An overview of different asymptotic models for anisotropic three-layer plates with soft adhesive

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Abstract

We give an overview of the possible asymptotic models for a layered plate with soft adhesive. More specifically, we study the mechanical behavior of an anisotropic non homogeneous linearly elastic three-layer plate with soft adhesive, including the inertia forces, by means of the asymptotic expansion method. By defining a small parameter ε, associated with the size and the stiffness of the intermediate layer, we derive various limit models and their corresponding limit problems, by varying the thickness and rigidity ratios of the adherents and the adhesive layers.

Keywords: Asymptotic expansions, layered plates, soft adhesive

1. Introduction

The modeling of complex structures obtained joining simpler elements with highly contrasted geometric and/or material characteristics represents a source of a variety of problems of practical importance in all fields of engineering. The geometrical complexity of a multilayer structure requires an

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effort to deduce simplified mathematical models: these models must take into account the presence of different sizes and stiffnesses among each constituent of the structure. In the present work, we focus our attention to a particular structural assembly consisting in two plates bonded together by a soft adhesive middle layer.

This paper attempts to give a complete spectrum of the possible reduced models for a generic three-layer plate with soft adhesive, comprising all the possible choices of thickness and rigidity ratios between the intermediate layer and the surrounding plate-like bodies. These models are derived by means of the asymptotic expansion method. The asymptotic methods allow to determine the so-called limit model without any a priori assumptions on the displacements and/or stress field of the resulting limit models, by considering only the geometrical and mechanical peculiarities of the structure, such as the small thickness or the elastic moduli ratios of the different layers constituting the multilayer assembly.

More specifically, we analyze the time-dependent mechanical behavior of an anisotropic non homogeneous linearly elastic three-layer plates with soft adhesive. By defining a small parameter ε, which will tend to zero, we suppose that the thickness of the upper and lower plate-like bodies depends linearly on ε, while the thickness of the middle layer has order of magnitude ε^n, $n \in \mathbb{N}$, $n \geq 1$. Moreover, we assume that the elastic coefficients of the top and bottom plates are independent of ε , while the elastic moduli of the adhesive varies with ε^p, $p \in \mathbb{N}$, $p \geq 1$. Then, we derive a series of limit models by taking into account all the possible choices of the magnitudes $\{n, p\}$.

The asymptotic analysis has been successfully employed not only to formally justify classical theories of beams, plates and shells (see, e.g., Ciarlet (1997)), in the framework of linear and nonlinear elasticity, but also to deduce rational simplified models of structural elements bonded together with a thin elastic interphase, which represents the most peculiar bonded joint between two media. The actual computation of the solution of this problem is quite difficult, even if numerical methods are employed: this is mostly due to the thinness of the adhesive, which requires a fine mesh and, hence, an increase of the degrees of freedom of the system. Moreover, the adhesive has usually a different rigidity with respect to the adherents and this causes numerical instabilities in the stiffness matrix. The previous difficulties can be overcome by introducing a reduced model of the adhesive which can be treated as an interface, by assuming, for instance, that the upper and lower bodies are linked by a continuous distributions of springs. This model has been initially proposed in the milestone paper by Goland and Reissner (1944).

Within the theory of elasticity, the asymptotic analysis of a thin elastic interphase between two elastic materials has been deeply investigated through the years, by varying the rigidity ratios between the thin inclusion and the surrounding materials and by considering different geometry features. It is worth mentioning the pioneering work by Acerbi et al. (1988) on the variational behavior of the elastic energy of a thin inclusion using Γ-convergence. Moreover, we refer to the contributions by Licht and Michaille (1997), Abdelmoula et al. (1998), Geymonat et al. (1999), Klarbring (1991), Klarbring and Movchan (1998) and Krasucki et al. (2004), for mathematical models for linear and nonlinear bonded joints with a soft thin adhesive and, also, to the papers Bessoud et al. (2009), Bessoud et al. (2011), Bessoud et al.

(2008), Lebon and Rizzoni (2010), Lebon and Rizzoni (2011) and Lebon and Zaittouni (2010), for the case of multimaterials with thin plate-like and shell-like inclusions with high rigidity. In those papers, existence and uniqueness of the solution of the limit problem and weak, strong and Γ-convergence results have been fully described.

The mechanics behind the junction of two plates has been studied in several works in a rigorous mathematical framework: for instance, G. Geymonat and F. Krasucki Geymonat and Krasucki (1997) and Zaittouni et al. Zaittouni et al. (2002) analyzed two Kirchhoff-Love isotropic plates joint together by a thinner isotropic adhesive, by varying the order of magnitude of the elastic moduli of the intermediate layer; more recently, Serpilli (2005) and Serpilli and Lenci (2008) analyze the mechanical behavior of three different two-dimensional isotropic layered strips through the asymptotic methods: namely, the case of comparable thicknesses and weak adhesive (analogous to the case $n = 1$ and $p = 2$, presented and commented in Section 3.2), the case of comparable thicknesses and comparable rigidities, and, finally, the case of a thinner and stiffer adhesive (these two cases are not treated in the present paper). Besides, in Serpilli and Lenci (2012), the authors study the linear dynamics of a two-dimensional three-layer strip, by characterizing the limit natural high, low and mean frequencies. In these papers the authors recover one-dimensional simplified models, starting from two-dimensional layered strips. While, in the present work, starting from a three-dimensional stack of plates, we derive two-dimensional limit models. Another important contribution is the paper by Åslund (2005), in which the author performs an asymptotic analysis starting from a three-dimensional geometrical configuration: by defining a small parameter ε, the author considers a three-layer plate-like body constituted by two top and bottom plates of thickness ε, bonded by superposition with an adhesive layer of thickness ε^2. The three layers are made of Saint-Venant-Kirchhoff materials and the Lamé's constants of the adhesive have order of magnitude ε^3 with respect to those of the upper and lower bodies. A distinguishing feature of the resulting limit model is that the shear forces dominate in the adhesive, whose membrane displacements depend on the gap of the membrane displacements at the interface between the upper and lower plates. The paper by Schmidt (2008) is a remarkable work that deserves to be mentioned: indeed, the author analyzes the mechanical behavior of two bonded plates with a thin soft isotropic adhesive via the asymptotic expansion methods and derives a two-dimensional surface model for this particular joint. Different cases of rigidity ratios between the adherents and the adhesive have been studied and, moreover, higher-order corrector terms of the asymptotic expansion have been characterize in order to improve and make an error estimate of the solution of the derived models. Finally, it is also noteworthy the paper by Licht (2007), in which the author considers two linearly elastic plates linked by a soft linearly elastic isotropic adhesive: the assembly is made by abutting or by superposition. The reduced models are derived by means of a two small parameters asymptotic analysis, with formal convergence results, and they correspond to bonding two Kirchhoff-Love plates by a mechanical constraint depending on the magnitude of the chosen parameters.

The layout of the paper is as follows. In Section 2, we define the statement of the problem and we perform the asymptotic analysis by defining the dependences on ε of the geometrical and mechanical quantities. In Section 3, we derive the asymptotic models by fixing $n = 1$ and by varying the magnitude $p \geq 1$. In Section 4, we deduce the asymptotic models by fixing $n \geq 2$ and by changing the magnitude $p \geq 1$. In Section 5, we discuss the obtained results in an extensive way and, finally, in Section 6, we give some concluding remarks to the paper.

2. Statement of the problem

In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, except m, n, p, and the Einstein's summation convention with respect to the repeated indices is adopted. Let $\omega \in \mathbb{R}^2$ be a smooth domain in the plane spanned by vectors $\mathbf{e}_\alpha$, let γ_0 be a measurable subset of the boundary γ of the set ω, such that length $\gamma_0 > 0$, and let $0 < \varepsilon < 1$ be a dimensionless *small* real parameter which will tend to zero. For each ε, we define

$$\begin{aligned}
&\Omega^{m,\varepsilon} := \omega \times \mathcal{I}^{m,\varepsilon},\ \Omega^{+,\varepsilon} := \omega \times \mathcal{I}^{+,\varepsilon},\ \Omega^{-,\varepsilon} := \omega \times \mathcal{I}^{-,\varepsilon},\\
&\mathcal{I}^{m,\varepsilon} := (-h^{m,\varepsilon}, h^{m,\varepsilon}),\quad \mathcal{I}^{+,\varepsilon} := (h^{m,\varepsilon}, h^{m,\varepsilon} + 2h^{+,\varepsilon}),\\
&\mathcal{I}^{-,\varepsilon} := (-h^{m,\varepsilon} - 2h^{-,\varepsilon}, -h^{m,\varepsilon}),\quad \mathcal{I}^{\varepsilon} := (-h^{m,\varepsilon} - 2h^{-,\varepsilon}, h^{m,\varepsilon} + 2h^{+,\varepsilon})\\
&\Gamma_0^\varepsilon := \gamma_0 \times \mathcal{I}^\varepsilon,\quad \Gamma_\pm^\varepsilon := \omega \times \{\pm(h^{m,\varepsilon} + 2h^{\pm,\varepsilon})\},\quad S_\pm^\varepsilon := \omega \times \{\pm h^{m,\varepsilon}\}.
\end{aligned}$$

Hence the boundary of the set $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$ is partitioned into the lateral surface $\gamma \times \mathcal{I}^\varepsilon$ and the upper and lower faces Γ_+^ε and Γ_-^ε, and the lateral surface is itself partitioned as $\gamma \times \mathcal{I}^\varepsilon = (\gamma_0 \times \mathcal{I}^\varepsilon) \cup (\gamma_1 \times \mathcal{I}^\varepsilon)$, where $\gamma_1 := \gamma - \gamma_0$. We note with $\Gamma_1^\varepsilon := \gamma_1 \times \mathcal{I}^\varepsilon$, $\Gamma_1^{\pm,m,\varepsilon} := \gamma_1 \times \mathcal{I}^{\pm,m,\varepsilon}$, with self-explanatory notation, and $\widehat{\Gamma}^\varepsilon := \Gamma_\pm^\varepsilon \cup \Gamma_1^{\pm,\varepsilon}$. The upper and lower plate-like domains $\Omega^{+,\varepsilon}$ and $\Omega^{-,\varepsilon}$ are called the *adherents*, while the intermediate

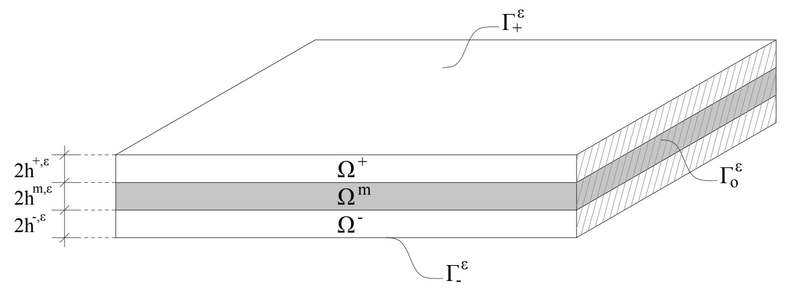

Figure 1: The reference configuration of the layered plate.

plate-like domain is called the *adhesive*.

We consider a three-layer plate occupying the reference configuration $\overline{\Omega}^{\varepsilon} \times [0,T]$ at a positive time $T > 0$, see Figure 1. We study the physical problem corresponding to the mechanical behavior of an anisotropic non homogeneous linearly elastic three-layer plate of thickness $2h^{\varepsilon} := 2h^{m,\varepsilon} + 2h^{+,\varepsilon} + 2h^{-,\varepsilon}$ and middle surface $\overline{\omega}$, with mass densities $\rho^{\pm,m,\varepsilon} > 0$. The sets $\Omega^{m,\varepsilon}$, $\Omega^{+,\varepsilon}$ and $\Omega^{-,\varepsilon}$ are filled by three anisotropic non homogeneous linearly elastic materials whose constitutive laws are defined as follows:

$$\sigma^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}) = C^{\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}),$$

where $\sigma^{\varepsilon}_{ij}$ represent the components of the Cauchy stress tensor, $e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) := \frac{1}{2}(\partial^{\varepsilon}_{\ell} u^{\varepsilon}_{k} + \partial^{\varepsilon}_{k} u^{\varepsilon}_{\ell})$ denote the components of the linearized strain tensor and $(C^{\varepsilon}_{ijk\ell})$ is the classical fourth order elasticity tensor. We assume that tensor $(C^{\varepsilon}_{ijk\ell})$ satisfies the classical symmetry and positivity properties. The plate is submitted to body forces $(f^{\varepsilon}_{i}) : \Omega^{\pm,\varepsilon} \times (0,T) \to \mathbb{R}^3$, acting in $\Omega^{\pm,\varepsilon}$, and surface forces $(g^{\varepsilon}_{i}) : \widehat{\Gamma}^{\varepsilon} \times (0,T) \to \mathbb{R}^3$, applied on $\widehat{\Gamma}^{\varepsilon}$. We suppose that the adhesive $\Omega^{m,\varepsilon}$ is not loaded. The initial conditions are posed in Ω^{ε}. Let $\mathbf{u}^{\varepsilon}_{0}$ and $\mathbf{u}^{\varepsilon}_{1}$ be, respectively, the displacement and the velocity at time $t = 0$; we have

$$\begin{cases} \mathbf{u}^\varepsilon(x^\varepsilon, 0) = \mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(x^\varepsilon, 0) = \dot{\mathbf{u}}^\varepsilon(0) = \mathbf{u}_1^\varepsilon & \text{in } \Omega^\varepsilon, \end{cases}$$

where $\dot{v} := \frac{\partial v}{\partial t}$ denotes the time derivative of function v. Let $\Sigma^\varepsilon \subset \partial\Omega^\varepsilon$, we introduce the following space of admissible displacements

$$V(\Omega^\varepsilon, \Sigma^\varepsilon) := \{\mathbf{v}^\varepsilon = (v_i^\varepsilon);\ \mathbf{v}^\varepsilon = \mathbf{0} \ \text{ on } \Sigma^\varepsilon\}.$$

The variational formulation of the problem defined over the variable domain Ω^ε takes the following form:

$$\begin{cases} \text{Find } \mathbf{u}^\varepsilon(t) \in V(\Omega^\varepsilon, \Gamma_0^\varepsilon),\ t \in (0,T), \ \text{ such that} \\ A^{-,\varepsilon}(\mathbf{u}^\varepsilon(t), \mathbf{v}^\varepsilon) + A^{+,\varepsilon}(\mathbf{u}^\varepsilon(t), \mathbf{v}^\varepsilon) + A^{m,\varepsilon}(\mathbf{u}^\varepsilon(t), \mathbf{v}^\varepsilon) = L^\varepsilon(\mathbf{v}^\varepsilon), \end{cases} \tag{1}$$

for all $\mathbf{v}^\varepsilon \in V(\Omega^\varepsilon, \Gamma_0^\varepsilon)$, with initial conditions $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon)$, where the bilinear forms $A^{\pm,\varepsilon}(\cdot,\cdot)$, $A^{m,\varepsilon}(\cdot,\cdot)$ and the linear form $L^\varepsilon(\cdot)$ are, respectively, defined by

$$A^{\pm,m,\varepsilon}(\mathbf{u}^\varepsilon(t), \mathbf{v}^\varepsilon) := \int_{\Omega^{\pm,m,\varepsilon}} \left\{ C_{ijk\ell}^{\pm,m,\varepsilon} e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon(t)) e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + \rho^{\pm,m,\varepsilon} \ddot{u}_i^\varepsilon(t) v_i^\varepsilon \right\} dx^\varepsilon,$$
$$L^\varepsilon(\mathbf{v}^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\widehat{\Gamma}^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon.$$

Remark 1. In order to guarantee the well-posedness of problem (1), suitable regularity properties have to be assumed for the unknowns, the initial data $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon)$ and applied loads f_i and g_i, (see, e.g., Ciarlet (1997)).

2.1. The rescaled problem and asymptotic expansions

In order to study the asymptotic behavior of the solution $\mathbf{u}^\varepsilon$ when ε tends to zero, we need to transform problem (1), posed on a variable domain Ω^ε, onto a problem posed on a fixed domain Ω (independent of ε). We suppose that the thicknesses of each layer of the plate admit the following dependences on ε: $h^{\pm,\varepsilon} := \varepsilon h^{\pm}$ and $h^{m,\varepsilon} := \varepsilon^n h^m$, $n \in \mathbb{N}$, $n \geq 1$, with $h^{\pm}$ and h^m independent of ε. The choice of n affects the order of magnitude between the thickness of the adhesive layer and the thicknesses on the upper and lower bodies. Accordingly, we let

$$
\begin{aligned}
&\Omega^m := \omega \times \mathcal{I}^m, \quad \Omega^+ := \omega \times \mathcal{I}^+, \quad \Omega^- := \omega \times \mathcal{I}^-, \quad \Omega = \Omega^+ \cup \Omega^m \cup \Omega^- \\
&\mathcal{I}^m := (-h^m, h^m), \quad \mathcal{I}^+ := (h^m, h^m + 2h^+), \\
&\mathcal{I}^- := (-h^m - 2h^-, -h^m), \quad \mathcal{I} := (-h^m - 2h^-, h^m + 2h^+) \\
&\Gamma_0 := \gamma_0 \times \mathcal{I},\ \Gamma_{\pm} := \omega \times \{\pm(h^m + 2h^{\pm})\},\ \widehat{\Gamma} := \Gamma_{\pm} \cup \Gamma_1^{\pm},\ S_{\pm} := \omega \times \{\pm h^m\}.
\end{aligned}
$$

Hence, we apply the usual change of variables (see, e.g., Ciarlet (1997) and Serpilli and Lenci (2008)):

$$
\pi^{\varepsilon} : \begin{cases}
x \equiv (\tilde{x}, x_3) \in \overline{\Omega}^+ \mapsto x^{\varepsilon} \equiv (\tilde{x}, \varepsilon^n h^m + \varepsilon x_3) \in \overline{\Omega}_{tr}^{+,\varepsilon}, \text{ with } \tilde{x} = (x_\alpha), \\
x \equiv (\tilde{x}, x_3) \in \overline{\Omega}^m \mapsto x^{\varepsilon} \equiv (\tilde{x}, \varepsilon^n x_3) \in \overline{\Omega}^{m,\varepsilon}, \\
x \equiv (\tilde{x}, x_3) \in \overline{\Omega}^- \mapsto x^{\varepsilon} \equiv (\tilde{x}, -\varepsilon^n h^m + \varepsilon x_3) \in \overline{\Omega}_{tr}^{-,\varepsilon},
\end{cases}
$$

where $\Omega_{tr}^{\pm,\varepsilon} := \{x \pm h^m \mathbf{e}_3,\ x \in \Omega^{\pm}\}$. In order to simplify the notation, we identify $\Omega_{tr}^{\pm,\varepsilon}$ with $\Omega^{\pm,\varepsilon}$. By using the bijection π^{ε}, one has

$$
\begin{aligned}
&\partial_\alpha^{\varepsilon} = \partial_\alpha \ \text{ and } \ \partial_3^{\varepsilon} = \tfrac{1}{\varepsilon}\partial_3 \ \text{ in } \Omega^{\pm}, \\
&\partial_\alpha^{\varepsilon} = \partial_\alpha \ \text{ and } \ \partial_3^{\varepsilon} = \tfrac{1}{\varepsilon^n}\partial_3 \ \text{ in } \Omega^m.
\end{aligned}
$$

In the sequel, only if necessary, we will note, respectively, with $\mathbf{v}^{\pm}$ and $\mathbf{v}^m$, the restrictions of functions $\mathbf{v}$ to $\Omega^{\pm}$ and Ω^m. With the unknown displacement field $\mathbf{u}^{\varepsilon}$, we associate the scaled unknown displacement field $\mathbf{u}(\varepsilon)$ defined by:

$$
u_\alpha^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon^2 u_\alpha(\varepsilon)(x, t) \text{ and } u_3^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon u_3(\varepsilon)(x, t) \quad \text{for all } x^{\varepsilon} = \pi^{\varepsilon} x \in \overline{\Omega}^{\varepsilon}.
$$

We likewise associate with any test functions $\mathbf{v}^\varepsilon$, the scaled test functions $\mathbf{v}$, defined by the scalings:

$$v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x) \text{ and } v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

Moreover, we suppose that the elastic coefficients of $\Omega^\pm$ are independent of ε, so that $C_{ijk\ell}^{\pm,\varepsilon} = C_{ijk\ell}^{\pm}$, while the elastic moduli of Ω^m depends on ε as follows

$$C_{ijk\ell}^{m,\varepsilon} = \varepsilon^p C_{ijk\ell}^m, \quad p \in \mathbb{N},\ p \geq 1.$$

Since $p \geq 1$, from a mechanical point of view, we are considering the case of a layered plate whose adhesive is softer with respect to the adherents. Let us assume that the data verify the following scaling assumptions:

$$\begin{array}{lll}
f_\alpha^\varepsilon(x^\varepsilon, t) = f_\alpha(x,t), & f_3^\varepsilon(x^\varepsilon, t) = \varepsilon f_3(x,t), & x \in \Omega^\pm,\ t \in (0,T), \\
g_\alpha^\varepsilon(x^\varepsilon, t) = g_\alpha(x,t), & g_3^\varepsilon(x^\varepsilon, t) = \varepsilon g_3(x,t), & x \in \Gamma_1^\pm,\ t \in (0,T), \\
g_\alpha^\varepsilon(x^\varepsilon, t) = \varepsilon g_\alpha(x,t), & g_3^\varepsilon(x^\varepsilon, t) = \varepsilon^2 g_3(x,t), & x \in \Gamma_\pm,\ t \in (0,T), \\
u_{0,\alpha}^\varepsilon(x^\varepsilon) = \varepsilon^2 u_{0,\alpha}(\varepsilon)(x), & u_{0,3}^\varepsilon(x^\varepsilon) = \varepsilon u_{0,3}(\varepsilon)(x), & x \in \Omega, \\
u_{1,\alpha}^\varepsilon(x^\varepsilon) = \varepsilon^2 u_{1,\alpha}(\varepsilon)(x), & u_{1,3}^\varepsilon(x^\varepsilon) = \varepsilon u_{1,3}(\varepsilon)(x), & x \in \Omega,
\end{array}$$

where functions f_i and g_i are independent of ε. In addition, we suppose that the mass densities satisfy:

$$\rho^{\pm,\varepsilon} = \varepsilon^2 \rho^\pm \quad \text{and} \quad \rho^{m,\varepsilon} = \varepsilon^{p+2} \rho^m,$$

where $\rho^{\pm,m} > 0$ are independent of ε. The above scaling of the mass densities allows to derive dynamical flexural models for the layered plate, (see. e.g., Ciarlet (1997) and Serpilli and Lenci (2012)). Moreover, the exponent list of the scaled mechanical quantities, namely unknowns and data, is such that the scaled energy functional remains bounded in the limit process when ε tends to zero, see Miara and Podio-Guidugli (2006).

For brevity, we will drop the dependence on the variable t in the variational equations. According to the previous hypothesis, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following rescaled problem

$$\begin{cases} \text{Find } \mathbf{u}(\varepsilon) \in V(\Omega, \Gamma_0),\ t \in (0,T), \quad \text{such that} \\ A^-(\mathbf{u}(\varepsilon), \mathbf{v}) + A^+(\mathbf{u}(\varepsilon), \mathbf{v}) + A^m_{n,p}(\mathbf{u}(\varepsilon), \mathbf{v}) = L(\mathbf{v}), \end{cases} \tag{2}$$

for all $\mathbf{v} \in V(\Omega, \Gamma_0)$, with initial conditions $(\mathbf{u}_0(\varepsilon), \mathbf{u}_1(\varepsilon))$, where the bilinear forms $A^\pm(\cdot,\cdot)$, $A^m_{n,p}(\cdot,\cdot)$ and the linear form $L(\cdot)$ are defined as follows

$$\begin{aligned} A^\pm(\mathbf{u}(\varepsilon), \mathbf{v}) := \int_{\Omega^\pm} \big\{ & C^\pm_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + \rho^\pm \ddot{u}_3(\varepsilon) v_3 + \varepsilon^2 \rho^\pm \ddot{u}_\alpha(\varepsilon) v_\alpha + \\ & + 2\varepsilon^{-1} C^\pm_{\alpha\beta\sigma 3} (e_{\sigma 3}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{\sigma 3}(\mathbf{v})) + \\ & + \varepsilon^{-2} \big(C^\pm_{\alpha\beta 33} (e_{33}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})) + \\ & \quad + 4 C^\pm_{\alpha 3\beta 3} e_{\beta 3}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) \big) + \\ & + 2\varepsilon^{-3} C^\pm_{\alpha 333} (e_{33}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) + e_{\alpha 3}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})) + \\ & + \varepsilon^{-4} C^\pm_{3333} e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) \big\}\, dx, \end{aligned}$$

$$
\begin{aligned}
A^m_{n,p}(\mathbf{u}(\varepsilon),\mathbf{v}) := \varepsilon^p \int_{\Omega^m} & \left\{\varepsilon^{n-1}\left(C^m_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + \rho^m \ddot{u}_3(\varepsilon) v_3\right) + \right.\\
& + \varepsilon^{n+1} \rho^m \ddot{u}_\alpha(\varepsilon) v_\alpha + \\
& + \varepsilon^{n-2} C^m_{\alpha\beta\sigma 3}(\partial_\sigma u_3(\varepsilon) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon))\partial_\sigma v_3) + \\
& + \varepsilon^{n-3} C^m_{\alpha 3\beta 3} \partial_\beta u_3(\varepsilon) \partial_\alpha v_3 + \\
& + \varepsilon^{-1} C^m_{\alpha\beta\sigma 3}(\partial_3 u_\sigma(\varepsilon) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon))\partial_3 v_\sigma) + \\
& + \varepsilon^{-2}\left(C^m_{\alpha\beta 33}(e_{33}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})) + \right.\\
& \qquad \left. + C^m_{\alpha 3\beta 3}(\partial_3 u_\beta(\varepsilon)\partial_\alpha v_3 + \partial_\beta u_3(\varepsilon)\partial_3 v_\alpha)\right) + \\
& + \varepsilon^{-3} C^m_{\alpha 333}(e_{33}(\mathbf{u}(\varepsilon))\partial_\alpha v_3 + \partial_\alpha u_3(\varepsilon) e_{33}(\mathbf{v})) + \\
& + \varepsilon^{-(n+1)} C^m_{\alpha 3\beta 3} \partial_3 u_\beta(\varepsilon) \partial_3 v_\alpha + \\
& + \varepsilon^{-(n+2)} C^m_{\alpha 333}(e_{33}(\mathbf{u}(\varepsilon))\partial_3 v_\alpha + \partial_3 u_\alpha(\varepsilon) e_{33}(\mathbf{v})) + \\
& \left. + \varepsilon^{-(n+3)} C^m_{3333} e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})\right\} dx,
\end{aligned}
$$

$$
L(\mathbf{v}) := \int_{\Omega^\pm} f_i v_i dx + \int_{\widehat{\Gamma}} g_i v_i d\Gamma.
$$

The aim of the present work is to study the behavior of the problem when ε tends to zero and to characterize the limit solution for each $n, p \geq 1$. From a practical point of view, this means that we are looking for a simplified model of the original problem whose solution is easier to compute and still a good approximation of the actual solution $\mathbf{u}(\varepsilon)$: mathematically, this means evaluate the $\lim_{\varepsilon\to 0}\mathbf{u}(\varepsilon)$.

Since in the rescaled problem (2) the parameter ε appears explicitly in polynomial form, we will look for the solution of the problem as a series of powers of ε:

$$
\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \dots \,. \tag{3}
$$

By substituting (3) into the rescaled problem (2) and by identifying the terms with identical power of ε, we obtain, as customary, a set of variational problems to be solved in order to characterize the limit displacement field $\mathbf{u}^0$ and its associated limit problem, for each $n, p \geq 1$. For the sake of brevity and in order to account of the real objectives of the present paper, we decide not to enter into the mathematical details and technicalities involving the solution of each variational subproblem and concentrate our attention on the resulting models and their comparison. In the sequel we will fix n, related to the thickness of the adhesive, and we will vary the exponent p related to stiffness of the adhesive. Each choice of the exponents $\{n, p\}$ will be associated with a different geometrical configuration of the three-layer plate, combined with a decreasing stiffness (since $p \geq 1$) of the adhesive layer with respect to the rigidity of the adherents: this will give rise to a series of different asymptotic models related to the specific choice of $\{n, p\}$.

3. Asymptotic models for $n = 1$

In this section we analyze the mechanical behavior of a three-layer plate, whose intermediate adhesive layer has a thickness comparable with the thicknesses of the adherents, see Figure 2.

We let the exponent $p \geq 1$ vary and we characterize the associated asymptotic models. In the case of $n = 1$, the rescaled bilinear form $A^m_{1,p}(\cdot,\cdot)$ can be rewritten in a very simple form, similar to $A^{\pm}(\cdot,\cdot)$. This will be very helpful

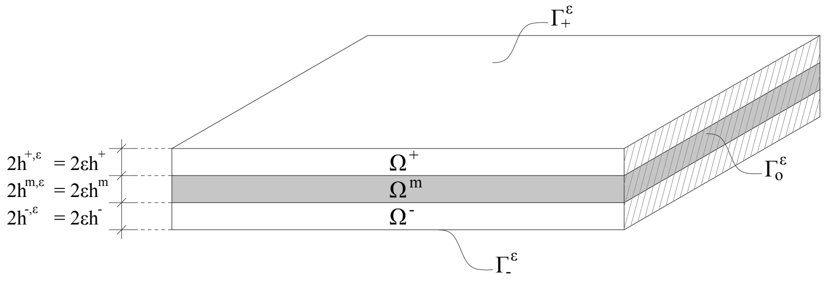

Figure 2: Layered plate with layers having comparable thicknesses.

for the consequent calculations. Indeed, one has

$$
\begin{aligned}
A^m_{1,p}(\mathbf{u}(\varepsilon),\mathbf{v}) := \varepsilon^p \int_{\Omega^m} \big\{ & C^m_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + \rho^m \ddot{u}_3(\varepsilon) v_3 + \varepsilon^2 \rho^m \ddot{u}_\alpha(\varepsilon) v_\alpha + \\
& + 2\varepsilon^{-1} C^m_{\alpha\beta\sigma 3} (e_{\sigma 3}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{\sigma 3}(\mathbf{v})) + \\
& + \varepsilon^{-2} \big(C^m_{\alpha\beta 33} (e_{33}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})) + \\
& \quad + 4 C^m_{\alpha 3 \beta 3} e_{\beta 3}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) \big) + \\
& + 2\varepsilon^{-3} C^m_{\alpha 333} (e_{33}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) + e_{\alpha 3}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v})) + \\
& + \varepsilon^{-4} C^m_{3333} e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) \big\} \, dx.
\end{aligned}
$$

3.1. Case $p = 1$

By choosing $p = 1$ in (2) and by identifying the terms with identical power of ε, we can characterize the solutions of the arising variational subproblems and, hence, the leading term of the asymptotic expansion $\mathbf{u}^0$.

Let us define the following geometrical quantities, corresponding, respectively, to the thicknesses and to the first and the second order moments of inertia associated with $\Omega^\pm$:

$$
J^\pm_{11} := \int_{\mathcal{I}^\pm} dx_3, \quad J^\pm_{12} = J^\pm_{21} := \int_{\mathcal{I}^\pm} x_3 dx_3, \quad J^\pm_{22} := \int_{\mathcal{I}^\pm} x_3^2 dx_3.
$$

Also, let

$$V_{KL}(\Omega,\Gamma_0) := \{\mathbf{v} = (v_i);\ e_{i3}(\mathbf{v}) = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

denote the space of the Kirchhoff-Love displacements, and

$$\begin{aligned} V_M(\omega,\gamma_0) &:= \{\mathbf{v}_H = (v_\alpha);\ v_\alpha(\tilde{x},x_3) = v_\alpha(\tilde{x}),\ \mathbf{v}_H = \mathbf{0} \text{ on } \gamma_0\}, \\ V_F(\omega,\gamma_0) &:= \{v_3;\ v_3(\tilde{x},x_3) = v_3(\tilde{x}),\ v_3 = \partial_\nu v_3 = 0 \text{ on } \gamma_0\}, \end{aligned}$$

denote the space of membrane and flexural displacements, respectively, defined on the middle plane of the layered plate ω. We recall that $\boldsymbol{\nu} = (\nu_\alpha)$ is the outer unit normal vector to γ and $\boldsymbol{\tau} = (-\nu_2,\nu_1)$ represents the unit tangent vector to γ.

By virtue of the asymptotic methods, we deduce that the limit displacement field $\mathbf{u}^0$ satisfies the Kirchhoff-Love kinematical assumptions, so that

$$\begin{cases} u_\alpha^{\pm,m,0}(\tilde{x},x_3) = \eta_\alpha(\tilde{x}) - x_3\partial_\alpha\eta_3(\tilde{x}),\ \boldsymbol{\eta}_H = (\eta_\alpha), \\ u_3^{\pm,m,0}(\tilde{x},x_3) = \eta_3(\tilde{x}), \end{cases}$$

and, thus, it belongs to the space $V_{KL}(\Omega,\Gamma_0)$. The limit displacement field $\mathbf{u}^0 \in V_{KL}(\Omega,\Gamma_0)$ is the solution of the following coupled limit problem:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in V_{KL}(\Omega,\Gamma_0),\ t \in (0,T),\ \text{such that} \\ \displaystyle\int_{\Omega^+} \left\{\widetilde{C}^{+}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + \rho^+ \ddot{u}_3^0 v_3\right\} dx + \\ \qquad + \displaystyle\int_{\Omega^-} \left\{\widetilde{C}^{-}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + \rho^- \ddot{u}_3^0 v_3\right\} dx = L(\mathbf{v}), \end{cases} \tag{4}$$

for all $\mathbf{v} \in V_{KL}(\Omega,\Gamma_0)$, where

$$\widetilde{C}^{\pm}_{\alpha\beta\sigma\tau} := C^{\pm}_{\alpha\beta\sigma\tau} - C^{\pm}_{\alpha\beta i3} d^{\pm}_{ij} C^{\pm}_{\sigma\tau j3},\ \text{with}\ (d^{\pm}_{ij}) := (C^{\pm}_{i3j3})^{-1}.$$

The reduced elasticity tensors $(\widetilde{C}^{\pm}_{\alpha\beta\sigma\tau})$ are symmetric and positive definite, see Caillerie (1984). The initial conditions are given by

$$\begin{cases} u_3^0(0) = u_{3,0}^0 = \eta_{3,0}, \\ \dot{u}_3^0(0) = u_{3,1}^0 = \eta_{3,1}. \end{cases}$$

As we can notice from Eq. (4) , the intermediate layer does not influence, from a mechanical point of view, the asymptotic behavior of the layered plate.

Let us suppose that the elastic materials are homogeneous, then the limit problem (4) can be rewritten in terms of the in-plane displacements $\boldsymbol{\eta}_H \in V_M(\omega, \gamma_0)$ and transversal displacement $\eta_3 \in V_F(\omega, \gamma_0)$, after an integration along the x_3-coordinate. We obtain, as customary, the following coupled problem defined over the middle plane of the plate ω:

$$\begin{cases} \text{Find } (\boldsymbol{\eta}_H, \eta_3) \in V_M(\omega, \gamma_0) \times V_F(\omega, \gamma_0), \; t \in (0,T), \quad \text{such that} \\ \displaystyle\int_\omega \left\{ \left(A^{11}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) - A^{12}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \right) e_{\alpha\beta}(\boldsymbol{\zeta}_H) + \tilde{\rho}\ddot{\eta}_3\zeta_3 + \right. \\ \qquad \left. + \left(-A^{12}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) + A^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \right) \partial_{\alpha\beta}\zeta_3 \right\} d\tilde{x} = \widetilde{L}(\boldsymbol{\zeta}), \end{cases} \tag{5}$$

for all $(\boldsymbol{\zeta}_H, \zeta_3) \in V_M(\omega, \gamma_0) \times V_F(\omega, \gamma_0)$, where

$$A^{\nu\mu}_{\alpha\beta\sigma\tau} := J^+_{\nu\mu}\widetilde{C}^+_{\alpha\beta\sigma\tau} + J^-_{\nu\mu}\widetilde{C}^-_{\alpha\beta\sigma\tau} = A^{\mu\nu}_{\alpha\beta\sigma\tau} \quad \text{and} \quad \tilde{\rho} := \rho^+ J^+_{11} + \rho^- J^-_{11},$$

and

$$\widetilde{L}(\boldsymbol{\zeta}) := \int_\omega (p_i\zeta_i - s_\alpha\partial_\alpha\zeta_3) d\tilde{x} + \int_{\gamma_1} (q_i\zeta_i - r_\alpha\partial_\alpha\zeta_3) d\gamma.$$

Moreover, the two-dimensional applied loads are

$$
\begin{aligned}
p_i &:= \int_{\mathcal{I}^+} f_i dx_3 + \int_{\mathcal{I}^-} f_i dx_3 + g_i^+ + g_i^-, \\
s_\alpha &:= \int_{\mathcal{I}^+} x_3 f_\alpha dx_3 + \int_{\mathcal{I}^-} x_3 f_\alpha dx_3 + (h^m + 2h^+) g_\alpha^+ - (h^m + 2h^-) g_\alpha^-, \\
q_i &:= \int_{\mathcal{I}^+} g_i dx_3 + \int_{\mathcal{I}^-} g_i dx_3, \\
r_\alpha &:= \int_{\mathcal{I}^+} x_3 g_\alpha dx_3 + \int_{\mathcal{I}^-} x_3 g_\alpha dx_3,
\end{aligned}
$$

where $g_i^\pm := g_i|_{\Gamma_\pm} = g_i(\tilde{x}, \pm(h^m + 2h^\pm))$ denote the restrictions of g_i to $\Gamma_\pm$.

We are now in position to rewrite the limit problem (5) in its differential form by using the Green's fomulae on ω. By posing

$$
\begin{aligned}
n_{\alpha\beta}(\boldsymbol{\eta}_H, \eta_3) &:= A^{11}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) - A^{12}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3, \\
m_{\alpha\beta}(\boldsymbol{\eta}_H, \eta_3) &:= -A^{12}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) + A^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3,
\end{aligned}
$$

which represent, respectively, the membrane stress tensor and the moment tensor of the plate, we obtain

$$\begin{cases} \textit{Field equations:} & \\ -\partial_\beta n_{\alpha\beta} = p_\alpha & \text{in } \omega \times (0,T), \\ \partial_{\alpha\beta} m_{\alpha\beta} + \tilde{\rho}\ddot{\eta}_3 = \tilde{p}_3 & \text{in } \omega \times (0,T), \\ \textit{Initial conditions:} & \\ \eta_3(0) = \eta_{3,0},\ \dot{\eta}_3(0) = \eta_{3,1} & \text{in } \omega \times (0,T), \\ \textit{Boundary conditions:} & \\ n_{\alpha\beta}\nu_\beta = q_\alpha & \text{on } \gamma_1 \times (0,T), \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = -r_\alpha\nu_\alpha & \text{on } \gamma_1 \times (0,T), \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = -\tilde{q}_3 & \text{on } \gamma_1 \times (0,T), \\ \eta_i = \partial_\nu\eta_3 = 0 & \text{on } \gamma_0 \times (0,T), \end{cases} \tag{6}$$

where $\tilde{p}_3 := p_3 + \partial_\alpha s_\alpha$ and $\tilde{q}_3 := q_3 - s_\alpha\nu_\alpha + \partial_\tau(r_\alpha\tau_\alpha)$.

Remark 2. It is interesting to notice that the simplified model reduces the three-layer plate into an equivalent single-layer plate, with a more complex constitutive behavior. Moreover, thanks to the particular scaling of the mass densities, we obtain a time-dependent flexural problem for the unknown η_3, while the dependence of the membrane displacement $\boldsymbol{\eta}_H$ upon the temporal variable t is only through the time-dependent functions f_α and g_α. Being a non-standard time-dependent problem, the membrane problem can be considered a *quasi-static problem*.

By the analysis of the limit problem, we can notice that it is strongly coupled and the reduced elastic coefficients are a combination of the membrane and flexural stiffnesses of the upper and lower plates. This coupling is due to the geometry of the layered plate and to the material reference frame.

Problem (5) can be decoupled into a membrane problem and a flexural problem, respectively, by considering, for instance, a symmetric layered plate from both a geometrical and mechanical point of view, i.e., $h^+ = h^- := h$, $\rho^+ = \rho^- := \rho$ and $\widetilde{C}^+_{\alpha\beta\sigma\tau} = \widetilde{C}^-_{\alpha\beta\sigma\tau} := \widetilde{C}_{\alpha\beta\sigma\tau}$. In this case, we have that $J^-_{12} = -J^+_{12}$ and, thus, coefficient $A^{12}_{\alpha\beta\sigma\tau}$ vanishes. Consequently, the limit problem reduces, respectively, to the following membrane problem

$$\begin{cases} \text{Find } \boldsymbol{\eta}_H \in V_M(\omega,\gamma_0),\ t \in (0,t), \quad \text{such that} \\ 4h \displaystyle\int_\omega \widetilde{C}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) e_{\alpha\beta}(\boldsymbol{\zeta}_H) d\tilde{x} = \int_\omega p_\alpha \zeta_\alpha d\tilde{x} + \int_{\gamma_1} q_\alpha \zeta_\alpha d\gamma, \end{cases}$$

for all $\boldsymbol{\zeta}_H \in V_M(\omega,\gamma_0)$, and flexural problem

$$\begin{cases} \text{Find } \eta_3 \in V_F(\omega,\gamma_0),\ t \in (0,t), \quad \text{such that} \\ \displaystyle\int_\omega \left\{ \frac{52}{3} h^3 \widetilde{C}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \partial_{\alpha\beta}\zeta_3 + 4\rho h \ddot{\eta}_3 \zeta_3 \right\} d\tilde{x} = \\ \qquad = \displaystyle\int_\omega (p_3\zeta_3 - s_\alpha \partial_\alpha \zeta_3) d\tilde{x} + \int_{\gamma_1} (q_3\zeta_3 - r_\alpha \partial_\alpha \zeta_3) d\gamma, \end{cases}$$

for all $\zeta_3 \in V_F(\omega,\gamma_0)$.

3.2. Cases $p = 2$ & $p = 3$

By choosing $p = 2$ or $p = 3$ in (2) and by identifying the terms with identical power of ε, we can characterize the solutions of the arising variational subproblems and, hence, the leading term of the asymptotic expansion $\mathbf{u}^0$. Concerning with $p = 2$, we obtain a generalization to the case of anisotropic plates of a previous result obtained in Serpilli and Lenci (2008), for layered elastic two-dimensional strips.

The obtained limit displacement field $\mathbf{u}^0$ verifies the following kinematical assumptions

$$\begin{cases} u_\alpha^{\pm,0}(\tilde{x}, x_3) = \eta_\alpha^\pm(\tilde{x}) - x_3 \partial_\alpha \eta_3(\tilde{x}), \\ u_3^{\pm,0}(\tilde{x}, x_3) = u_3^{m,0}(\tilde{x}, x_3) = \eta_3(\tilde{x}), \end{cases}$$

where $\eta_\alpha^\pm$ denote the two independent membrane displacements of $\Omega^\pm$. In the sequel, for the case $p = 2$, thanks to the particular form of the limit problem, we will explicitly characterize the expression of the in-plane displacements $u_\alpha^{m,0}$ in terms of $\eta_\alpha^\pm$. We notice that $\mathbf{u}^{\pm,0} \in V_{KL}(\Omega^\pm, \Gamma_0)$ have the form of a Kirchhoff-Love displacement field. The limit problem takes the following form:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in V_1(\Omega, \Gamma_0),\ t \in (0,T), \quad \text{such that} \\ \displaystyle\int_{\Omega^+} \left\{ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,+}) e_{\alpha\beta}(\mathbf{v}) + \rho^+ \ddot{u}_3^{0,+} v_3 \right\} dx + \\ \quad \displaystyle + \int_{\Omega^-} \left\{ \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,-}) e_{\alpha\beta}(\mathbf{v}) + \rho^- \ddot{u}_3^{0,-} v_3 \right\} dx + \\ \quad \displaystyle + \chi_2(p) \int_{\Omega^m} 4 \widetilde{C}^m_{\alpha 3 \beta 3} e_{\beta 3}(\mathbf{u}^{0,m}) e_{\alpha 3}(\mathbf{v}) dx = L(\mathbf{v}), \end{cases}$$

for all $\mathbf{v} \in V_1(\Omega, \Gamma_0)$, where $\widetilde{C}^m_{\alpha 3 \beta 3} := C^m_{\alpha 3 \beta 3} - \frac{C^m_{\alpha 333} C^m_{\beta 333}}{C^m_{3333}}$, and the space of admissible displacements is defined by

$$V_1(\Omega, \Gamma_0) := \{ \mathbf{v} = (v_i);\ \mathbf{v}^\pm \in V_{KL}(\Omega^\pm, \Gamma_0),\ \partial_3 v_3^m = 0,$$
$$\mathbf{v} = \mathbf{0} \text{ on } \Gamma^0,\ \mathbf{v}^\pm = \mathbf{v}^m \text{ on } S^\pm \}.$$

The function $\chi_q : p \in \mathbb{N} \mapsto [0, 1]$, depending on the choice of the exponent p, is an indicator function such that

$$\chi_q(p) := \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise}, \end{cases}$$

where $q = 2$ in this case. This function helps to distinguish between the two cases of study and their associated limit models.

Let us consider the case $p = 2$, with $\chi_2(2) = 1$. The limit problem can be simplified if one considers the structure of the bilinear form defined on Ω^m, which involves the derivatives along x_3. Indeed, by choosing test functions v_i with compact support in Ω^m, with $v_3 = 0$ in Ω^m, one has

$$\int_{\Omega^m} 2\widetilde{C}^m_{\alpha 3\beta 3} e_{\beta 3}(\mathbf{u}^{0,m}) \partial_3 v_\alpha dx = 0.$$

The previous variational problem implies the existence of a constant function $z_\alpha = z_\alpha(\tilde{x})$, such that $2\widetilde{C}^m_{\alpha 3\beta 3} e_{\beta 3}(\mathbf{u}^{0,m}) = z_\alpha$, and, thus,

$$\partial_3 u^{0,m}_\alpha = \widetilde{d}^m_{\alpha\beta} z_\beta - \partial_\alpha u^{0,m}_3,$$

with $(\widetilde{d}^m_{\alpha\beta}) := (\widetilde{C}^m_{\alpha 3\beta 3})^{-1}$. By integrating expression above along x_3 between $-h^m$ and h^m and by imposing the continuity of the displacements on $S^\pm$, we get that $z_\alpha = \frac{1}{2h^m}\widetilde{C}^m_{\alpha 3\beta 3}[\![\eta_\beta]\!]$, and $u^{0,m}_\alpha$ becomes an affine function of x_3:

$$u^{0,m}_\alpha(\tilde{x}, x_3) = \langle \eta_\alpha \rangle(\tilde{x}) + \frac{x_3}{2h^m}\left([\![\eta_\alpha]\!] - 2h^m \partial_\alpha \eta_3\right)(\tilde{x}), \tag{7}$$

where $\langle f \rangle := \frac{1}{2}(f|_{S^+} + f|_{S^-})$ and $[\![f]\!] := f|_{S^+} - f|_{S^-}$ represent, respectively, the mean value and the jump function between the values of f at the interfaces $S^\pm$. In this case the middle layer behaves as an elastic interface of stiffness $K^m_{\alpha\beta} := \frac{\widetilde{C}^m_{\alpha 3\beta 3}}{2h^m}$, whose displacement is an appropriate linear combination of the membrane displacements of the adherents.

Considering the case of homogeneous materials, by virtue of relation (7), the limit problem can be reformulated in a reduced form defined on the middle plane ω, where the membrane displacements $\boldsymbol{\eta}^\pm_H = (\eta^\pm_\alpha)$ and the transversal displacements η_3 are the primary unknowns. So, we get

$$\left\{\begin{array}{l} \text{Find } (\boldsymbol{\eta}_H^+, \boldsymbol{\eta}_H^-, \eta_3) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0),\ t \in (0,T),\ \text{such that} \\ \displaystyle\int_\omega \Big\{ \Big(-J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) + A^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) \partial_{\alpha\beta}\zeta_3 + \tilde{\rho}\ddot{\eta}_3\zeta_3 + \\ \quad + \Big(J_{11}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^+) + \chi_2(p) K^m_{\alpha\beta} [\![\eta_\beta]\!] [\![\zeta_\alpha]\!] + \\ \quad + \Big(J_{11}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^-) \Big\} d\tilde{x} = \widetilde{L}(\boldsymbol{\zeta}), \end{array}\right. \tag{8}$$

for all $(\boldsymbol{\zeta}_H^+, \boldsymbol{\zeta}_H^-, \zeta_3) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0)$. Let

$$n^\pm_{\alpha\beta}(\boldsymbol{\eta}_H^\pm, \eta_3) := J_{11}^\pm \widetilde{C}^\pm_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^\pm) - J_{12}^\pm \widetilde{C}^\pm_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3,$$
$$\widehat{m}_{\alpha\beta}(\boldsymbol{\eta}_H^+, \boldsymbol{\eta}_H^-, \eta_3) := -J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) + A^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3,$$

be the membrane stress tensors and the moment tensor of the equivalent plate, respectively. The limit variational problem (8) is equivalent to the following differential problem (coupled membrane-flexural problem):

$$\left\{\begin{array}{ll} \textit{Field equations:} & \\ -\partial_\beta n^+_{\alpha\beta} + \chi_2(p) K^m_{\alpha\beta} [\![\eta_\beta]\!] = p^+_\alpha & \text{in } \omega \times (0,T), \\ -\partial_\beta n^-_{\alpha\beta} - \chi_2(p) K^m_{\alpha\beta} [\![\eta_\beta]\!] = p^-_\alpha & \text{in } \omega \times (0,T), \\ \partial_{\alpha\beta}\widehat{m}_{\alpha\beta} + \tilde{\rho}\ddot{\eta}_3 = \tilde{p}_3 & \text{in } \omega \times (0,T), \\ \textit{Initial conditions:} & \\ \eta_3(0) = \eta_{3,0},\ \dot{\eta}_3(0) = \eta_{3,1} & \text{in } \omega \times (0,T), \\ \textit{Boundary conditions:} & \\ n^\pm_{\alpha\beta}\nu_\beta = q^\pm_\alpha & \text{on } \gamma_1 \times (0,T), \\ \widehat{m}_{\alpha\beta}\nu_\alpha\nu_\beta = -r_\alpha\nu_\alpha & \text{on } \gamma_1 \times (0,T), \\ \partial_\alpha \widehat{m}_{\alpha\beta}\nu_\beta + \partial_\tau(\widehat{m}_{\alpha\beta}\nu_\alpha\tau_\beta) = -\tilde{q}_3 & \text{on } \gamma_1 \times (0,T), \\ \eta^\pm_\alpha = \eta_3 = \partial_\nu\eta_3 = 0 & \text{on } \gamma_0 \times (0,T). \end{array}\right.$$

Remark 3. Let us consider the case of $p = 2$. The previous system shows that the interlayer behaves as an elastic interphase of stiffness $K^m_{\alpha\beta}$ which reacts to the difference of the membrane displacements at the interface between the top and bottom layers. The membrane and flexural behaviors of the plate are strongly coupled due to the heterogeneity of the layered plate and the different layer thicknesses. However, by taking into account the static case of a symmetric three-layer plate ($h^+ = h^-$, $J^+_{11} = J^-_{11} := J_{11}$, $J^+_{12} = -J^-_{12} := J_{12}$, $J^+_{22} = J^-_{22} := J_{22}$ and $\widetilde{C}^+_{\alpha\beta\sigma\tau} = \widetilde{C}^-_{\alpha\beta\sigma\tau} = \widetilde{C}_{\alpha\beta\sigma\tau}$), by defining a new variable $\psi_\alpha := \eta^+_\alpha - \eta^-_\alpha = [\![\eta_\alpha]\!]$, we can rewrite the differential system in a decoupled form, as follows (see, also, Serpilli and Lenci (2008)):

$$\begin{cases} \frac{J_{11}J_{22}-J_{12}^2}{J_{12}}\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\alpha\beta\sigma}\psi_\tau + 2K^m_{\alpha\beta}\partial_\beta\psi_\alpha = \tilde{p}_3 + \frac{J_{22}}{J_{12}}\partial_\alpha(p^+_\alpha - p^-_\alpha), \\ \widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma\tau}\eta_3 = \frac{1}{J_{12}}\left\{J_{11}\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma}\psi_\tau + 2K^m_{\alpha\beta}\psi_\beta - (p^+_\alpha - p^-_\alpha)\right\}, \\ \widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma}\eta^+_\tau = \frac{1}{J_{11}}\left\{K^m_{\alpha\beta}\psi_\beta + J_{11}\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma}\psi_\tau + p^-_\alpha\right\}, \\ \widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma}\eta^-_\tau = \frac{1}{J_{11}}\left\{-K^m_{\alpha\beta}\psi_\beta - J_{11}\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\beta\sigma}\psi_\tau + p^+_\alpha\right\}. \end{cases}$$

Remark 4. If we consider the choice $p = 3$, the indicator function $\chi_2(3) = 0$ and, thus, the mechanical constraint involving the membrane displacements of the adherents, appearing in (8), namely $K^m_{\alpha\beta}[\![\eta_\beta]\!][\![\zeta_\alpha]\!]$, vanishes. In the present case, no information about $u^{0,m}_\alpha$ can be deduced from the variational formulation of the limit problem. From a mechanical point of view, this means that the adhesive middle layer does not affect the membrane behavior of the top and bottom plates, whose in-plane deformations are totally independent and separate from one another.

3.3. Case $p \geq 4$

If we choose $p \geq 4$ in (2), we can easily characterize the limit displacement field $\mathbf{u}^0$ and its associated limit problem.

In this case we obtain that the limit displacement field $\mathbf{u}^0$ satisfies

$$\begin{cases} u_\alpha^{\pm,0}(\tilde{x}, x_3) = \eta_\alpha^{\pm}(\tilde{x}) - x_3 \partial_\alpha \eta_3^{\pm}(\tilde{x}), \\ u_3^{\pm,0}(\tilde{x}, x_3) = \eta_3^{\pm}(\tilde{x}), \end{cases}$$

where $\eta_i^{\pm}$ denote the independent membrane and transversal displacements of $\Omega^{\pm}$. Thus, $\mathbf{u}^{\pm,0} \in V_{KL}(\Omega^{\pm}, \Gamma_0)$ is a Kirchhoff-Love displacement. The limit problem reads as follows

$$\begin{cases} \text{Find } \mathbf{u}^0 \in V_2(\Omega, \Gamma_0),\ t \in (0,T), \quad \text{such that} \\ \displaystyle\int_{\Omega^+} \left\{ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,+}) e_{\alpha\beta}(\mathbf{v}) + \rho^+ \ddot{u}_3^{0,+} v_3 \right\} dx + \\ \quad + \displaystyle\int_{\Omega^-} \left\{ \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,-}) e_{\alpha\beta}(\mathbf{v}) + \rho^- \ddot{u}_3^{0,-} v_3 \right\} dx + \\ \quad + \chi_4(p) \displaystyle\int_{\Omega^m} C^m_{3333} e_{33}(\mathbf{u}^{0,m}) e_{33}(\mathbf{v}) dx = L(\mathbf{v}), \end{cases}$$

for all $\mathbf{v} \in V_2(\Omega, \Gamma_0)$, where

$$V_2(\Omega, \Gamma_0) := \left\{ \mathbf{v} = (v_i);\ \mathbf{v}^{\pm} \in V_{KL}(\Omega^{\pm}, \Gamma_0),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ v_3^{\pm} = v_3^m \text{ on } S^{\pm} \right\}.$$

Considering $p = 4$, with $\chi_4(4) = 1$, and the homogeneous case, by virtue of the structure of the bilinear form defined on Ω^m, involving the x_3-derivatives, we can characterize the transversal displacement $u_3^{m,0}$. By choosing test functions v_3 with compact support in Ω^m, we obtain that

$$\int_{\Omega^m} C^m_{3333} e_{33}(\mathbf{u}^{0,m}) e_{33}(\mathbf{v}) dx = 0,$$

which implies the existence of a constant function $c = c(\tilde{x})$ such that $C^m_{3333}\partial_3 u_3^{m,0} = c$. Now, by integrating this expression between $-h^m$ and h^m, and by imposing the continuity of the displacements on $S^\pm$, we obtain, as customary,

$$u_3^{0,m}(\tilde{x}, x_3) = \langle \eta_3 \rangle(\tilde{x}) + \frac{x_3}{2h^m}[\![\eta_3]\!](\tilde{x}). \tag{9}$$

In this case the adhesive behaves as an elastic interphase of stiffness $K^m_{33} := \frac{C^m_{3333}}{2h^m}$, reacting to the gap of the transversal displacements of the adherents. No information arises on $u_\alpha^{m,0}$, meaning that the membrane behaviors of the upper and lower plate can be considered completely independent. The limit problem can be simplified by using (9), as follows

$$\begin{cases} \text{Find } (\boldsymbol{\eta}_H^+, \boldsymbol{\eta}_H^-, \eta_3^+, \eta_3^-) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0) \times V_F(\omega,\gamma_0) \ \text{ s. t.} \\ \displaystyle\int_\omega \Big\{ \Big(J_{11}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^+ \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^+) + \rho^+ J_{11}^+ \ddot{\eta}_3^+ \zeta_3^+ + \\ \quad + \Big(-J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) + J_{22}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^+ \Big) \partial_{\alpha\beta}\zeta_3^+ + \chi_4(p) K^m_{33} [\![\eta_3]\!][\![\zeta_3]\!] + \\ \quad + \Big(J_{11}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^- \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^-) + \rho^- J_{11}^- \ddot{\eta}_3^- \zeta_3^- + \\ \quad + \Big(-J_{12}^- \widetilde{C}^{\cdot}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) + J_{22}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^- \Big) \partial_{\alpha\beta}\zeta_3^- \Big\} \, d\tilde{x} = \widetilde{L}(\boldsymbol{\zeta}), \end{cases} \tag{10}$$

for all $(\boldsymbol{\zeta}_H^+, \boldsymbol{\zeta}_H^-, \zeta_3^+, \zeta_3^-) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0) \times V_F(\omega,\gamma_0)$, $t \in (0,T)$.

The variational problem (10) corresponds to the following coupled differential problem. By defining

$$\begin{aligned} n^\pm_{\alpha\beta}(\boldsymbol{\eta}_H^\pm, \eta_3^\pm) &:= J_{11}^\pm \widetilde{C}^\pm_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^\pm) - J_{12}^\pm \widetilde{C}^\pm_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^\pm, \\ m^\pm_{\alpha\beta}(\boldsymbol{\eta}_H^\pm, \eta_3^\pm) &:= -J_{12}^\pm \widetilde{C}^{\cdot}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^\pm) + J_{22}^\pm \widetilde{C}^\pm_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3^\pm, \end{aligned}$$

we have

$$\begin{cases} \textit{Field equations:} & \\ -\partial_\beta n^+_{\alpha\beta} = p^+_\alpha & \text{in } \omega \times (0,T), \\ -\partial_\beta n^-_{\alpha\beta} = p^-_\alpha & \text{in } \omega \times (0,T), \\ \partial_{\alpha\beta} m^+_{\alpha\beta} + \chi_4(p) K^m_{33} [\![\eta_3]\!] + \rho^+ J^+_{11} \ddot{\eta}^+_3 = \tilde{p}^+_3 & \text{in } \omega \times (0,T), \\ \partial_{\alpha\beta} m^-_{\alpha\beta} - \chi_4(p) K^m_{33} [\![\eta_3]\!] + \rho^- J^-_{11} \ddot{\eta}^-_3 = \tilde{p}^-_3 & \text{in } \omega \times (0,T), \\ \textit{Initial conditions:} & \\ \eta^\pm_3(0) = \eta^\pm_{3,0},\ \dot{\eta}^\pm_3(0) = \eta^\pm_{3,1} & \text{in } \omega \times (0,T), \\ \textit{Boundary conditions:} & \\ n^\pm_{\alpha\beta}\nu_\beta = q^\pm_\alpha & \text{on } \gamma_1 \times (0,T), \\ m^\pm_{\alpha\beta}\nu_\alpha\nu_\beta = -r^\pm_\alpha \nu_\alpha & \text{on } \gamma_1 \times (0,T), \\ \partial_\alpha m^\pm_{\alpha\beta}\nu_\beta + \partial_\tau (m^\pm_{\alpha\beta}\nu_\alpha\tau_\beta) = -\tilde{q}^\pm_3 & \text{on } \gamma_1 \times (0,T), \\ \eta^\pm_i = \partial_\nu \eta^\pm_3 = 0 & \text{on } \gamma_0 \times (0,T). \end{cases}$$

Remark 5. For all other exponents $p > 4$, the middle layer mechanically disappears and the adherents behaves as two independent Kirchhoff-Love plates. The mechanical interpretation is straightforward: the adhesive becomes too weak to bear loads and, hence, it cannot be perceived in the final simplified model.

4. Asymptotic models for $n \geq 2$

In this section we analyze the mechanical behavior of a three-layer plate, in which the adhesive layer is considered to be *thinner* and *softer* with respect to the adherents, see Figure 3.

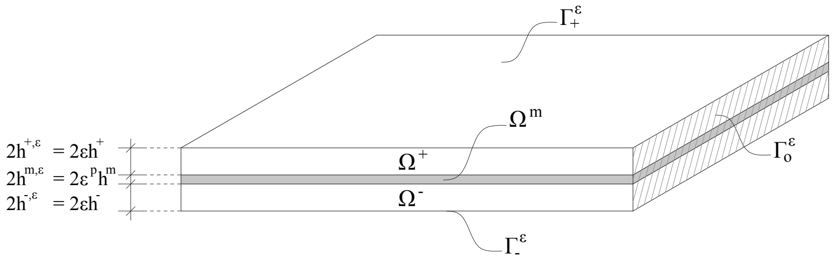

Figure 3: Layered plate with thin adhesive layer.

The asymptotic procedure allows to derive a set of limit models with different mechanical behaviors, depending on the choice of the exponents $\{n, p\}$, with $n \geq 2$ and $p \geq 1$.

4.1. Case $1 \leq p < n + 1$

Let us consider the case of $1 \leq p < n+1$ in the rescaled problem (2). First of all, we denote with

$$\bar{J}_{11}^{\pm} = J_{11}^{\pm}, \quad \bar{J}_{12}^{\pm} = \bar{J}_{21}^{\pm} := \int_{\mathcal{I}^{\pm}} (x_3 \mp h^m) dx_3, \quad \bar{J}_{22}^{\pm} := \int_{\mathcal{I}^{\pm}} (x_3 \mp h^m)^2 dx_3,$$

the modified first and the second order moments of inertia associated with $\Omega^{\pm}$. Note that $\bar{J}_{12}^{\pm} = J_{12}^{\pm} \mp h^m J_{11}^{\pm}$ and $\bar{J}_{22}^{\pm} = J_{22}^{\pm} + (h^m)^2 J_{11}^{\pm} \mp 2h^m J_{12}^{\pm}$.

The limit model kinematics verifies the following relations:

$$\begin{cases} u_{\alpha}^{\pm,0}(\tilde{x}, x_3) = \eta_{\alpha}(\tilde{x}) - (x_3 \mp h^m)\partial_{\alpha}\eta_3(\tilde{x}), \\ u_{\alpha}^{m,0}(\tilde{x}, x_3) = \eta_{\alpha}(\tilde{x}), \\ u_3^{\pm,0}(\tilde{x}, x_3) = u_3^{m,0}(\tilde{x}, x_3) = \eta_3(\tilde{x}). \end{cases}$$

Clearly, $\mathbf{u}^{\pm,0} \in V_{KL}(\Omega^{\pm}, \Gamma_0)$ verify the Kirchhoff-Love kinematical assumptions, while $\mathbf{u}^{m,0}$ is independent of the through-the-thickness coordinate x_3.

The variational formulation of the limit problem takes the following form:

$$\left\{\begin{array}{l}\text{Find } \mathbf{u}^0 \in V_3(\Omega, \Gamma_0),\ t \in (0,T),\ \text{such that} \\ \displaystyle\int_{\Omega^+} \left\{ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + \rho^+ \ddot{u}^0_3 v_3 \right\} dx + \\ \qquad + \displaystyle\int_{\Omega^-} \left\{ \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + \rho^- \ddot{u}^0_3 v_3 \right\} dx = L(\mathbf{v}), \end{array}\right. \tag{11}$$

for all $\mathbf{v} \in V_3(\Omega, \Gamma_0)$, where

$$V_3(\Omega, \Gamma_0) := \{ \mathbf{v} = (v_i);\ \mathbf{v}^\pm \in V_{KL}(\Omega^\pm, \Gamma_0)\ ,$$
$$\partial_3 \mathbf{v}^m = \mathbf{0},\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ \mathbf{v}^\pm = \mathbf{v}^m \text{ on } S^\pm \} \,.$$

Considering homogeneous materials, the limit problem (11) can be rewritten in terms of the in-plane displacements $\boldsymbol{\eta}_H \in V_M(\omega, \gamma_0)$ and transversal displacement $\eta_3 \in V_F(\omega, \gamma_0)$, after an integration along the x_3-coordinate. We obtain, as customary, the following coupled problem defined over the middle plane of the plate ω:

$$\left\{\begin{array}{l}\text{Find } (\boldsymbol{\eta}_H, \eta_3) \in V_M(\omega, \gamma_0) \times V_F(\omega, \gamma_0),\ t \in (0,T), \ \ \text{such that} \\ \displaystyle\int_\omega \big\{ \big(\bar{A}^{11}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) - \bar{A}^{12}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3 \big)\, e_{\alpha\beta}(\boldsymbol{\zeta}_H) + \tilde{\rho} \ddot{\eta}_3 \zeta_3 + \\ \qquad + \big(-\bar{A}^{12}_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H) + \bar{A}^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3 \big)\, \partial_{\alpha\beta} \zeta_3 \big\}\, d\tilde{x} = \bar{L}(\boldsymbol{\zeta}), \end{array}\right.$$

for all $(\boldsymbol{\zeta}_H, \zeta_3) \in V_M(\omega, \gamma_0) \times V_F(\omega, \gamma_0)$, where

$$\bar{A}^{\nu\mu}_{\alpha\beta\sigma\tau} := \bar{J}^+_{\nu\mu} \widetilde{C}^+_{\alpha\beta\sigma\tau} + \bar{J}^-_{\nu\mu} \widetilde{C}^-_{\alpha\beta\sigma\tau} = \bar{A}^{\mu\nu}_{\alpha\beta\sigma\tau},$$

and

$$\bar{L}(\boldsymbol{\zeta}) := \int_\omega (\bar{p}_i \zeta_i - \bar{s}_\alpha \partial_\alpha \zeta_3) d\tilde{x} + \int_{\gamma_1} (\bar{q}_i \zeta_i - \bar{r}_\alpha \partial_\alpha \zeta_3) d\gamma.$$

Moreover, the modified two-dimensional applied loads are

$$
\begin{aligned}
&\bar{p}_i = p_i, \quad \bar{q}_i = q_i, \\
&\bar{s}_\alpha := \int_{\mathcal{I}^+} (x_3 - h^m) f_\alpha dx_3 + \int_{\mathcal{I}^-} (x_3 + h^m) f_\alpha dx_3 + 2h^+ g_\alpha^+ - 2h^- g_\alpha^-, \\
&\bar{r}_\alpha := \int_{\mathcal{I}^+} (x_3 - h^m) g_\alpha dx_3 + \int_{\mathcal{I}^-} (x_3 + h^m) g_\alpha dx_3.
\end{aligned}
$$

We omit the differential form of the limit problem, since it admits a similar formulation as the one obtained in Sect. 3.1, (see Eq. (6)).

Remark 6. It is important to remark that the form of the limit problem (11) is analogous to the one obtained in Sect. 3.1, for the case $n = p = 1$, (see Eq. (4)). However, the involved kinematics are quite different: in the present case, the displacements of the adhesive depends just on the in-plane coordinates x_α, i.e., $\partial_3 \mathbf{v}^m = \mathbf{0}$, while, for what concerns with case of Sect. 3.1, $\mathbf{v}^m$ is a Kirchhoff-Love displacement, meaning that $e_{i3}(\mathbf{v}^m) = 0$.

4.2. Cases $p = n + 1$ & $p = n + 2$

By choosing $p = n + 1$ or $p = n + 2$ in (2) and by identifying the terms with identical power of ε, we can characterize the solutions of the arising variational subproblems and, hence, the leading term of the asymptotic expansion $\mathbf{u}^0$. If $n = 2$ and $p = 3$, we derive an analogous model of the one obtained in Åslund (2005), for bonded nonlinearly elastic plates.

The limit displacement field $\mathbf{u}^0$ verifies the following kinematical assumptions

$$
\begin{cases}
u_\alpha^{\pm,0}(\tilde{x}, x_3) = \eta_\alpha^\pm(\tilde{x}) - x_3 \partial_\alpha \eta_3(\tilde{x}), \\
u_3^{\pm,0}(\tilde{x}, x_3) = u_3^{m,0}(\tilde{x}, x_3) = \eta_3(\tilde{x}),
\end{cases}
$$

where $\eta_\alpha^\pm$ denote the two independent membrane displacements of $\Omega^\pm$. In the sequel, thanks to the particular form of the limit problem, we will explicitly characterize the expression of the in-plane displacements $u_\alpha^{m,0}$ in terms of $\eta_\alpha^\pm$, for the case $p = n+1$. Note that $\mathbf{u}^{\pm,0} \in V_{KL}(\Omega^\pm, \Gamma_0)$. The limit problem takes the following form:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in V_4(\Omega, \Gamma_0),\ t \in (0,T), \quad \text{such that} \\ \displaystyle\int_{\Omega^+} \left\{ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,+}) e_{\alpha\beta}(\mathbf{v}) + \rho^+ \ddot{u}_3^{0,+} v_3 \right\} dx + \\ \quad + \displaystyle\int_{\Omega^-} \left\{ \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\mathbf{u}^{0,-}) e_{\alpha\beta}(\mathbf{v}) + \rho^- \ddot{u}_3^{0,-} v_3 \right\} dx + \\ \quad + \chi_{n+1}(p) \displaystyle\int_{\Omega^m} \widetilde{C}^m_{\alpha 3 \beta 3} \partial_3 u_\beta^{0,m} \partial_3 v_\alpha dx = L(\mathbf{v}), \end{cases}$$

for all $\mathbf{v} \in V_4(\Omega, \Gamma_0)$, where

$$V_4(\Omega, \Gamma_0) := \{ \mathbf{v} = (v_i);\ \mathbf{v}^\pm \in V_{KL}(\Omega^\pm, \Gamma_0),\ \partial_3 v_3^m = 0, \\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ \mathbf{v}^\pm = \mathbf{v}^m \text{ on } S^\pm \}.$$

By means of the same procedure adopted in Sect. 3.2, taking into account homogeneous materials, thanks to the structure of the bilinear form defined on Ω^m, we can characterize the membrane displacement $u_\alpha^{0,m}$ as an affine function of x_3, depending on $\eta_\alpha^\pm$, for the case $p = n+1$. Thus, one has

$$u_\alpha^{0,m}(\tilde{x}, x_3) = \langle \eta_\alpha \rangle (\tilde{x}) + \frac{x_3}{2h^m} \left([\![\eta_\alpha]\!] - 2h^m \partial_\alpha \eta_3 \right)(\tilde{x}).$$

Hence, the limit problem can be reformulated in an alternative form, in which

$\boldsymbol{\eta}_H^+, \boldsymbol{\eta}_H^-, \eta_3$ represent the primary unknowns:

$$\begin{cases} \text{Find } (\boldsymbol{\eta}_H^+, \boldsymbol{\eta}_H^-, \eta_3) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0),\ t \in (0,T),\ \text{such that} \\ \displaystyle\int_\omega \Big\{ \Big(-J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) + A^{22}_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) \partial_{\alpha\beta}\zeta_3 + \\ \quad + \Big(J_{11}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^+) - J_{12}^+ \widetilde{C}^+_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^+) + \\ \quad + \Big(J_{11}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta}_H^-) - J_{12}^- \widetilde{C}^-_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \Big) e_{\alpha\beta}(\boldsymbol{\zeta}_H^-) + \\ \quad + \chi_{n+1}(p) K^m_{\alpha\beta} \left([\![\eta_\beta]\!] - 2h^m \partial_\beta \eta_3 \right)\left([\![\zeta_\alpha]\!] - 2h^m \partial_\alpha \zeta_3 \right) + \tilde{\rho}\ddot{\eta}_3 \zeta_3 \Big\} \, d\tilde{x} = \widetilde{L}(\boldsymbol{\zeta}), \end{cases} \tag{12}$$

for all $(\boldsymbol{\zeta}_H^+, \boldsymbol{\zeta}_H^-, \zeta_3) \in V_M(\omega,\gamma_0) \times V_M(\omega,\gamma_0) \times V_F(\omega,\gamma_0)$. The limit variational problem (12) is equivalent to the following differential problem (coupled membrane-flexural problem):

$$\begin{cases} \textit{Field equations:} \\ -\partial_\beta n^+_{\alpha\beta} + \chi_{n+1}(p) K^m_{\alpha\beta}([\![\eta_\beta]\!] - 2h^m \partial_\beta \eta_3) = \bar{p}^+_\alpha & \text{in } \omega \times (0,T), \\ -\partial_\beta n^-_{\alpha\beta} - \chi_{n+1}(p) K^m_{\alpha\beta}([\![\eta_\beta]\!] - 2h^m \partial_\beta \eta_3) = \bar{p}^-_\alpha & \text{in } \omega \times (0,T), \\ \partial_{\alpha\beta}\widehat{m}_{\alpha\beta} + 2\chi_{n+1}(p) h^m K^m_{\alpha\beta} \partial_\alpha([\![\eta_\beta]\!] - 2h^m \partial_\beta \eta_3) + \tilde{\rho}\ddot{\eta}_3 = \tilde{\bar{p}}_3 & \text{in } \omega \times (0,T), \\ \textit{Initial conditions:} \\ \eta_3(0) = \eta_{3,0},\ \dot{\eta}_3(0) = \eta_{3,1} & \text{in } \omega \times (0,T), \\ \textit{Boundary conditions:} \\ n^\pm_{\alpha\beta}\nu_\beta = \bar{q}^\pm_\alpha & \text{on } \gamma_1 \times (0,T), \\ \widehat{m}_{\alpha\beta}\nu_\alpha\nu_\beta = -\bar{r}_\alpha \nu_\alpha & \text{on } \gamma_1 \times (0,T), \\ \partial_\alpha \widehat{m}_{\alpha\beta}\nu_\beta + \partial_\tau(\widehat{m}_{\alpha\beta}\nu_\alpha\tau_\beta) + \\ \qquad + 2\chi_{n+1}(p) h^m K^m_{\alpha\beta}([\![\eta_\beta]\!] - 2h^m \partial_\beta \eta_3)\nu_\alpha = -\tilde{\bar{q}}_3 & \text{on } \gamma_1 \times (0,T), \\ \eta^\pm_\alpha = \eta_3 = \partial_\nu \eta_3 = 0 & \text{on } \gamma_0 \times (0,T). \end{cases}$$

where $\tilde{\bar{p}}_3 := \bar{p}_3 + \partial_\alpha \bar{s}_\alpha$ and $\tilde{\bar{q}}_3 := \bar{q}_3 - \bar{s}_\alpha \nu_\alpha + \partial_\tau(\bar{r}_\alpha \tau_\alpha)$.

4.3. Case $p \geq n+3$

Considering the case of $p \geq n+3$, we deduce the same expressions of the limit displacement field and its associated limit problem, obtained in Sect. 3.3 for $n=1$ and $p \geq 4$. The kinematics of the layered plate is characterized by two independent membrane displacements $\eta_\alpha^\pm$ and by two independent transversal displacements $\eta_3^\pm$ for the top and bottom plates. Concerning with the case $p=n+3$, the transversal displacements are linked together by a linear constraint in Ω^m, while, for the case $p>n+3$, the adherents are two independent Kirchhoff-Love plates, deforming separately from one another.

5. Discussion on the results

This section is aimed at summarizing the previously obtained results and comparing the different asymptotic models for a three-layer plate with soft adhesive, comprising all the possible choice of thickness and rigidity ratios between the adhesive and the adherents.

The obtained limit mechanical behaviors depend strongly on the relative ratios between the thickness and the stiffness properties of the plate constituents. A common feature, clearly visible in all models for each choice of thickness ratio, is represented by the behavior of the layered plate after a consecutive reduction of the middle layer rigidity. By fixing $n \geq 1$, the gradual decrease of the adhesive stiffness ($p \geq 1$) entails the mechanical separation of the adherents which occurs in successive steps: at the beginning, we have perfect adhesion among the different layers and the plate behaves as a monolithic single-layer plate; then, the membrane separation starts (involving the membrane displacements $\eta_\alpha^\pm$): once the in-plane separation has taken place, we achieve a transversal separation of the upper and lower plates (involving the transversal displacements $\eta_3^{\pm}$), finally behaving as two separate independent Kirchhoff-Love plates.

In the sequel we give a brief recap of the obtained results, distinguishing the main mechanical behaviors for a particular choice of thickness ratio, namely $n \geq 1$. As already mentioned above, we can identify three different scenarios:

i) *Perfect adhesion:* $1 \leq p < n+1$. Concerning with this particular choice of rigidity ratio, we obtain that the layered plate behaves as an equivalent single-layer plate, whose interlayer does not influence the global mechanical behavior of the plate assembly. The main distinction between the case $n = 1$ & $p = 1$ (Section 3.1) and $n \geq 2$ & $1 \leq p < n+1$ (Section 4.1) relies in the different limit kinematics, while the structure of the limit problem remains slightly the same (see Eqs. (4)-(11)). More specifically, for what concerns with the first case, the plate behaves as a homogenized Kirchhoff-Love single-layer plate, as shown in Figure 4; according to the second case, we still have perfect adhesion but the membrane and transversal displacements of the intermediate layer are both independent of the x_3-coordinate, see Figure 5: this is mainly due to the *thinness* of the adhesive with respect to the adherents thickness.

ii) *In-plane separation:* $p = n+1$ & $p = n+2$. According to this choice of the exponents p, the limit kinematics is characterized by two different membrane displacements for the top and bottom plates, noted with $\eta_\alpha^{\pm}$, still sharing the

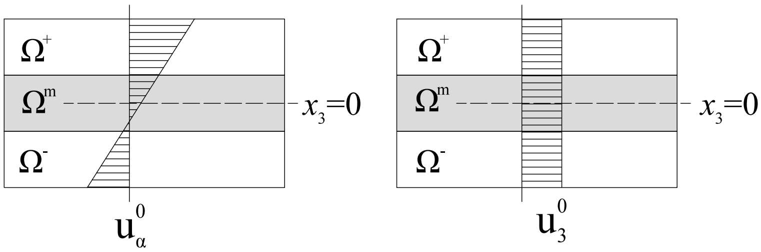

Figure 4: Limit kinematics for $n = 1$ & $p = 1$.

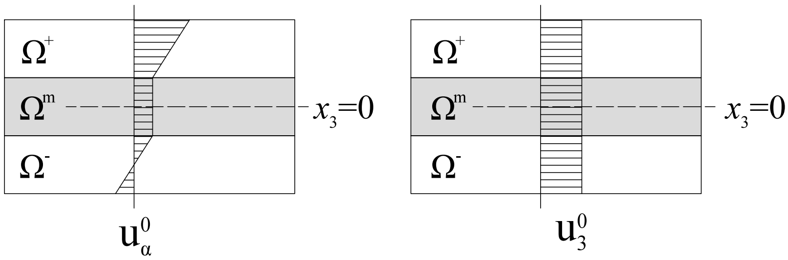

Figure 5: Limit kinematics for $n \geq 2$ & $1 \leq p < n + 1$.

same transversal displacement. When $p = n + 1$, the intermediate layer behaves as an elastic interphase (linear in-plane springs) reacting to the gap between the values of the in-plane displacements at the interface (see Figure 6). While, when $p = n + 2$, the adhesive layer does not affect anymore the overall membrane behavior. Indeed, we encounter a discontinuity of the membrane displacements at the interfaces with the upper and lower layers (see Figure 7) and, thus, the in-plane separation is achieved.

iii) *Transversal separation:* $p \geq n + 3$. Once the membrane separation occurred, meaning that the membrane displacements for the top and bottom plates are completely independent from one another, then the transversal separation takes place. In this case, we obtain two different transversal displacements $\eta_3^{\pm}$ for the adherents, influencing the slopes of the normal fibers to

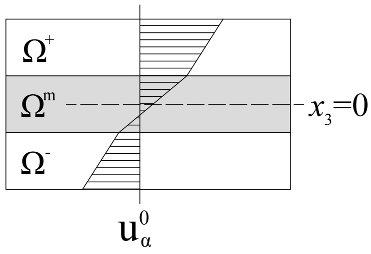

Figure 6: Limit kinematics for $n \geq 1$ & $p = n + 1$.

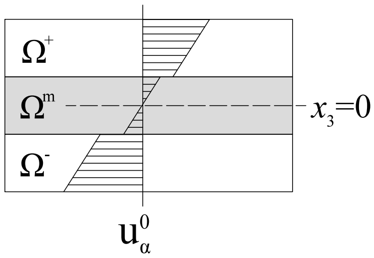

Figure 7: Limit kinematics for $n \geq 1$ & $p = n + 2$.

the middle plane of the upper and lower layers, respectively. No distinctions have been highlighted between the case $n = 1$ and $n \geq 2$. For $p = n + 3$, the upper and lower plates are still connected together by an internal constraint on the independent transversal displacements (linear transversal through-the-thickness springs), as shown in Figure 8. While, for $p > n + 3$, the adherents deform separately as two autonomous Kirchhoff-Love plates, as in Figure 9.

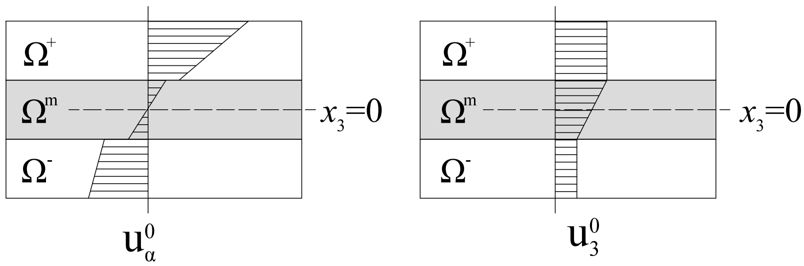

Figure 8: Limit kinematics for $n \geq 1$ & $p = n + 3$.

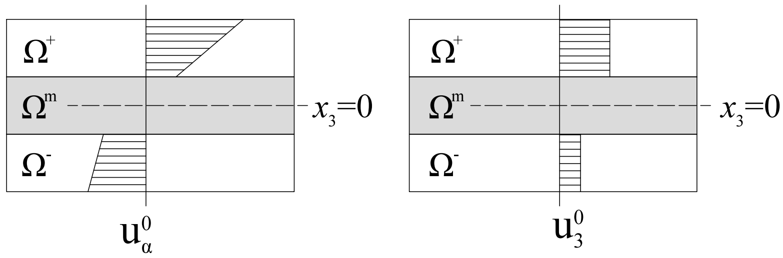

Figure 9: Limit kinematics for $n \geq 1$ & $p > n + 3$.

In Figure 10, we present a summarizing behavior chart in which the three phases, described above, are well distinguished with their related limit kinematics for the case $n = 1$ and $n = 2$.

6. Concluding remarks

In the present paper we attempt to give a complete overview of the asymptotic models for a three-layer plate with soft adhesive. The analysis has been conducted by considering general hypothesis of anisotropy and non homogeneity of the involved materials and, also, by taking into account the inertia forces. By means of the asymptotic expansion method, we derive a series of limit models by changing the thickness/rigidity ratios of the adherents and the adhesive.

We remark that, by decreasing the stiffness of adhesive, we encounter three main phenomena: the perfect adhesion, in which the layered plate behaves as an equivalent single-layer plate, whose middle layer does not influence the global mechanical behavior of the structure; the in-plane separation, according to which the adherents admit two different membrane displacement; finally, the transversal separation, occurring after the membrane

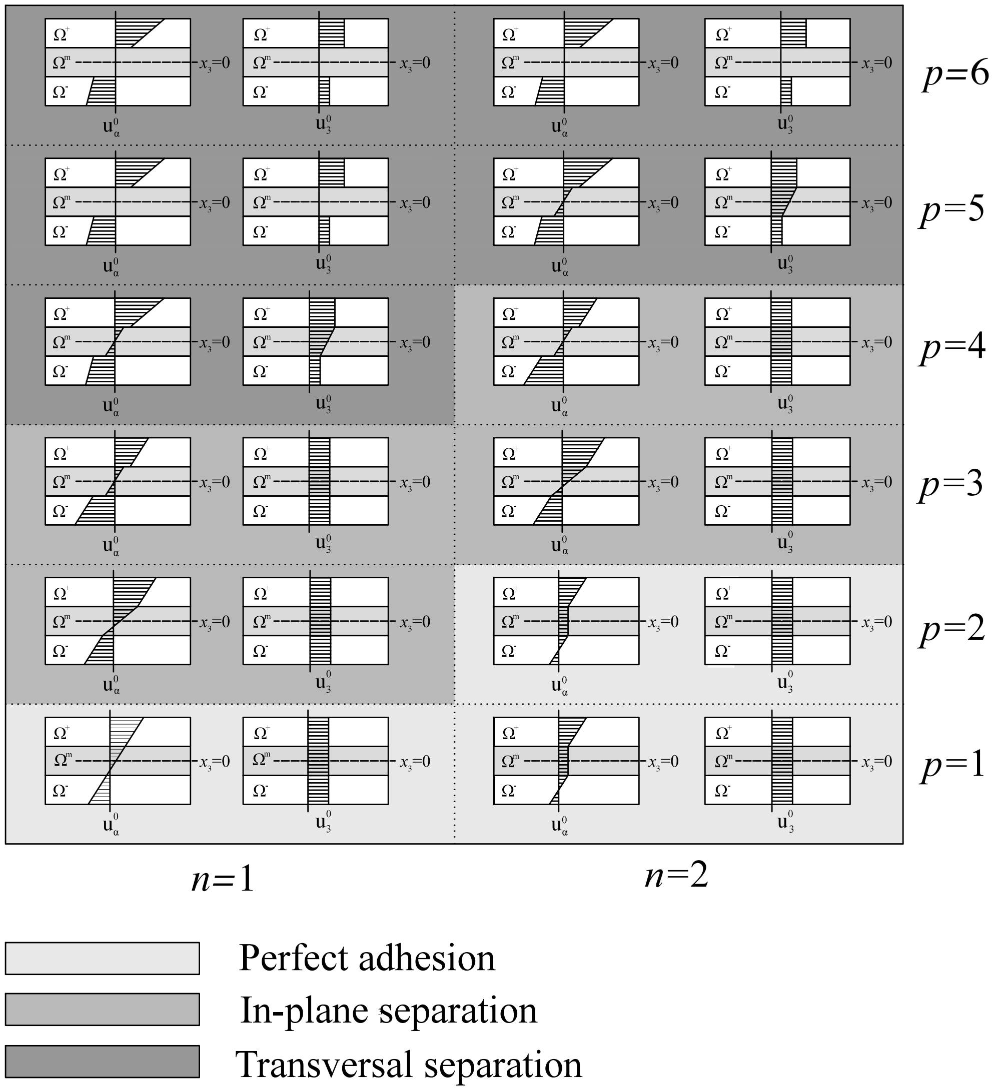

Figure 10: Behavior chart: increasing n entails decreasing the interlayer thickness, while increasing p corresponds to decreasing the interlayer stiffness.

separation, in which the adhesive *disappears* from a mechanical point of view and the upper and lower plates behaves as two independent Kirchhoff-Love plates.

As future developments, in order to have an overall look on the asymptotic models for layered thin plates, we want to extend our analysis to the case of a three-layer plate whose intermediate layer is stronger or with similar rigidity with respect to the adherents.

Acknowledgment

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