stress direction from any plastic flow direction, as required from Eq. 16. Thus, the Cauchy stress for Yld2000-2D can be obtained as:

$$\sigma = \frac{\sigma_Y}{\sigma_T} \Im(\hat{\mathbf{n}}_p) \quad (41)$$

The more points are used to generate the interpolant the more accuracy is obtained in the interpolation. However, a large number of points increases the computational time too. The approximation error as a function of the number of points N_0 used to create the interpolant was assessed using a Monte Carlo simulation that checks 10^4 randomly generated $\hat{\mathbf{n}}_p$. The graph of Figure 3b demonstrates that when $N_0 > 10^4$ such error is negligible. In the following computations 2500 points were used to generate the interpolant (see Figure 3a). This choice is a suitable compromise between precision and computational time, as demonstrated in the next sections.

3.3 | Computational time

One advantage of the direct method, if compared with the classical integration schemes that involve iterations, is the fast computational time. This aspect has an important practical impact, in fact, sometimes, the advantage of using inverse methods in plasticity is limited by the long computational time required to derive the stress from the strain [39]. To assess the computational efficiency, a benchmark between the proposed direct stress integration method and the standard Euler-Backward with multi-stage return mapping algorithm reported in [35, 40] was conducted. The machine employed for the analysis was equipped with an Intel® Core™ i7-7700HQ CPU @ 2.80 GHz, 16 MB RAM, and all benchmark routines were tested on a single core without parallelization.

Matlab® was used to compute the stress from an array of strain increments. Each component of the strain increment tensor in the array was generated using a random function so that the resultant equivalent strain was 1%. The number of points of the array were varied to assess the computational efficiency of the stress reconstruction routine.

The two integration algorithms were used for both Hill48 and YLD2000-2D material models. Figure 4a shows the results in a double-logarithmic diagram. As the number of points increases, a significant computational time reduction is observed using the direct method; with 10^6 points the algorithm is around 1000 times faster than the radial return. The use of interpolation functions for YLD2000-2D does not undermine the algorithm efficiency, which is always much faster than the corresponding standard Euler-Backward counterpart.

The main reason for this behaviour is the Matlab's capability of performing fast matrix operations (sum, product, etc.) without using loop cycles. The implementation of the direct method in Matlab can be indeed carried out without introducing loop cycles but only matrix/vector operations.

To check the algorithm performance with a different programming language, the same benchmark was conducted

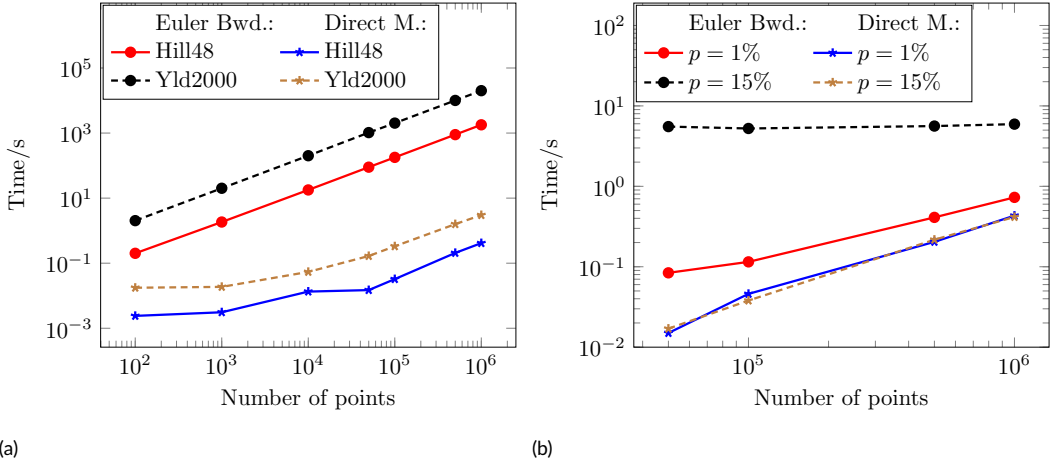


FIGURE 4 Computational time evaluation using a) Matlab® and b) Fortran95 (for Hill48 only).

using Fortran95. In this case, all matrix operations were implemented using loop cycles, which Fortran95 runs much faster than Matlab. Looking at Figure 4b, if the same equivalent plastic strain increment with $p = 1\%$ is used, the direct method is around 5 times faster than the standard one. However, if the size of the strain increment increases (using an equivalent strain $p = 15\%$), the direct method is almost unaffected while the standard method is around 50-100 times slower. At large strains, indeed, in order to ensure convergence, the Euler-Backward method needs to use substepping and more iterations, increasing necessarily the computational time. However, it must be highlighted here that the strain increment cannot be increased excessively to avoid errors in the stress integration, see also discussion in Section 4.2.

4 | RESULTS AND DISCUSSION

The accuracy of the proposed algorithm for stress reconstruction in actual applications was evaluated comparing the stress field obtained from a finite element analysis. The case study is a numerical model of a tensile test on a double notched specimen (Figure 5), which provides heterogeneous stress/strain fields [21, 41]. The implicit commercial code ABAQUS-Standard®, assuming plane stress state condition, was used to build the models.

The hardening behaviour of the material was described with Swift's law:

$$\sigma_Y = K_H (\epsilon_0 + p)^{n_H}, \quad (42)$$

where σ_Y is the yield stress, p the equivalent plastic strain and $\xi^H = [K_H, \epsilon_0, n_H]$ the hardening constitutive parameters. Two FE models were built up using the material models reported in the previous section, i.e. Hill48 and Yld2000-2D, the latter implemented with the UMAT user-subroutine, following the approach described in [35, 40]. The characteristics of the FE model and the adopted constitutive parameters are listed in Table 1. The parameters were chosen to reproduce the behaviour of an anisotropic steel-like sheet metal [28].

Boundary conditions replicating a tensile test in displacement control were set fixing the nodes on the lower

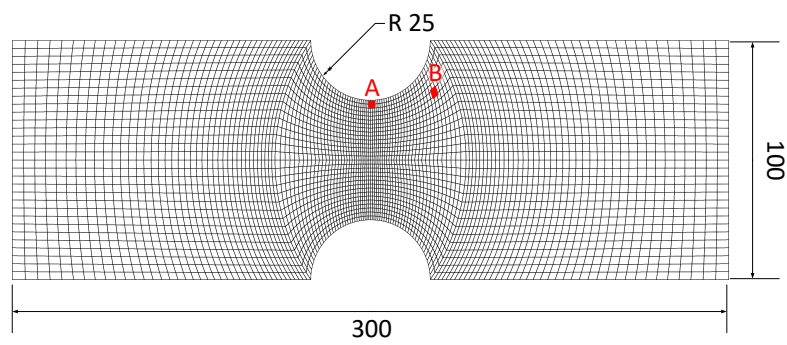


FIGURE 5 Specimen geometry and used mesh of the FE model. Dimensions are in mm.

TABLE 1 FE model characteristics and used constitutive parameters.

Elastic constants		Swift's law parameters		FEM characteristics					
E	200 GPa	K_H	1000 MPa	Element type		CPS4R			
ν	0.3	ϵ_0	0.05	Elements number		4200			
		n_H	0.5	Nodes number		4361			
Anisotropy parameters									
	R_0	R_{45}	R_{90}						
Hill48	2.2	1.5	1.8						
	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	a
Yld2000-2d	1.11	1.35	1.21	1.11	1.07	0.96	1.21	1.15	8

side of the specimen and applying to the upper side ones a total vertical displacement of 45 mm. In order to better assess the response of the algorithm to different strain increments and to evaluate the stress reconstruction during the transition between elastic and plastic regimes, the total displacement was divided into two phases:

- to look at first yielding and small strain increments, 30 steps were used to reach a displacement of 1 mm;
- to look at large strain increments and the effect of strain localization, 30 steps were used to accomplish the remaining 44 mm.

The FE model was built adopting CPS4R element, characterized by 4 nodes, bilinear shape functions and reduced integration method with hourglass control, which provides one integration point for each element. For each loading increment, the strain, stress and equivalent plastic strain were extracted from the integration point of every element and used to perform the verifications illustrated in the next sections.

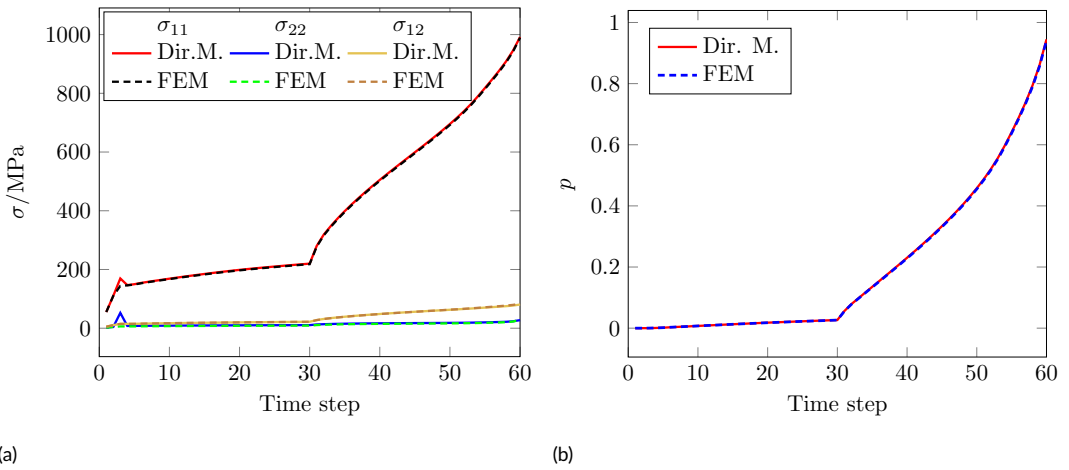


FIGURE 6 Comparison between stress reconstructed with the direct method and FE results, at integration point A: a) stress components b) equivalent plastic strain.

4.1 | Stress comparison at integration points

A first appraisal of proposed algorithm accuracy was conducted comparing, in specific Gauss points, the stress reconstructed with the direct method starting from the FE strain and the stress obtained from the ABAQUS® simulation, evaluated using the standard integration algorithms. The Hill48 model was used for this analysis. As illustrated in Figure 5, two Gauss points were selected: Point A situated at the centre of the specimen, where the maximum deformation occurs, and Point B situated in the outer edge of the plastically deformed zone, where less plastic flow is encountered during deformation and unloading with consequent transition from plasticity to elasticity is observed in the final stage of the test when the stress localizes towards the centre.

With respect to Point A, Figure 6a shows the comparison in terms of stress components and Figure 6b in terms of equivalent plastic strain. An almost perfect agreement is found except for step 3, where first yielding occurs and the approximation $\Delta \epsilon^p \approx \Delta \epsilon$ is not valid because the elastic part of the strain tensor is not negligible. However, the

direct integration scheme is able to correct this error in the subsequent stage. A perfect match is observed for the equivalent plastic strain.

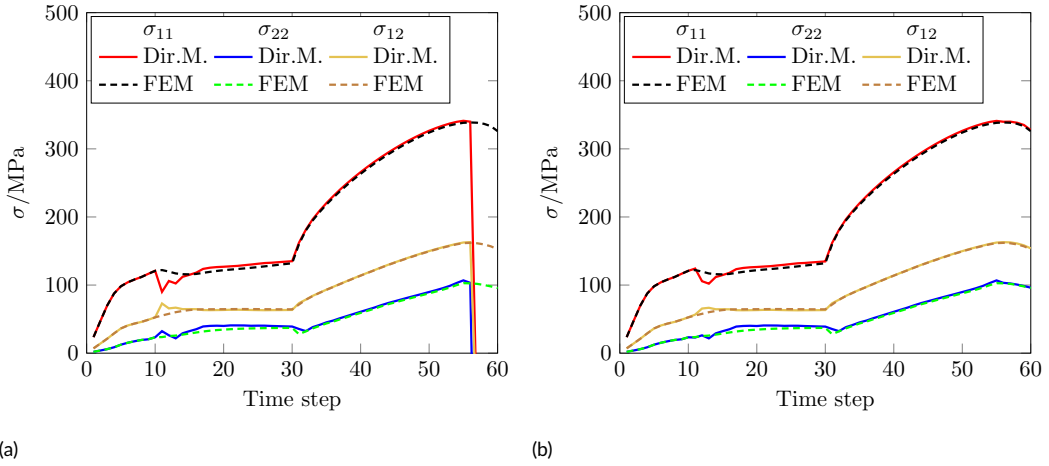


FIGURE 7 Comparison between stress reconstructed with the direct method and FE at integration point B: a) without residual threshold; b) with residual threshold ($R_{th} = \sigma_Y^0/5$).

A different behaviour is observed in Point B, as shown in Figure 7a. The direct integration algorithm is not able to reconstruct stress properly in two critical zones: the first, corresponding to time steps 11-12, whose differences are related again to early stage of plastic deformation; the latter at time step 56, where unloading started. As observed in Point A, the error in the initial yielding is recovered when the plastic flow becomes completely developed, instead, in the unloading phase, after step 56 the reconstruction goes entirely wrong.

The error is due to an erroneous check of the trial stress at the beginning of the procedure, see Figure 2. This mismatch can be solved introducing an appropriate residual threshold R_{th} in the yielding criterion of Eq. 22, viz:

$$\sigma_T(\sigma^{\text{trial}}, \xi_i^Y) > \sigma_Y(p^k) + R_{th} \quad (43)$$

The residual R_{th} was here chosen as a fraction of the initial yield stress σ_Y^0 . Figure 7b shows the stress reconstruction in Point B using the modified Eq. 43, with $R_{th} = \sigma_Y^0/5$. A good agreement is now found after the unloading while no variations are observed in the initial stage.

In terms of equivalent plastic strain, a good agreement is always found in Point B, see the graph of Figure 8a. In this case, the introduction of a residual threshold R_{th} does not lead to significant differences in terms of accumulated equivalent plastic strain. This is not surprising because, during the unloading, the accumulation of plastic strain is zero.

The analysis was so far limited to two indicative measurement points. In order to evaluate the global accuracy of the proposed algorithm, the internal work of the whole model was used as reference quantity. For each step i , the internal work W_i is obtained as:

$$W_i = \int_V \sigma_i : \epsilon_i dV \quad (44)$$

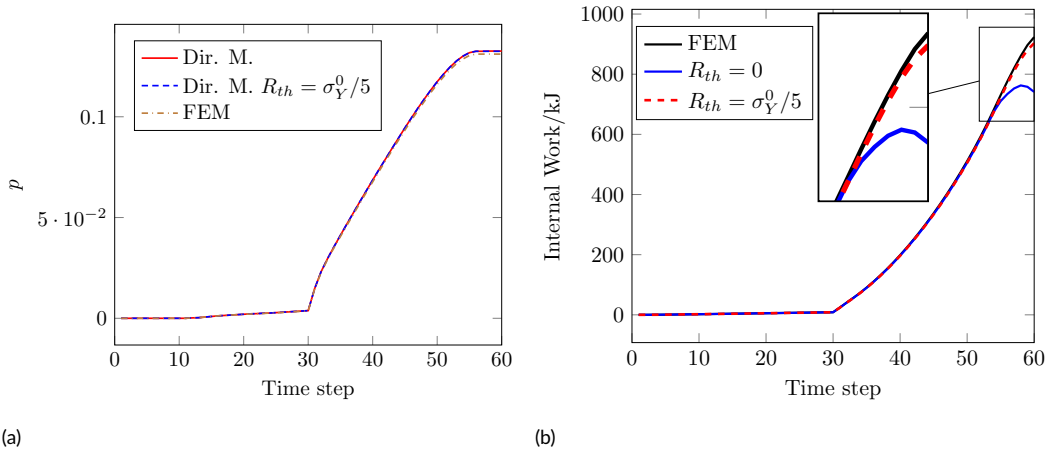


FIGURE 8 a) Comparison between equivalent plastic strain computed with the direct method and FE at integration point B and b) global comparison in terms of internal work using different R_{th} values.

where V is the volume of the specimen. Eq. 44 was evaluated using the stress obtained from the FE model and the one computed with the direct method. The internal work gives a better insight on the global impact of observed differences in stress computation.

Figure 8b reports the calculation of W_i along all the 60 steps of simulation for the notched specimen with Hill48. A perfect correspondence is found till time step 54, then there is a mismatching due to unloading if the residual threshold is not used. When $R_{th} = \sigma_Y^0/5$ a good agreement is also found in the final part of the test. It is worth noting that the stress oscillations observed in points A and B at the initial yielding phase play a marginal role if analysed in terms of global response.

The verification shown in this section was conducted using Hill48, however, the same outcomes were observed with Yld2000-2D and will not be discussed here. In order to give an example of the stress field reconstructed with the direct method, Figure 9 shows the stress components obtained at the last step of the FE model. In particular, Figure 9a refers to the Hill48 model with a material axes orientation of 0° and Figure 9b to the Yld2000-2D model with a material orientation of 30° . The orientation was changed to show that the direct method works properly independently from the anisotropic direction chosen.

In both graphs, the first row represents the reference stress field from FEM, the second row the stress field reconstructed using the direct method and the third row the difference between the two. A good agreement is found and the highest differences are observed in the external parts of the specimen, where the level of plastic deformation is very low. However, the difference is always rather low, almost negligible where the plastic flow is well developed, with a maximum error of few MPa. The white areas in the figures represent zones where the specimen remains always in the elastic range and, therefore, the stress is not computed using the proposed routine. This qualitative comparison confirms what already demonstrated from the global comparison of Figure 8b and locally in Figure 6-8a.

Figure 10 illustrates the equivalent plastic strain maps corresponding to the stress maps of Figure 9. In both examples, the value of the strain in the central zone of the specimen is larger than 0.25, involving large strain plasticity. As final remark, Figure 11 plots the stress-triaxiality map at three different load steps of the test. The stress-triaxiality, defined as the ratio of the hydrostatic stress to the equivalent stress, is a scalar value that characterizes the type of stress state in one point. As illustrated in Figure 11, because of geometrical and material non-linearities, the triaxiality

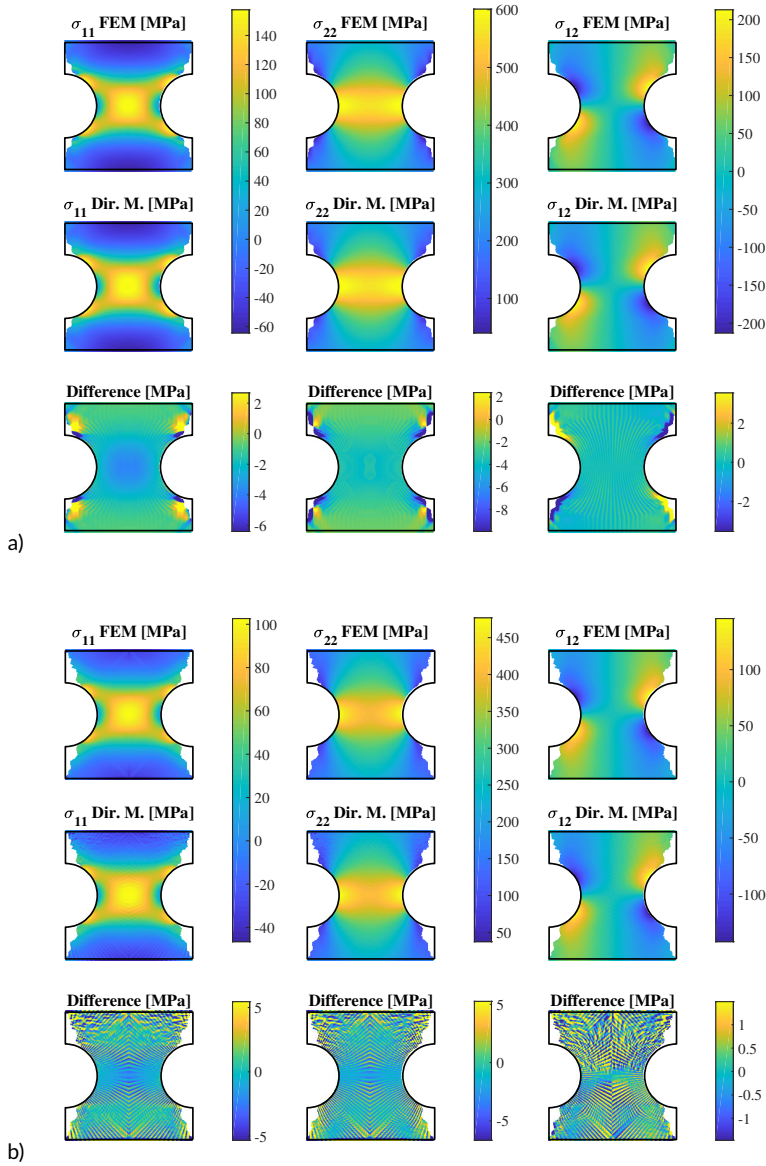


FIGURE 9 Comparison of the Cauchy stress components for two cases: a) Hill48 with material axes orientation of 0° , b) Yld2000-2D with material axes orientation of 30° .

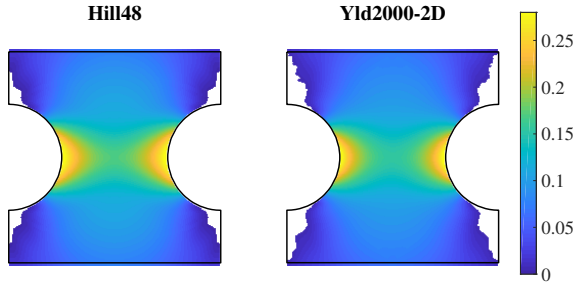


FIGURE 10 Equivalent plastic strain contour plots corresponding to the stress state reported in Figure 9, for Hill48 and Yld2000-2D, respectively.

map changes during the tests and, consequently, the material points undergo non-proportional loading. The studied test case, therefore, demonstrates that the proposed method can be successfully applied to non-proportional loading paths. Nonetheless, further verifications are necessary to validate it for more complex loading histories (e.g. reverse loading or combined tests).

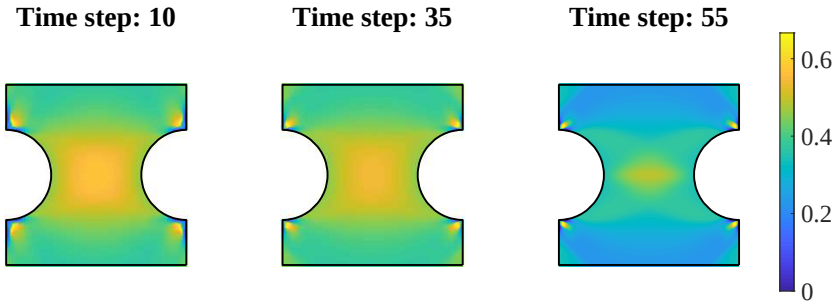


FIGURE 11 Stress-triaxiality map at three load steps for Hill48 model.

4.2 | Influence of the size of step increments

Another important advantage of the direct method is the ability to handle large strain increments. This property can have important industrial applications to recover the stress field from the strain field measured during experiments. In fact, in several engineering cases, it is not possible to follow the strain history with a high temporal resolution, e.g. in high-speed application or in forming processes where the deformation can be measured only at the end of the process.

Figure 12 shows the stress field reconstructed using 10 increments of the 60 available from the FE model and then using only the final step. The first row shows the reconstructed stress and the second row its difference with the reference one. Only the σ_{22} component is illustrated and the same constitutive models of Figure 9 are investigated, i.e. Hill48 (a) and Yld2000-2D with 30° orientation (b).

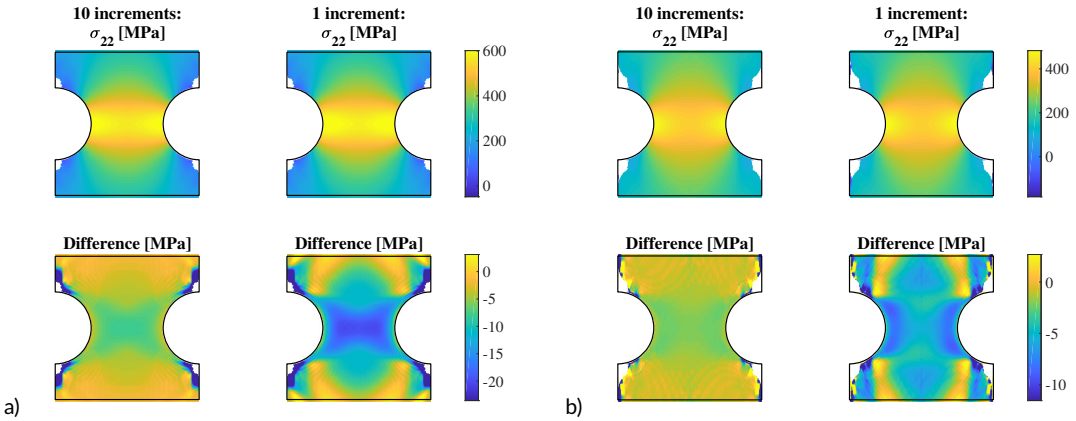


FIGURE 12 Stress reconstruction obtained using few increments, 10 and 1, respectively. The σ_{22} component was illustrated for Hill48 model (a) and Yld2000-2D model with 30° orientation (b).

For both material models, the proposed algorithm is able to reconstruct reasonably well the stress fields. Even with only one strain increment, the reconstruction is fairly good with a maximum error below 5%. The maximum error is observed at the centre of the specimen, where the deformation is larger and the loading history has a larger impact.

Notwithstanding, in the double-notched specimen the deformation history is rather simple and the loading path does not change significantly. In case of severe non-proportional loading (e.g. abrupt change of the loading direction), the one-step stress reconstruction is presumably not applicable. This topic can be investigated in future works.

4.3 | Influence of noise

The reconstruction of stress from strain data obtained experimentally with full-field measurements (e.g. DIC technique) is necessarily affected by noise uncertainty. In order to evaluate the robustness of the direct method when handling noisy data, the following analysis was carried out. Noisy strain maps ϵ_N were generated according to:

$$\epsilon_N = \epsilon_{FEM} + k_N Y \quad (45)$$

where ϵ_{FEM} indicates the FE strain data, Y is an array of random numbers generated according to a standard normal distribution and k_N is the standard deviation of the noise level. Two levels of noise are considered, with a standard deviation of 10^{-4} and 10^{-3} , respectively. The first represents a normal noise level in DIC measurement [42] where the latter a particularly high level, used to check the limit of the proposed algorithm.

Figure 13 shows the maps of the reconstructed stress (σ_{22} component) and, below, the difference between the reconstructed stress and the reference value obtained from the FE model.

Looking at the reconstructed stress, the direct method is able to reproduce the stress field also in case of high-level noise. In this case, the larger error is observed far from the central zone, where the level of plastic strain is lower and, accordingly, the signal-to-noise ratio is low.

The standard Euler backward radial-return algorithm was also employed to reconstruct the stress from the same

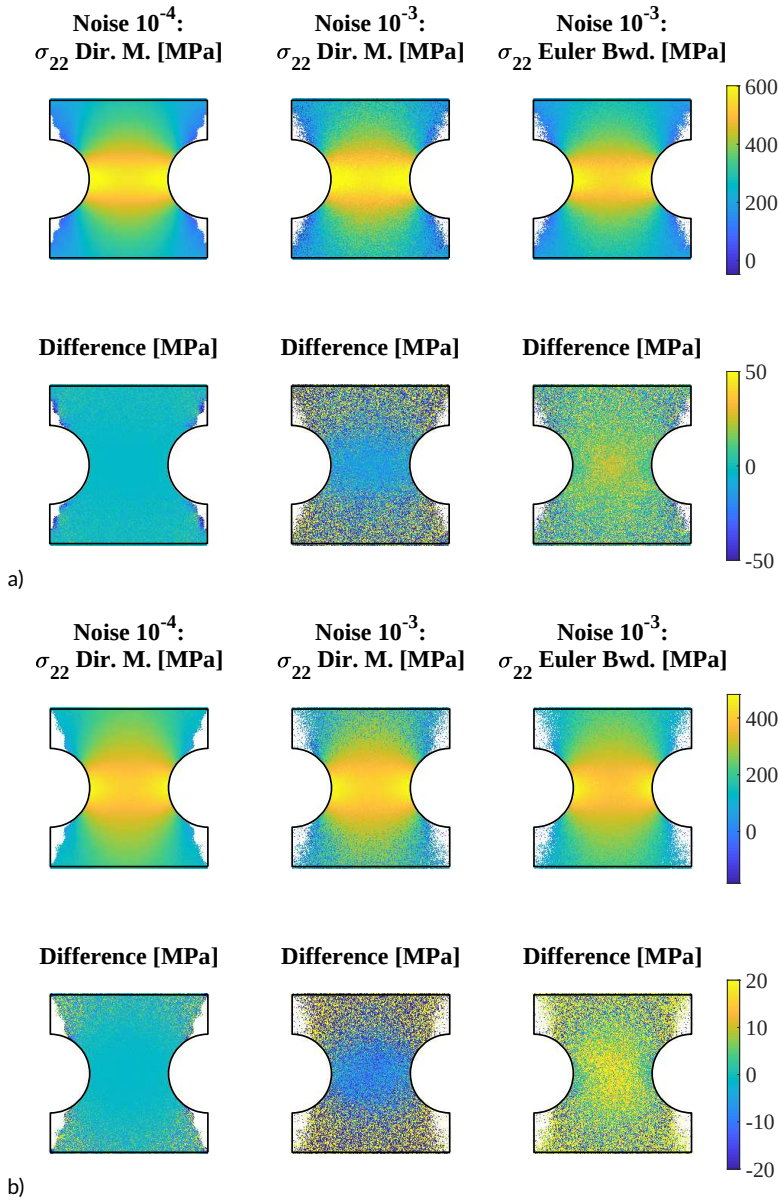


FIGURE 13 Stress integration from strain field affected by noise, using Hill48 model (a) and Yld2000-2D (b). A comparison between the proposed method and the standard Euler Backward algorithm is also presented for the high noise.

noisy data. Only the stress maps concerning the high level of noise were reported in Figure 13 because for the low level there were no significant differences in terms of reconstruction error. The Euler backward algorithm produces a worse reconstruction than the direct method in the central zone where the strain increments are larger, while it works similarly in the peripheral zones, where the level of plastic strain is low. This demonstrates how the proposed method is particularly well suited to reconstruct the stress in case of large strain increments.

From this analysis, it also turns out that the constitutive model plays a role in the sensitivity to noise of the stress reconstruction. For the same level of noise, the Hill48 model is more affected than Yld2000-2D. This is probably due to the shape of the yield locus and to the particular constitutive parameters used (see Table 1). In any case, this consideration does not modify the general conclusions.

5 | CONCLUSION

In this paper, a direct method to derive the stress field from strain data is presented. The method can be applied to any plasticity model with a convex yield surface. The proposed method was implemented on two constitutive models widely used in anisotropic plasticity, i.e. Hill48 and Yld2000-2D. An FE model of a tensile test on double-notched specimen was used as a benchmark. The main outcomes can be summarized as follows:

- the model works very well for large strain increments and brings to a remarkable reduction of the computational time;
- for small strain increments in the transition from elastic to plastic and vice versa, when the elastic strain is comparable to the plastic one, the model produces some inaccuracy and a classical radial-return approach is preferable;
- if a large part of the investigated body is under a developed plastic flow, the inaccuracy described in the previous point is negligible in terms of the global internal work;
- the method is able to reconstruct the stress with a reasonable accuracy also if only few steps of the loading history are available;
- the reconstruction works sufficiently also in conditions of strain field affected by high noise.

The principal application of this method is to inverse identification techniques, e.g. the virtual fields method, however, it can be adopted whenever a numerical method is required to derive the stress in plasticity, with possible applications in industrial processes (e.g. metal forming), testing procedures and other fields of experimental mechanics which involve severe plastic deformation.

Appendix

In Section 3, the incremental deformation theory based on minimum plastic work path (proportional logarithmic strain path) was adopted. Let us consider a time increment Δt , the deformation gradient $\mathbf{F}$ can be divided into the rotation and stretch tensors ($\mathbf{R}$ and $\mathbf{U}$) using the polar decomposition:

$$\mathbf{F}(t) = \mathbf{R}(t)\mathbf{U}(t) \quad \text{for } t_0 \leq t \leq t_0 + \Delta t \quad (46)$$

when decomposed into principal values and directions, $\mathbf{U}$ is defined as

$$\mathbf{U}(t) = \mathbf{Q}(t)\boldsymbol{\eta}(t)\mathbf{Q}(t) \quad (47)$$

where $\boldsymbol{\eta}$ is a diagonal tensor consisting in principal values and $\mathbf{Q}$ is an orthogonal tensor consisting of principal directions of $\mathbf{U}$. If the principal directions remain constant during the increment, the principal material lines of $\mathbf{U}$ coincide with those of $\mathbf{U}(t_0 + \Delta t)$ obtained for the current placement at time $t_0 + \Delta t$, i.e.:

$$\begin{aligned} \mathbf{U}(t) &= \mathbf{Q}(t_0 + \Delta t)\boldsymbol{\eta}(t)(t)\mathbf{Q}(t_0 + \Delta t) \\ \mathbf{U}(t) &= \mathbf{Q}(t_0 + \Delta t)\boldsymbol{\eta}(t)(t)\mathbf{Q}(t_0 + \Delta t) \end{aligned} \quad (48)$$

The spatial velocity gradient $\mathbf{L} = \mathbf{F}\mathbf{F}^{-1}$, defined in Eq. 4, can be decomposed as $\mathbf{L} = \mathbf{D} + \mathbf{W}$ where $\mathbf{W}$ is called the spin tensor. Using the polar decomposition (Eq. 46), it follows:

$$\mathbf{W} = \mathbf{R}\mathbf{R}^{-1} + \mathbf{R}\left(\mathbf{U}\mathbf{U}^{-1}\right)_a\mathbf{R}^T \quad \text{and} \quad \mathbf{D} = \mathbf{R}\left(\mathbf{U}\mathbf{U}^{-1}\right)_s\mathbf{R}^T \quad (49)$$

where the subscripts a and s represent the anti-symmetric and symmetric part of the tensor, respectively. Since the principal material directions are kept constant during the deformation of the increment, the principal lines of $\mathbf{U}$ and $\mathbf{U}$ coincide (Eq. 48) and, accordingly, $\mathbf{U}\mathbf{U}^{-1}$ is symmetric. Thus, Eq. 49 becomes:

$$\mathbf{W} = \mathbf{R}\mathbf{R}^{-1} \quad \text{and} \quad \mathbf{D} = \mathbf{R}\left(\mathbf{U}\mathbf{U}^{-1}\right)\mathbf{R}^T \quad (50)$$

The spin tensor $\mathbf{W}$ only depends on the rotation tensor $\mathbf{R}$ and the deformation is completely decoupled. A rotationless strain rate tensor $\tilde{\mathbf{D}}$ is now defined as:

$$\tilde{\mathbf{D}} = \mathbf{R}^T\mathbf{D}\mathbf{R} = \mathbf{U}\mathbf{U}^{-1} \quad (51)$$

the tensor $\tilde{\mathbf{D}}$ represent the value $\mathbf{D}$ measured with respect to the coordinate system that rotated by $\mathbf{R}$. The principal directions of $\tilde{\mathbf{D}}$ are stationary and aligned with the principal material lines of $\mathbf{U}(t_0 + \Delta t)$. According to the definition of $\tilde{\mathbf{D}}$, the following equation is valid:

$$\int_{t_0}^{t_0+\Delta t} \tilde{\mathbf{D}} \, dt = \ln \mathbf{U}|_{t_0}^{t_0+\Delta t} = \Delta \mathbf{E} \quad (52)$$

where $\mathbf{E}$ is the Hencky material logarithmic strain, i.e. $\mathbf{E} = \ln \mathbf{U}$. According to the incremental deformation theory, the minimum plastic work conditions are satisfied when deformation satisfies the following relations:

$$\begin{aligned}\Delta \mathbf{E} &= \alpha(t) \Delta \mathbf{E}(t_0 + \Delta t) \\ \tilde{\mathbf{D}} &= \alpha(t) \Delta \mathbf{E}(t_0 + \Delta t)\end{aligned}\quad (53)$$

where $\alpha(t)$ is a function of time, satisfying $\int_{t_0}^{t_0+\Delta t} \alpha(t) dt = 1$ and α always positive. The logarithmic strain increment $\Delta \mathbf{E}$ was used in Section 3 to identify the direction of the plastic flow and to integrate numerically the equivalent plastic strain. Since $\mathbf{E}$ is a Lagrangian quantity that describe the deformation in the material coordinate system, the stress reconstructed is also in the material coordinate system. The reconstructed stress tensor can be easily reported in the global system using the rotation tensor

$$\boldsymbol{\sigma}_{(\text{global})} = \mathbf{R} \boldsymbol{\sigma} \mathbf{R}^T \quad (54)$$

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