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(Article begins on next page)

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On the development of an implicit high-order Discontinuous Galerkin method for DNS and implicit LES of turbulent flows

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Abstract

In recent years Discontinuous Galerkin (DG) methods have emerged as one of the most promising high-order discretization techniques for CFD. DG methods have been successfully applied to the simulation of turbulent flows by solving the Reynolds averaged Navier–Stokes (RANS) equations with first-moment closures. More recently, due to their favourable dispersion and dissipation properties, DG discretizations have also been found very well suited for the Direct Numerical Simulation (DNS) and Implicit Large Eddy Simulation (ILES) of turbulent flows.

The growing interest in the implementation of DG methods for DNS and ILES is motivated by their attractive features. In particular, these methods can easily achieve high-order accuracy on arbitrarily shaped elements and are perfectly suited to *hp*-adaptation techniques. Moreover, their compact stencil is independent of the degree of polynomial approximation and is thus well suited for implicit time discretization and for massively parallel implementations.

In this paper we focus on recent developments and applications of an implicit high-order DG method for the DNS and ILES of both compressible and incompressible flows. High-order spatial and temporal accuracy has been achieved using the same numerical technology in both cases. Numerical inviscid flux formulations are based on the exact solution of Riemann problems (suitably perturbed in the incompressible case), and viscous flux discretizations rely on the BR2 scheme. Several types of high-order (up to order six) implicit schemes, suited also for DAEs, can be employed for accurate time integration. In particular, linearly implicit Rosenbrock-type Runge–Kutta schemes have been used for all the simulations presented in this work.

The massively separated incompressible flow past a sphere at $Re_D = 1000$, with transition to

turbulence in the wake region, is considered as a DNS test case, while the potential of the ILES is demonstrated by computing the compressible transitional flow at $Re_c = 60000$, $M_\infty = 0.1$ and $\alpha = 8^\circ$, around the Selig-Donovan 7003 airfoil. The computed solutions are compared with experimental data and numerical results available in the literature, showing good agreement.

27 *Keywords:* Discontinuous Galerkin discretization, implicit high-order accurate time
28 integration, linearly implicit Rosenbrock-type Runge-Kutta schemes, DNS, ILES, compressible
29 and incompressible flows

30 1. Introduction

31 Research work on high-order Discontinuous Galerkin (DG) methods applied to CFD is still
32 very active due to the features that make such methods quite attractive. In particular, DG methods
33 can easily achieve high-order accuracy on arbitrarily shaped elements and are perfectly suited to
34 *hp*-adaptation techniques. Moreover, the optimal compactness of DG discretization schemes is
35 independent of the degree of polynomial approximation and is thus well suited for implicit time
36 discretization and for massively parallel implementations of the schemes.

37 DG methods were already successfully applied to the simulation of compressible and incom-
38 pressible turbulent flows, by solving the Reynolds averaged Navier–Stokes (RANS) equations
39 with first-moment closures [1, 2, 3, 4, 5, 6, 7, 8, 9]. Thanks to their favourable dispersion and dis-
40 sipation properties, DG methods were also found to be well suited for the Direct Numerical Simu-
41 lation (DNS) and the Large Eddy Simulation (LES) of turbulent flows, [10, 11, 12, 13, 14, 15, 16].
42 More recently, investigating on the behaviour of DG methods applied to LES, several authors
43 have found that such methods appear to have dissipation properties which are perfectly suited for
44 an Implicit LES (ILES) of turbulent flows, *i.e.*, for a LES where the dissipation of the numerical
45 scheme, behaving like a spectral cut-off filter, plays the role of subgrid-scale (SGS) models, like
46 the Smagorinsky model, proper of “classical” LES approaches. This is the case of the works of
47 Refs. [17, 15], where the authors analyse the accuracy of high-order DG methods for the coarse-

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48 grid (“under-resolved”) simulation of turbulent flows, and of Refs. [13, 16] where the authors
49 explicitly refer to their approaches as ILES.

50 The aim of this work is to begin investigating the potential of a common computational frame-
51 work, based on the DG method, to address the DNS and ILES of both compressible and incom-
52 pressible turbulent flows. The governing equations are the Navier–Stokes equations, whereby
53 the turbulent kinetic energy dissipation, at the appropriate spatial and temporal scales, should be
54 provided by the numerical dissipation of the scheme. High-order spatial and temporal accuracy
55 has been achieved using the same numerical technology for both compressible and incompress-
56 ible flows. The numerical inviscid flux formulations are based on the exact solution of local
57 Riemann problems, suitably perturbed in the incompressible case, see [18, 19], while the vis-
58 cous flux discretizations rely on the BR2 scheme, see [20]. Several types of high-order (up to
59 order six) implicit schemes can be used for the accurate time integration, *e.g.* Modified Extended
60 BDF (MEBDF), Two Implicit Advanced Step-point (TIAS) [21, 22, 23], which allow to exploit
61 the benefits of the high-order discretizations both in space and time. The performance of these
62 schemes have been recently investigated, and, according to a preliminary analysis [24], the lin-
63 early implicit Rosenbrock-type Runge–Kutta schemes result as the most promising in terms of
64 both accuracy and efficiency. The main attractive feature of this class of schemes is the need to
65 solve only one linear system per stage for each time step, *i.e.*, the Jacobian matrix is assembled
66 and factored only once. In this work we employ Rosenbrock-type schemes that preserve accu-
67 racy also when applied to systems of Differential Algebraic Equations (DAEs) as those arising
68 from our discretization of the incompressible fluid dynamics governing equations.

69 The capabilities of the DG solver here presented are demonstrated by computing two well-
70 known test cases: the DNS of the massively separated incompressible flow past a sphere at
71 $Re_D = 1000$, with transition to turbulence in the wake region, and the ILES of the transitional
72 flow around the Selig-Donovan (SD) 7003 airfoil at $Re_c = 60000$, $M_\infty = 0.1$ and $\alpha = 8^\circ$.

73 The simulation of transonic flows with shock-waves, where DG methods require to introduce
74 some form of stabilization to control the numerical oscillation when flow discontinuities occur
75 inside cells, *e.g.* [25], is beyond the scope of this paper.

76 The rest of the paper is organized as follows. A quick overview of the governing equations is
 77 given in Sec. 2, while Sec. 3 describes the space and time discretizations. Numerical results are
 78 discussed in Sec. 4 and conclusions are given in Sec. 5. Finally, Appendix A reports some notes
 79 on flow statistics computation in the context of an implicit modal DG framework.

80 2. Governing equations

Using Einstein notation, the Navier–Stokes equations for compressible flows read

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}(\rho u_j H) = \frac{\partial}{\partial x_j} [u_i \tau_{ij} - q_j], \quad (3)$$

81 where E and H are total energy and total enthalpy, respectively. The pressure, stress tensor and
 82 heat flux vector are given by

$$p = (\gamma - 1)\rho (E - u_k u_k / 2), \quad (4)$$

$$\tau_{ij} = 2\mu \left[S_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right], \quad (5)$$

$$q_j = -\frac{\mu}{\text{Pr}} \frac{\partial h}{\partial x_j}, \quad (6)$$

where γ is the ratio of gas specific heats, S_{ij} is the mean strain-rate tensor

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

85 and Pr is the molecular Prandtl number.

For incompressible flows, we consider the set of governing equations

$$\frac{\partial u_j}{\partial x_j} = 0, \quad (7)$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}, \quad (8)$$

where the density has been assumed to be uniform and equal to one. The stress tensor τ_{ij} is again computed from Eq. (5), assuming a constant μ and keeping the velocity divergence in this equation.

3. Space and time discretization

3.1. The DG discrete setting

Let $\mathcal{T}_h = \{K\}$ denote a mesh of the domain $\Omega \in \mathbb{R}^d, d \in \{2, 3\}$ consisting of non-overlapping arbitrarily shaped elements K such that

$$\overline{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} \overline{K}. \quad (9)$$

Following the idea to define discrete polynomial spaces in physical coordinates, see, *e.g.*, [20, 18, 26, 27, 28, 29], we consider DG approximations based on the space

$$\mathbb{P}_d^k(\mathcal{T}_h) \stackrel{\text{def}}{=} \left\{ v_h \in L^2(\Omega) \mid v_h|_K \in \mathbb{P}_d^k(K), \forall K \in \mathcal{T}_h \right\}, \quad (10)$$

where k is a non-negative integer and $\mathbb{P}_d^k(K)$ denotes the restriction to K of the polynomial functions of d variables and total degree $\leq k$. To build a satisfactory basis for the space (10) we rely on the procedure presented in [30], see also [31, 32], allowing to obtain orthonormal and hierarchical basis functions by means of the modified Gram-Schmidt (MGS) algorithm. The starting set of basis functions for the MGS algorithm are the monomials defined over each elementary space $\mathbb{P}_d^k(K)$, $K \in \mathcal{T}_h$, in a reference frame relocated in the element barycenter and aligned with the principal axes of inertia of K . For the sake of presenting the DG discretization, we introduce

the set \mathcal{F}_h of the mesh faces $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^b$, where \mathcal{F}_h^b collects the faces located on the boundary of Ω_h and for any $F \in \mathcal{F}_h^i$ there exist two elements $K^+, K^- \in \mathcal{T}_h$ such that $F \in \partial K^+ \cap \partial K^-$. Moreover, for all $F \in \mathcal{F}_h^b$, \mathbf{n}_F denotes the unit outward normal to Ω_h , whereas, for all $F \in \mathcal{F}_h^i$, \mathbf{n}_F^- and \mathbf{n}_F^+ are the unit outward normals pointing to K^+ and K^- , respectively.

Since a function $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$ is double valued over an internal face $F \in \mathcal{F}_h^i$, we introduce the jump $[[\cdot]]$ and average $\{\cdot\}$ trace operators, that is

$$[[v_h]] \stackrel{\text{def}}{=} v_{h|K^+} \mathbf{n}_F^+ + v_{h|K^-} \mathbf{n}_F^-, \quad \{v_h\} \stackrel{\text{def}}{=} \frac{v_{h|K^+} + v_{h|K^-}}{2}, \quad (11)$$

and consider them to act componentwise when applied to vector functions. Finally, the DG discretization of second-order viscous terms employs the lifting operators \mathbf{r}_F and \mathbf{r} . For all $F \in \mathcal{F}_h$, we define the local lifting operator $\mathbf{r}_F : [L^2(F)]^d \rightarrow [\mathbb{P}_d^k(\mathcal{T}_h)]^d$, such that, for all $\mathbf{v} \in [L^2(F)]^d$,

$$\int_{\Omega} \mathbf{r}_F(\mathbf{v}) \cdot \boldsymbol{\tau}_h d\mathbf{x} = - \int_F \{\boldsymbol{\tau}_h\} \cdot \mathbf{v} dF \quad \forall \boldsymbol{\tau}_h \in [\mathbb{P}_d^k(\mathcal{T}_h)]^d. \quad (12)$$

The global lifting operator \mathbf{r} is then defined as

$$\mathbf{r}(\mathbf{v}) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \mathbf{r}_F(\mathbf{v}). \quad (13)$$

3.2. DG discretization of the Navier–Stokes equations

The Navier–Stokes equations for the m variables in d dimensions, Eqs. (1)-(2)-(3) or (7)-(8), can be written in compact form as

$$\mathbf{P}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \mathbf{F}_c(\mathbf{w}) + \nabla \cdot \mathbf{F}_v(\mathbf{w}, \nabla \mathbf{w}) = \mathbf{0}, \quad (14)$$

where $\mathbf{w} \in \mathbb{R}^m$ is the unknown solution vector, $\mathbf{F}_c, \mathbf{F}_v \in \mathbb{R}^m \otimes \mathbb{R}^d$ are the inviscid and viscous flux functions, and $\mathbf{P}(\mathbf{w}) \in \mathbb{R}^m \otimes \mathbb{R}^m$ is a transformation matrix. Employing the conservative variables $\mathbf{w}_c = [\rho, \rho u_i, \rho E]^T$ for compressible flows and the primitive variables $\mathbf{w}_p = [p, u_i]^T$ for incompressible flows, \mathbf{P} reduces to the identity matrix ($\mathbf{P} = \mathbf{I}$) in the former case and to the

120 difference between the identity and a single-entry matrix ($\mathbf{P} = \mathbf{I} - \mathbf{J}^{11}$) in the latter.

121 Alternatives to the set \mathbf{w}_c for compressible flows have been proposed by several authors for
 122 accuracy and robustness purposes. Primitive variables (p, u_i, T) are widely used for the preconditioning of the governing equations of low Mach number flows [33, 34, 35], and for the design of
 123 numerical schemes suited for both compressible and incompressible flows [36]. Such choice also
 124 simplifies the implementation of implicit contributions to the Jacobian matrix related to viscous
 125 terms and boundary conditions.

127 In this work, following the approach proposed by Bassi et al. in [37, 23], the positivity of the
 128 thermodynamic variables at a discrete level is ensured by employing a set of primitive variables
 129 where pressure and temperature have been replaced by their *logarithms*, $\tilde{p} = \log(p)$ and $\tilde{T} = \log(T)$. Essentially, we substitute in the governing equations the variables (p, T) with $(e^{\tilde{p}}, e^{\tilde{T}})$
 130 and use a polynomial approximation for \tilde{p} and \tilde{T} in the DG discretization. Using then, for
 131 compressible flows, the set of variables $\mathbf{w} = [\tilde{p}, u_i, \tilde{T}]^T$, the transformation matrix $\mathbf{P}(\mathbf{w})$ reads
 132

$$\mathbf{P}(\mathbf{w}) = \begin{bmatrix} \rho_{\tilde{p}} & 0 & 0 & 0 & \rho_{\tilde{T}} \\ \rho_{\tilde{p}}u_1 & \rho & 0 & 0 & \rho_{\tilde{T}}u_1 \\ \rho_{\tilde{p}}u_2 & 0 & \rho & 0 & \rho_{\tilde{T}}u_2 \\ \rho_{\tilde{p}}u_3 & 0 & 0 & \rho & \rho_{\tilde{T}}u_3 \\ \rho_{\tilde{p}}H + \rho h_{\tilde{p}} - e^{\tilde{p}} & \rho u_1 & \rho u_2 & \rho u_3 & \rho_{\tilde{T}}H + \rho h_{\tilde{T}} \end{bmatrix}, \quad (15)$$

133 where

$$\rho = e^{(\tilde{p}-\tilde{T})}, \quad \hat{e} = \frac{e^{\tilde{T}}}{\gamma - 1}, \quad (16)$$

134

$$h_{\tilde{p}} = \left. \frac{\partial h}{\partial \tilde{p}} \right|_{\tilde{T}} = \hat{e}_{\tilde{p}} + \frac{e^{\tilde{p}}}{\rho} - \frac{\rho_{\tilde{p}}}{\rho^2} e^{\tilde{p}}, \quad h_{\tilde{T}} = \left. \frac{\partial h}{\partial \tilde{T}} \right|_{\tilde{p}} = \hat{e}_{\tilde{T}} - \frac{\rho_{\tilde{T}}}{\rho^2} e^{\tilde{p}}, \quad (17)$$

135 and assuming an ideal gas

$$\rho_{\tilde{p}} = \left. \frac{\partial \rho}{\partial \tilde{p}} \right|_{\tilde{T}} = \rho, \quad \rho_{\tilde{T}} = \left. \frac{\partial \rho}{\partial \tilde{T}} \right|_{\tilde{p}} = -\rho, \quad (18)$$

136

$$\hat{e}_{\bar{p}} = \frac{\partial \hat{e}}{\partial \bar{p}} \Big|_{\bar{T}} = 0, \quad \hat{e}_{\bar{T}} = \frac{\partial \hat{e}}{\partial \bar{T}} \Big|_{\bar{p}} = \hat{e}, \quad (19)$$

137

$$h_{\bar{p}} = 0, \quad h_{\bar{T}} = \hat{e}_{\bar{T}} - \frac{e_{\bar{p}}}{\rho}. \quad (20)$$

138 By multiplying Eq. (14) by an arbitrary smooth test function $\mathbf{v} = \{v_1, \dots, v_m\}$, and integrating
139 by parts, we obtain the weak formulation

$$\int_{\Omega} \mathbf{v} \cdot \left(\mathbf{P}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} \right) d\mathbf{x} - \int_{\Omega} \nabla \mathbf{v} : \mathbf{F}(\mathbf{w}, \nabla \mathbf{w}) d\mathbf{x} + \int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n} : \mathbf{F}(\mathbf{w}, \nabla \mathbf{w}) d\sigma = \mathbf{0}, \quad (21)$$

140 where \mathbf{F} is the sum of the inviscid and viscous flux functions and \mathbf{n} is the unit vector normal to
141 the boundary.

To discretize Eq. (21) we replace the solution \mathbf{w} and the test function \mathbf{v} with a finite element approximation \mathbf{w}_h and a discrete test function \mathbf{v}_h , respectively, where \mathbf{w}_h and \mathbf{v}_h belong to the space $\mathbf{V}_h \stackrel{\text{def}}{=} [\mathbb{P}_d^k(\mathcal{T}_h)]^m$. For each of the m equations of system (21), and without loss of generality, we choose the set of test and shape functions in any element K coincident with the set $\{\phi\}$ of N_{dof}^K orthogonal and hierarchical basis functions in that element. With this choice each component $w_{h,j}$, $j = 1, \dots, m$, of $\mathbf{w}_h \in \mathbf{V}_h$ can be expressed, in terms of the elements of the global vector \mathbf{W} of unknown degrees of freedom, as $w_{h,j} = \phi_l W_{j,l}$, $l = 1, \dots, N_{dof}^K$, $\forall K \in \mathcal{T}_h$. Then, the DG discretization of the Navier–Stokes equations consists in seeking, for $j = 1, \dots, m$, the elements of \mathbf{W} such that

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \phi_i P_{j,k}(\mathbf{w}_h) \phi_l \frac{dW_{k,l}}{dt} d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial \phi_i}{\partial x_n} F_{j,n}(\mathbf{w}_h, \nabla_h \mathbf{w}_h + \mathbf{r}(\llbracket \mathbf{w}_h \rrbracket)) d\mathbf{x} \\ + \sum_{F \in \mathcal{F}_h} \int_F \llbracket \phi_i \rrbracket_n \widehat{F}_{j,n}(\mathbf{w}_h^\pm, (\nabla_h \mathbf{w}_h + \eta_F \mathbf{r}_F(\llbracket \mathbf{w}_h \rrbracket))^\pm) d\sigma = 0, \end{aligned} \quad (22)$$

142 for $i = 1, \dots, N_{dof}^K$. In Eq. (22) repeated indices imply summation over the ranges $k = 1, \dots, m$,
143 $l = 1, \dots, N_{dof}^K$, $n = 1, \dots, d$.

144 The DG discretization of the viscous fluxes is based on the BR2 scheme, proposed in [20] and
145 theoretically analyzed in [38] and [39]. According to this scheme, the viscous numerical flux is

146 given by

$$\widehat{\mathbf{F}}_v(\mathbf{w}_h^\pm, (\nabla_h \mathbf{w}_h + \eta_F \mathbf{r}_F(\llbracket \mathbf{w}_h \rrbracket))^\pm) \stackrel{\text{def}}{=} \{\mathbf{F}_v(\mathbf{w}_h, \nabla_h \mathbf{w}_h + \eta_F \mathbf{r}_F(\llbracket \mathbf{w}_h \rrbracket))\} \quad (23)$$

147 where the stability parameter η_F is defined according to [39].

148 The inviscid numerical flux is computed from the solution of local Riemann problems in the
 149 normal direction at each integration point on elements faces. For compressible flows, we use ei-
 150 ther the exact Riemann solver of Gottlieb and Groth, [40], or, alternatively, the van Leer flux vec-
 151 tor splitting method as modified by Hänel et al., [41]. For incompressible flows, we employ the
 152 approach proposed by Bassi et al. in [18], whereby the inviscid numerical flux $\widehat{\mathbf{F}}_c(\mathbf{w}_h^\pm) = \mathbf{F}_c(\mathbf{w}_h^*)$
 153 is computed from the exact solution \mathbf{w}_h^* of local Riemann problems suitably modified by means
 154 of an artificial compressibility perturbation, c^2 . The details of the procedure for the determination
 155 of the state $\mathbf{w}_h^* = \mathbf{w}(\mathbf{w}_h^\pm, c^2)$ are thoroughly discussed in Appendix A of [18] for the case of the
 156 incompressible Euler equations as well as for the Stokes and Oseen equations. We remark that
 157 in this approach the artificial compressibility is only introduced locally to allow the evaluation
 158 of the convective numerical flux and no pressure time derivative is added to the DG discretized
 159 continuity equation, thus preserving the time accuracy of the method. According to the results
 160 of numerical experiments presented in [18] and [7], the value of the artificial compressibility
 161 parameter c^2 can be chosen in the range $[0.01, 100]$, without affecting the numerical accuracy.
 162 For the incompressible flow computations presented in this work the value was set equal to 0.1.
 163 The DG discretization is best suited for a weak enforcement of boundary conditions, [20, 19, 23].
 164 This can easily be attained by properly defining boundary states and their gradients, which are
 165 used, directly or together with the internal states and their gradients, to compute the numerical
 166 inviscid and viscous fluxes and the lifting operators for all $F \in \mathcal{F}_h^b$. The boundary states and
 167 their derivatives must be defined according to the boundary types and, together with the internal
 168 states, enter in the Riemann solvers and ensure that the computed numerical fluxes are consistent
 169 with the physical ones.

170 3.3. Accurate time integration

171 Numerical integration of Eq. (22) by means of suitable Gauss quadrature rules leads to a
172 system of nonlinear ODEs, or DAEs for incompressible flows, that can be written as

$$\mathbf{M}_P(\mathbf{W}) \frac{d\mathbf{W}}{dt} + \mathbf{R}(\mathbf{W}) = \mathbf{0}, \quad (24)$$

173 where $\mathbf{R}(\mathbf{W})$ is the vector of residuals and $\mathbf{M}_P(\mathbf{W})$ is a global block diagonal matrix arising from
174 the discretization of the first integral in Eq. (22). When using the set of the compressible \mathbf{w}_c and
175 incompressible \mathbf{w}_p flow variables together with the set of orthonormal basis functions outlined
176 in Sec. 3.1, the matrix \mathbf{M}_P reduces to the identity matrix and to a modified identity matrix, with
177 zeros in the diagonal positions corresponding to the pressure degrees of freedom, respectively.
178 However, for different sets of variables, the transformation matrix \mathbf{P} couples the degrees of free-
179 dom of the variables \mathbf{w}_h within each block of \mathbf{M}_P , so that this matrix can not be diagonal, even
180 using a set of orthogonal basis functions. According to [23], implicit and accurate time inte-
181 gration of Eq. (24) can be efficiently performed by means of linearly implicit Rosenbrock-type
182 Runge–Kutta schemes. We opted for this class of temporal schemes for four attractive features:
183 (i) they have excellent stability properties; (ii) they are self-starting; (iii) they can use variable
184 time steps; (iv) the Jacobian matrix needs to be assembled and factored only once per time step.
185 Properly dealing with the solution dependent block diagonal matrix $\mathbf{M}_P(\mathbf{W})$, see [23], and ac-
186 cording to the implementation-oriented formulation reported by Hairer and Wanner [42], the
187 time integration of the DG space discretized equations with an s -stage Rosenbrock scheme can
188 be written as

$$\mathbf{W}^{n+1} = \mathbf{W}^n + \sum_{j=1}^s m_j \mathbf{Y}_j, \quad (25)$$

$$\left(\frac{\mathbf{I}}{\gamma \Delta t} + \tilde{\mathbf{J}} \right)^n \mathbf{Y}_i = -\tilde{\mathbf{R}} \left(\mathbf{W}^n + \sum_{j=1}^{i-1} a_{ij} \mathbf{Y}_j \right) + \sum_{j=1}^{i-1} \frac{c_{ij}}{\Delta t} \mathbf{Y}_j, \quad i = 1, \dots, s, \quad (26)$$

190 where, omitting the dependence on \mathbf{W} for notational convenience,

$$\mathbf{J} = \frac{\partial \mathbf{R}}{\partial \mathbf{W}}, \quad \tilde{\mathbf{R}} = \mathbf{M}_P^{-1} \mathbf{R}, \quad \tilde{\mathbf{J}} = \frac{\partial \tilde{\mathbf{R}}}{\partial \mathbf{W}} = \mathbf{M}_P^{-1} \left(\mathbf{J} - \frac{\partial \mathbf{M}_P}{\partial \mathbf{W}} \tilde{\mathbf{R}} \right), \quad (27)$$

191 and m_i, a_{ij}, c_{ij} are real coefficients.

Recasting Eq. (26) so as to avoid the cumbersome product $\mathbf{M}_\mathbf{p}^{-1}\mathbf{J}$ in Eq. (27), we obtain the final form used for the implementation [23]

$$\mathbf{W}^{n+1} = \mathbf{W}^n + \sum_{j=1}^s m_j \mathbf{Y}_j, \quad (28)$$

$$\left(\frac{\mathbf{M}_\mathbf{p}}{\gamma \Delta t} + \mathbf{J} - \frac{\partial \mathbf{M}_\mathbf{p}}{\partial \mathbf{W}} \tilde{\mathbf{R}} \right)^n \mathbf{Y}_i = -\mathbf{M}_\mathbf{p}^n \left[\tilde{\mathbf{R}} \left(\mathbf{W}^n + \sum_{j=1}^{i-1} a_{ij} \mathbf{Y}_j \right) - \sum_{j=1}^{i-1} \frac{c_{ij}}{\Delta t} \mathbf{Y}_j \right], \quad i = 1, \dots, s, \quad (29)$$

192 which also allows to quantify the computational overhead due to the solution dependent block
 193 diagonal matrix $\mathbf{M}_\mathbf{p}$. We remark that our implementation relies on an analytical derivation of the
 194 Jacobian matrix \mathbf{J} that takes fully account of the dependence of the vector of the residuals on the
 195 unknowns and their derivatives, including the implicit treatment of lifting operators and bound-
 196 ary conditions. The linear problems in Eq. (29) can be efficiently solved in parallel by means of
 197 the matrix-explicit or the matrix-free GMRES method included in the PETSc library [43, 44].
 198 Preconditioning by using the block Jacobi method with one block per process, each solved with
 199 ILU(0), or the Additive Schwarz Method (ASM) is usually employed to make GMRES conver-
 200 gence acceptable for complex problems.

201 In the following, to keep the notation as simple as possible, we will refer to the Rosenbrock
 202 schemes as RO q - s , where q is the order of convergence and s is the number of stages. In this work
 203 we employ the schemes RO2-2 of Iannelli and Baker, [45], RO3-3 (ROS3P) of Lang and Ver-
 204 wer, [46], and RO5-8 (RODAS5(4)-Rod5.1) of Di Marzo, [47]. All these schemes are designed
 205 to preserve their accuracy with DAEs.

206 The authors in [23] investigated the temporal accuracy and performance of several Rosenbrock
 207 schemes (up to order six) for the computation of both compressible and incompressible flows.
 208 Their numerical investigation demonstrated that higher-order schemes are much more efficient
 209 than lower-order ones if low levels of time integration error, as those required by DNS and
 210 LES, are being sought. In the same paper the coefficients of all the considered schemes are also
 211 tabulated. An example of an indirect proof of order assessment, based on the error introduced by
 212 the time integration scheme in fulfilling the discretized incompressibility constraint, is given for

213 a complex three dimensional problem in [48].

214 A simple approach to dense output, based on a fifth-degree polynomial interpolation of the
215 solution within the time step, and where coefficients are obtained by imposing the solution and
216 its first and second time derivatives at the time step extrema, is reported in Appendix A.1. This
217 interpolation technique is suitable for any implicit time integration scheme that relies on an exact
218 Jacobian matrix evaluation.

219 4. Numerical results

220 In this section we demonstrate the reliability of the proposed coupling between a high-order
221 Rosenbrock scheme and a high-order DG space discretization applied to DNS and ILES. To this
222 purpose, two well-established flow problems, one compressible and one incompressible, have
223 been computed and results presented below.

224 The former concerns the DNS of the incompressible flow past a sphere at a diameter-based
225 Reynolds number of $Re_D = 1000$. For this test case the simulations have been performed with
226 two different temporal schemes and polynomial degree of the DG space approximation. The
227 latter is the ILES of the transitional flow around the SD7003 airfoil at a chord-based Reynolds
228 number of $Re_c = 60000$. In this case we computed the solution on two meshes with different
229 polynomial degrees and a very highly accurate time integration scheme, *i.e.* the RO5-8, to assess
230 the influence of p - and h - refinement.

231 Both the test cases have been run in parallel, using up to 4096 CPU cores, on the CINECA
232 IBM-BlueGene/Q supercomputer named FERMI.

233 4.1. Flow past a sphere at $Re_D = 1000$

234 In this section the assessment of our DG solver for the DNS of an incompressible transitional
235 flow is reported. Few papers have been published on DG methods applied to DNS, see, *e.g.* [10,
236 49, 50, 12], and, up to authors knowledge, they are restricted to compressible flows.

237 The incompressible flow past a sphere has been computed at a diameter-based $Re_D = 1000$
238 on the computational mesh made of 57600 20-node hexahedral elements (quadratic edge repre-
239 sentation) and first cell height $\Delta n/D = 0.005$ shown in Figure 1. The computations have been

240 performed with a free-stream velocity aligned with the x_3 axis on exactly the same cylindrical
 241 domain used by Tomboulides and Orszag [51]. The cylinder radius is $4.5D$, the inflow boundary
 242 is located $4.5D$ upstream from the sphere, and the outflow boundary is placed $25D$ downstream
 243 from the sphere, thus a long wake can be accurately computed. In fact, at this flow conditions
 the laminar to turbulent transition takes place in the sphere wake.

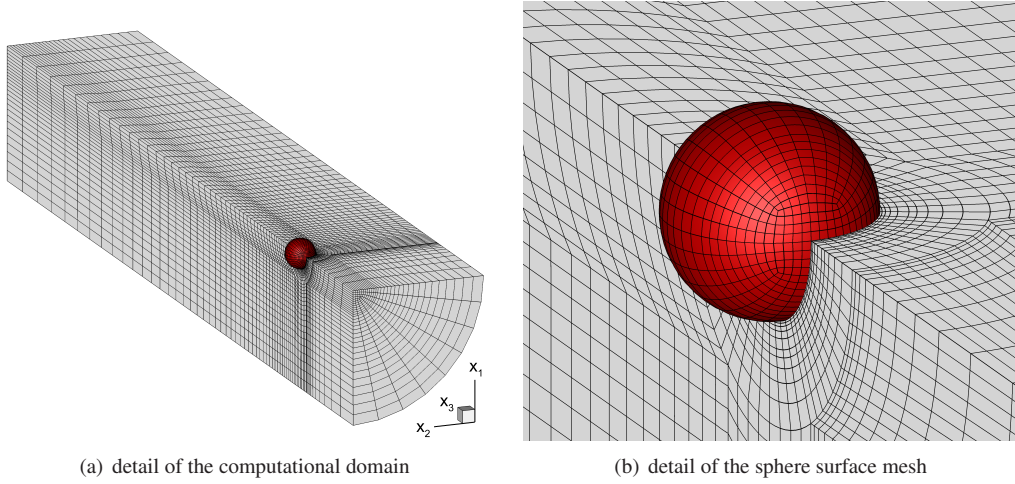


Figure 1: Flow past a sphere - Computational mesh with 57600 20-node hexahedral elements (quadratic edges)

244

245 The test case has been computed by using the RO2-2 scheme of Iannelli and Baker, [45], and
 246 the RO3-3 scheme of of Lang and Verwer, [46], coupled with \mathbb{P}^3 and \mathbb{P}^4 DG space approxima-
 247 tions, respectively. The solutions have been advanced in time with a fraction $f = 4e-3$ of the
 248 convective time, *i.e.*, $\Delta t = f(D/u_\infty)$ and initialized by a \mathbb{P}^3 $\text{Re}_D = 250$ steady solution. Tur-
 249 bulence statistics have been computed for $130D/u_\infty$ time units after the appearance of distorted
 250 hairpin vortices in the wake region.

251 Both the RO2-2-DG \mathbb{P}^3 and the RO3-3-DG \mathbb{P}^4 time averaged and root mean square distribu-
 252 tions of velocity downstream of the sphere along the x_3 axis are in good agreement with the
 253 experimental results of Wu and Faeth [52] obtained for $\text{Re}_D = 960$, see Figure 2. The results are
 254 also in satisfactory agreement with the numerical computations of Tomboulides and Orszag [51].
 255 Figure 3 shows the time averaged and root mean square radial distributions of u_3 at different x_3
 256 locations. Although noticeable differences are observed between the \mathbb{P}^3 and \mathbb{P}^4 solutions the

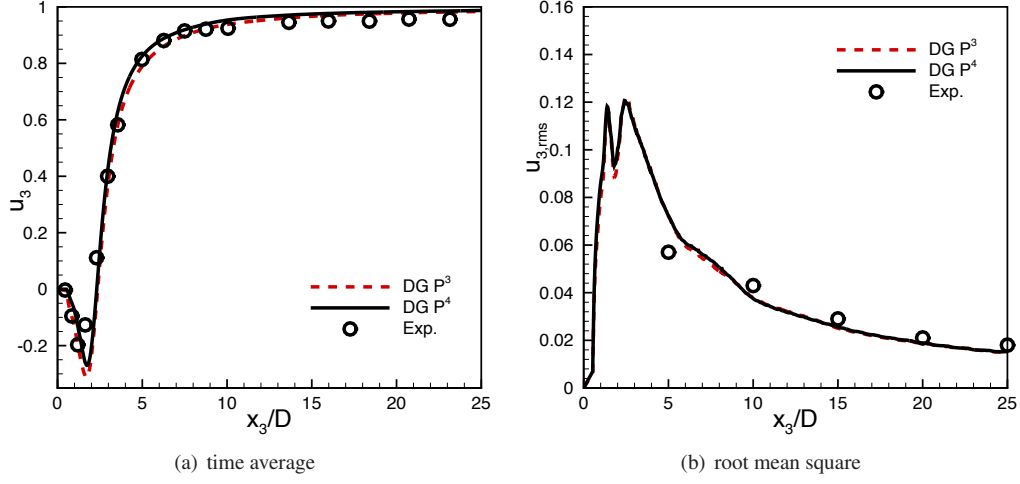


Figure 2: Flow past a sphere - Time averaged and root mean square distributions of u_3 along the x_3 axis downstream of the sphere (experimental data of Wu and Faeth [52] obtained by digitalizing the paper of Tomboulides and Orzsag [51])

higher-order results are similar to the computations of Tomboulides and Orzsag [51]. We remark

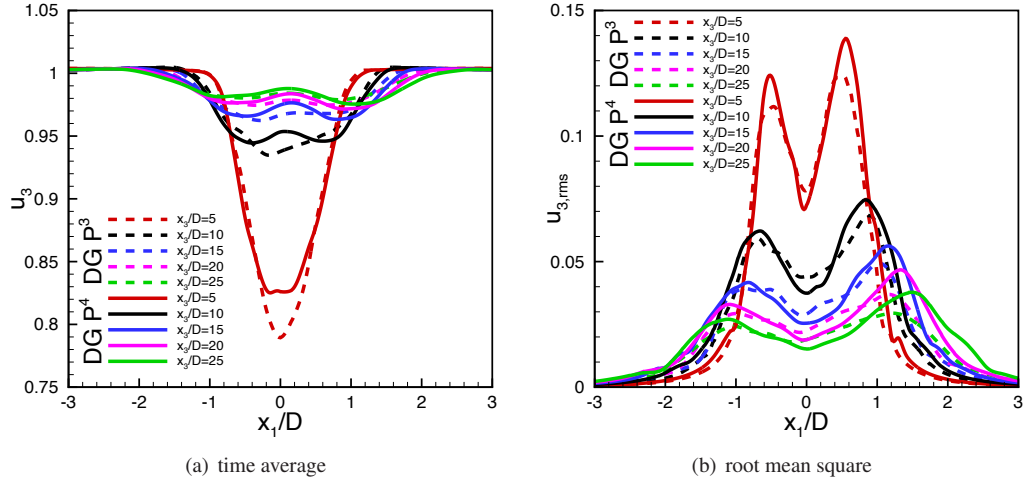


Figure 3: Flow past a sphere - Time averaged and root mean square distributions of u_3 along the radial direction at different x_3 locations

257

258 that, due to the limited simulation time dedicated to compute the turbulence statistics, our results
 259 are not perfectly symmetrical with respect to the x_3 axis. The continuation of our computations
 260 is the subject of ongoing work.

261 At these flow conditions the wake is essentially axisymmetric in an averaged sense, as con-

262 firmed by the energy power spectra at four points located at the same x_3 and radial $r = (x_1^2 + x_2^2)^{1/2}$
 263 position but with different azimuthal coordinates, $\theta = \tan^{-1}(x_2/x_1)$, see Figure 4(a). The energy
 264 power spectrum is here computed as $(\text{PSD}(u_1) + \text{PSD}(u_2) + \text{PSD}(u_3))/2$, where u_i is the tempo-
 265 ral signal of the i -component of the velocity. For a point located downstream from the sphere
 266 and not laying on the wake axis, Figure 4(b) displays the influence of the solution accuracy on
 267 the spectrum. We remark that, although the decay spectrum has the expected slope of $-5/3$ for
 268 both the solutions, while for low frequencies the results are almost indistinguishable, for higher
 269 frequencies the RO3-3-DG \mathbb{P}^4 spectrum is slightly shifted to the right side. This difference is
 270 noticeably smaller than what we observed for the ILES application reported in the following
 271 section, see Figure 14. A similar behaviour is also observed for the energy spectra shown in a
 272 recent paper of Chapelier et al. [12] on DG applied to DNS, where the authors computed the
 273 Taylor-Green vortex and channel flow using a compressible explicit in time solver. Our spectra
 274 confirm that close to the sphere the results have to be considered as a DNS, however, far from the
 275 sphere, where for computational efficiency reasons the grid size has been slightly increased, it is
 276 possible that not all the turbulent scales have been fully resolved, thus explaining the differences
 277 between the RO2-2-DG \mathbb{P}^3 and RO3-3-DG \mathbb{P}^4 solutions shown in Figure 3 and Figure 4(b).

278 Figure 4(c) shows the power spectra at four points on the x_3 axis at different distances down-
 279 stream from the sphere. At $x_3/D = 1.5$, within the recirculation zone in the separated region,
 280 the spectrum does not follow the $-5/3$ law, thus confirming that separation occurs in the laminar
 281 regime and then develops in a turbulent wake, see also Figure 4(d). The time averaged drag co-
 282 efficient, and the Strouhal number corresponding to the shedding frequency of the hairpin vortex
 283 shown in Figure 5, are tabulated in Table 1. The computed values are in good agreement with
 284 published data and show the beneficial effect of increasing the solution accuracy.

285 4.2. Transitional flow around the SD7003 airfoil

286 The ILES of the transitional compressible flow around the SD7003 airfoil, one of the test
 287 cases proposed within the “International Workshop on High-Order CFD Methods” [57], is here
 288 considered. As ILES does not include any explicit SGS model and the discretization itself acts
 289 like a subgrid-scale model, the flow equations are simply the DG space discretized compressible

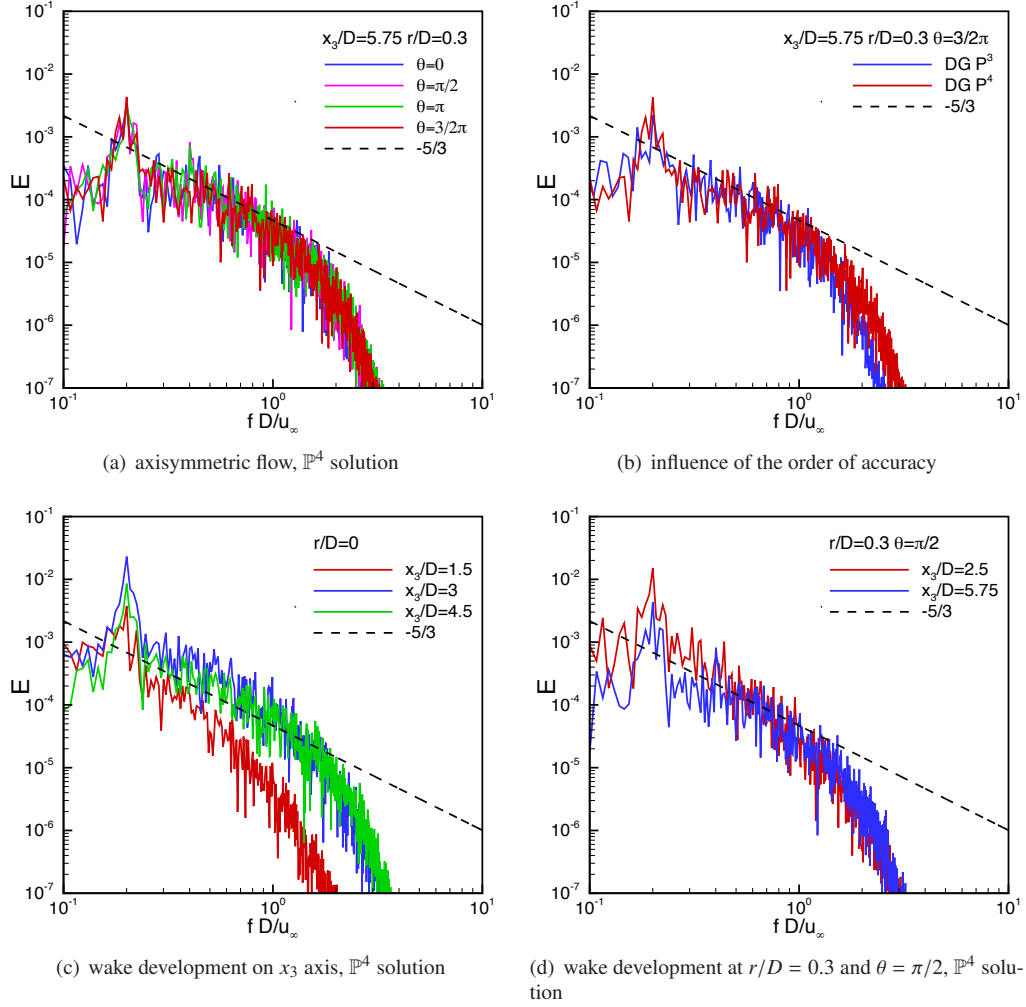


Figure 4: Flow past a sphere - Energy power spectra for different locations in the wake and different polynomial degrees

Navier–Stokes equations integrated in time by means of a very high-order temporal scheme. The main difficulties of this challenging test case are related to the complex physical features of an unsteady flow characterized by a laminar separation, *i.e.* the formation of a transitional shear layer followed by turbulent reattachment on the suction side of the wing by virtue of the enhanced momentum transport.

To investigate the influence of p - and h - refinement on ILES, the flow around the SD7003 airfoil has been computed for the conditions $M_\infty = 0.1$, $\alpha = 8^\circ$ and chord-based $Re_c = 60000$

	C_D	St
RO2-2-DG \mathbb{P}^3	0.471	0.201
RO3-3-DG \mathbb{P}^4	0.475	0.200
Hindenlang et al. [53]	0.48	–
Ploumhans et al. [54]	0.48	–
Tomboulides and Orszag [51]	–	0.195
Poon et al. [55]	0.46	0.20
Sakamoto and Haniu [56] (Exp.)	–	0.195–0.205

Table 1: Flow past a sphere - Time averaged drag coefficient and Strouhal number

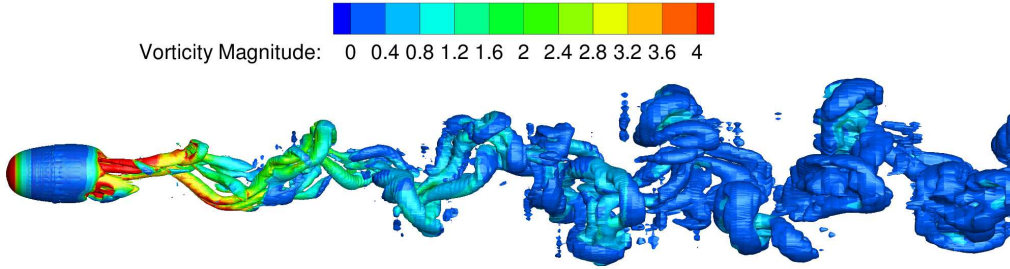


Figure 5: Flow past a sphere - Instantaneous $\lambda_2 = 1e-3$ isosurface of λ_2 -criterion coloured with the non-dimensional vorticity magnitude

297 with different DG space approximations on a coarse mesh of 20064 50-node hexahedral elements
 298 (quartic edge representation) and on a uniformly refined mesh of 160512 20-node hexahedral
 299 elements (quadratic edge representation) with a first cell height of $\Delta n/c = 2.9e-4$ and of $\Delta n/c =$
 300 $1.45e-4$, respectively. The computational grids are shown in Figure 6. This low Mach number
 301 value has been chosen to compare our solutions with the computational results presented in the
 302 literature, *e.g.* [58, 59]. A no-slip isothermal wall condition with $T_{wall}/T_\infty = 1.002$ is imposed
 303 at the airfoil surface, while a periodic boundary condition over a width $s/c = 0.2$ is set in the
 304 spanwise direction to emulate an infinite wing. At the farfield boundary, located $\sim 100c$ from
 305 the wing, characteristic-based conditions have been imposed. The test case has been computed
 306 with the RO5-8 time-integration scheme of Di Marzo, see [47], coupled with \mathbb{P}^3 and \mathbb{P}^4 DG
 307 space approximations on the coarse mesh, and with \mathbb{P}^2 and \mathbb{P}^3 approximations on the fine mesh.
 308 The number of degrees of freedom per equation (DOFs) of these solutions is reported in Table 2
 309 together with the results available in the literature. On the coarse mesh the solutions have been

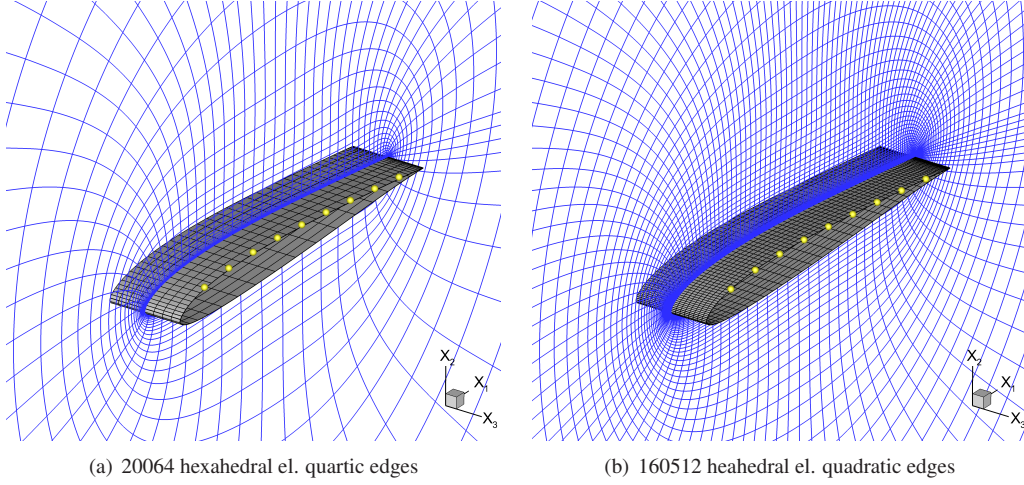


Figure 6: SD7003 - Computational meshes, the yellow spheres indicate the chordwise locations of the measured velocity profiles

310 advanced in time with a time step equal to a fraction $f = 4.4\text{e-}3$ of the convective time, *i.e.*,
 311 $\Delta t = f(c/u_\infty)$, while on the fine mesh the values $f = 4.4\text{e-}3$ and $f = 2.2\text{e-}3$ have been used
 312 for the \mathbb{P}^2 and \mathbb{P}^3 space approximation, respectively. All the computations have been initialized
 313 from a p -sequence of lower-order, not fully statistically converged solutions. Figures 7 and 8
 314 show the contours of the mean pressure coefficient and of the mean x_1 -component of velocity
 315 around the airfoil. A detailed view of the flow field and of detachment and reattachment points
 316 of the laminar separation bubble (LSB) is shown in Figure 9. The term “mean” denotes solutions
 317 averaged both in time and in the spanwise direction. A rigorous yet practical approach to the
 318 spatial averaging in a modal DG framework is reported in Appendix A.2.

319 To assess the influence of p - and h - refinement on our computations we assume as reference
 320 the \mathbb{P}^3 solution obtained on the fine mesh with the halved time step. The dimensions of the
 321 laminar separation bubble reported in Table 2, along with the mean skin friction and pressure
 322 coefficient distributions shown in Figure 11, highlight the noticeable difference between the LSB
 323 size predicted on the coarse grid by the \mathbb{P}^3 approximation and the other more accurate solutions.
 324 We remark that our C_p and C_f distributions are in fair agreement with the incompressible LES
 325 of Catalano and Tognaccini obtained using second-order central differences in streamwise and
 326 wall-normal directions, and Fourier collocations in the spanwise direction, on a grid with 864

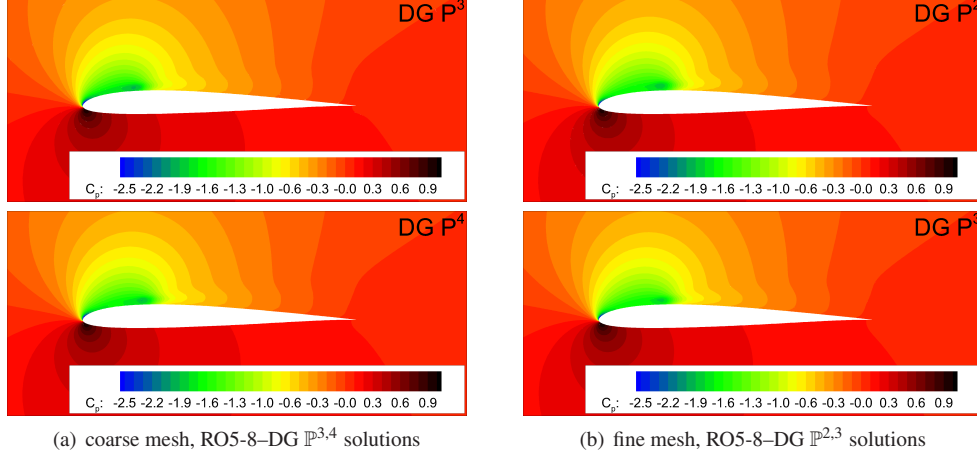


Figure 7: SD7003 - Contours of mean pressure coefficient

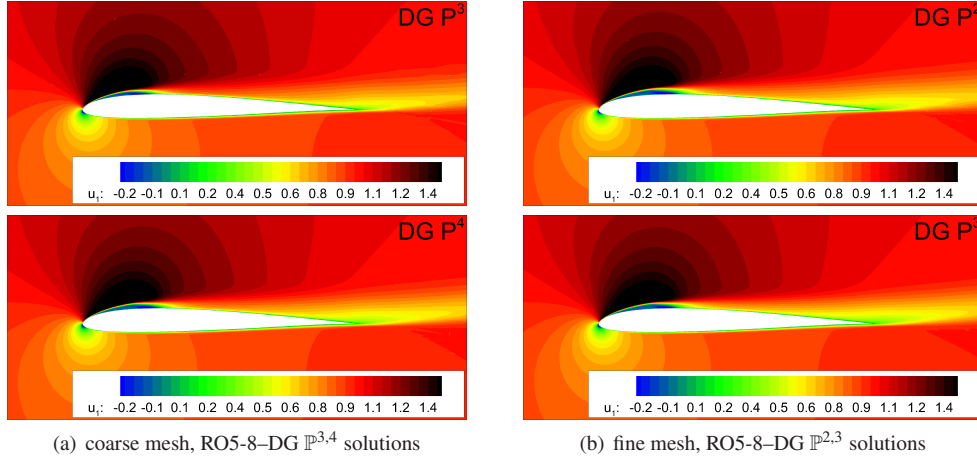


Figure 8: SD7003 - Contours of mean x_1 -component of velocity

cells in the streamwise, 208 in the crosswise, and 48 in the spanwise direction [60]. The lack of spatial resolution of our \mathbb{P}^3 approximation is confirmed by the significant difference between the predicted mean velocity profiles and the reference, see Figure 10. The profiles have been extracted along vertical lines at the chordwise locations indicated in Figure 6. We remark that, although having less DOFs per equation than the \mathbb{P}^2 solution on the fine mesh, roughly the 56% less, the \mathbb{P}^4 results on the coarse grid are closer to the reference, both in terms of LSB dimensions and mean velocity profiles, thus suggesting that increasing the degree of the

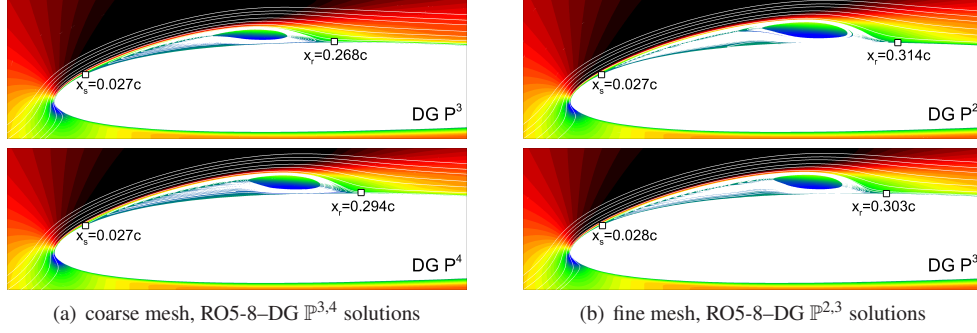


Figure 9: SD7003 - Contours of mean x_1 -component of velocity, detail of the laminar separation bubble with indication of the separation and reattachment points

	DOFs	LSB details				Aerodynamic loads		
		x_s/c	x_r/c	L/c	H/c	C_D	C_L	C_m
RO5-8-DG \mathbb{P}^3 coarse	401280	0.027	0.268	0.241	0.017	0.0423	0.9615	-0.0233
RO5-8-DG \mathbb{P}^4 coarse	702240	0.027	0.294	0.267	0.021	0.0454	0.9534	-0.0224
RO5-8-DG \mathbb{P}^2 fine	1605120	0.027	0.314	0.287	0.022	0.0471	0.9548	-0.0235
RO5-8-DG \mathbb{P}^3 fine	3210240	0.028	0.303	0.275	0.021	0.0457	0.9441	-0.0223
DGSEM \mathbb{P}^3 [59]	4.26M	0.027	0.310	-	-	0.045	0.923	-
DGSEM \mathbb{P}^7 [59]	4.55M	0.030	0.336	-	-	0.050	0.932	-
Comp. FD $\mathcal{O}(6)$ [58]	53.4M	0.031	0.303	0.272	0.020	0.0447	0.917	-0.0187
SBP-SAT $\mathcal{O}(4)$ [61]	4.48M	0.037	0.200	-	-	0.034	0.968	-

Table 2: SD7003 - Details of the laminar separation bubble and mean aerodynamic loads. x_s and x_r are the separation and reattachment points coordinates, L and H the separation bubble length and height

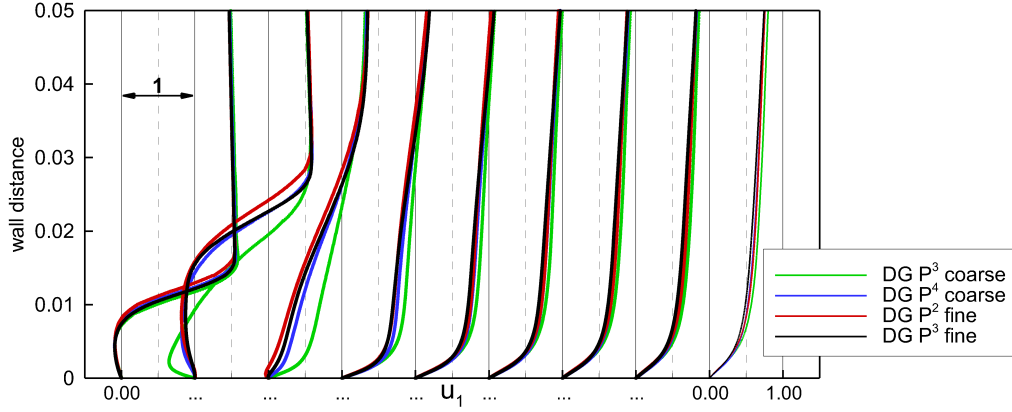


Figure 10: SD7003 - Profiles of mean x_1 -component of velocity at chordwise locations $x_1/c = \{0.1, \dots, 0.9\}$, coarse mesh RO5-8-DG $\mathbb{P}^{3,4}$ and fine mesh RO5-8-DG $\mathbb{P}^{2,3}$ solutions

polynomial DG space approximation is better than refining the mesh. The mean drag, lift and moment coefficients of the airfoil are reported in Table 2.

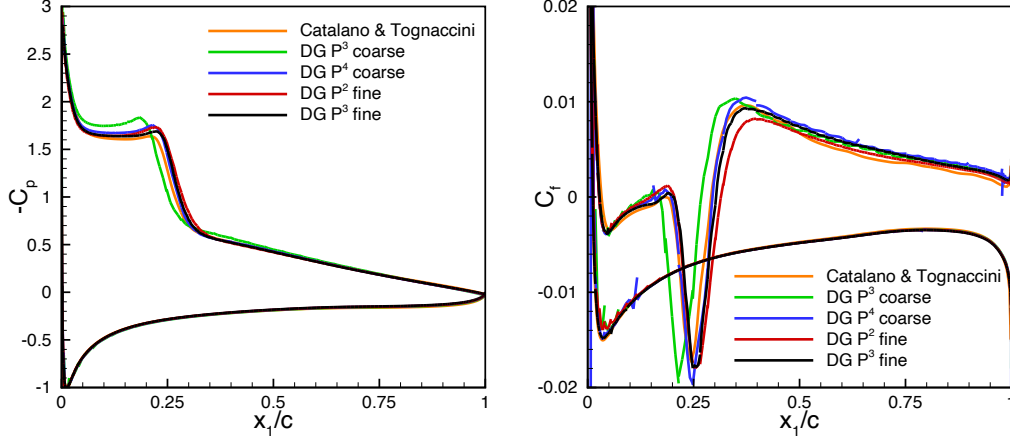


Figure 11: SD7003 - Mean C_p and C_f distributions, coarse mesh RO5-8–DG $\mathbb{P}^{3,4}$ and fine mesh RO5-8–DG $\mathbb{P}^{2,3}$ solutions

The same beneficial effect of a higher-degree polynomial approximation on the solution is observed for the instantaneous $Q = 500$ isosurface of Q -criterion displayed in Figure 13, and for the contours of spanwise-averaged Reynolds stress component $\langle u'_1 u'_1 \rangle$ in Figure 12. In fact, the \mathbb{P}^4 solution on the coarse mesh, compared to the \mathbb{P}^3 on the same mesh and also to the \mathbb{P}^2 on the finer grid, is able to solve much better the small vortical structures above the airfoil and to provide a distribution of the mean squared fluctuation of x_1 -velocity closer to the reference result and noticeably less diffused than the \mathbb{P}^2 computation. Figure 14 displays the influence of p - or h - refinement on the energy spectrum for different chordwise locations. The spectra have been computed as $1/n_k \sum_{k=1}^{n_k} ((\text{PSD}(u_{1,k}) + \text{PSD}(u_{2,k}) + \text{PSD}(u_{3,k}))/2)$, where n_k is the number of equally spaced points used for averaging in the spanwise direction, and $u_{i,k}$ is the temporal signal of the i -component of the velocity at the k^{th} spanwise position. The n_k value was set equal to 104 and 2048 for the computations on the coarse and on fine grid, respectively. We remark that for both the meshes the onset of dissipation is pushed towards higher frequencies by the higher-order solutions. Although having far less DOFs the cut-off level of the coarse \mathbb{P}^4 solution is comparable and even better than the \mathbb{P}^2 solution on the fine mesh, thus confirming that, at least for this test case, p -refinement is preferable to h -refinement in terms of global solution accuracy.

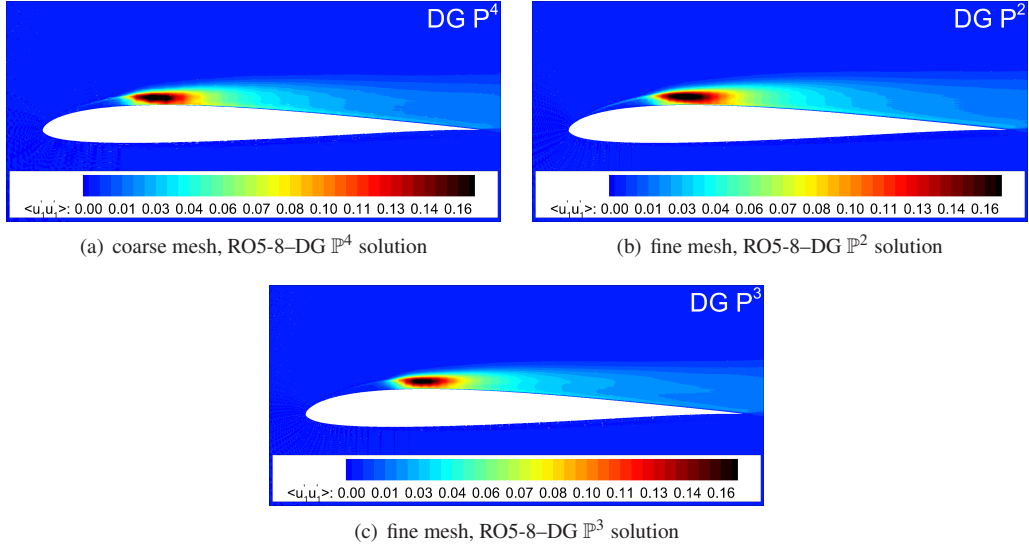


Figure 12: SD7003 - Contours of spanwise-averaged Reynolds stress $\langle u'_1 u'_1 \rangle$

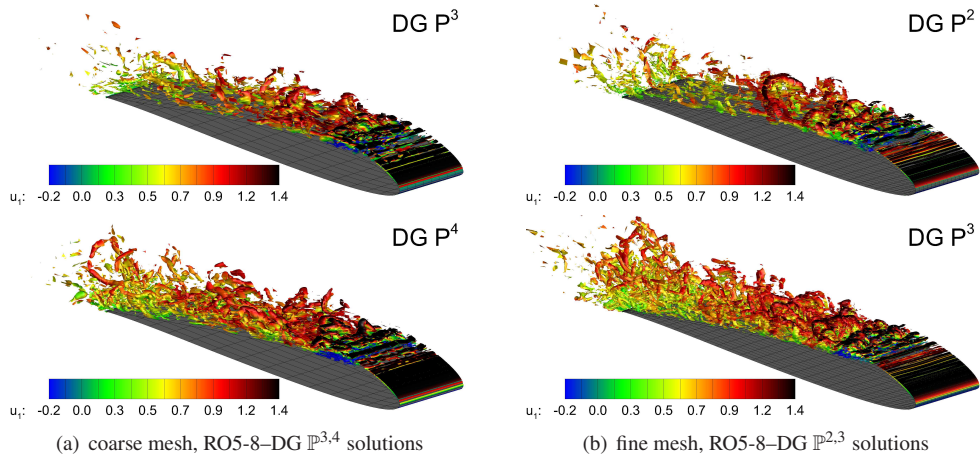


Figure 13: SD7003 - Instantaneous $Q = 500$ isosurface of Q -criterion

Due to the great computational resources required for a thorough analysis, the evaluation of the computational cost related to the higher-order DG space solutions in the DNS and LES context has been left for future work.

The presence of a challenging flow feature such as the laminar separation bubble makes this test case very well suited for assessing the accuracy of different LES approaches but, unfortu-

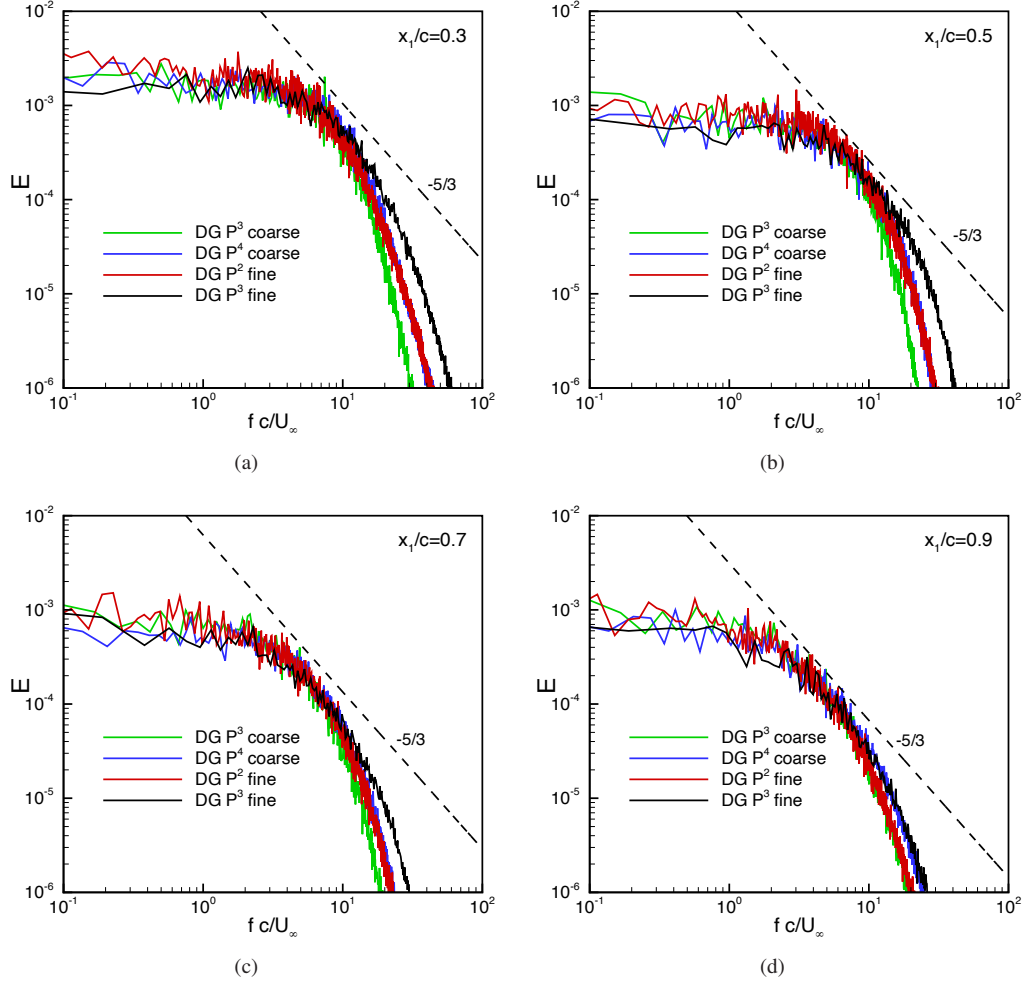


Figure 14: SD7003 - Energy power spectra at chordwise locations $x_1/c = \{0.3, 0.5, 0.7, 0.9\}$

357 nately, no reference DNS database is available and the several published LES results show a
 358 certain dispersion. Nevertheless the results here presented are reasonably in good agreement
 359 with those reported by other authors in [60, 58, 59] and confirm the suitability of high-order DG
 360 methods for this very promising approach to the solution of turbulent flows.

361 5. Conclusions

362 In this paper the potential of our DG solver in the context of the DNS and ILES has been
363 investigated. Thanks to its great flexibility, the proposed implementation allows to handle, within
364 a unified numerical framework, both compressible and incompressible flows. The employed DG
365 method also allows to compute very high-order accurate solutions, both in space and time, on
366 curved, possibly hybrid, computational grids, without being affected by the quality of the mesh
367 elements and the domain complexity.

368 Our numerical computations have been compared with experimental and numerical results
369 available in the literature, showing a good agreement and confirming the great accuracy and the
370 good dissipation and dispersion properties of DG method. This paper also proves the reliabil-
371 ity and robustness of our implementation in dealing with large problems on massively parallel
372 architectures, here up to 4096 cores.

373 Ongoing work is devoted to further investigate DNS and ILES on canonical test cases, *e.g.*
374 channelflow, to assess whether the DNS- and ILES-DG methods can really compete with state-
375 of-the-art high-accuracy solvers and SGS models. The assessment of code scalability on very
376 large problems, ten of thousands cores, and the study of hybrid MPI/OpenMP programming
377 paradigms is left for future work.

378 Acknowledgements

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382 Bergamo”, for the availability of high performance computing and support. We also acknowledge
383 Dr. P. Catalano and Dr. R. Tognaccini for providing us their SD7003 numerical results for
384 comparison purpose.

385 Appendix A. Notes on flow statistics computation

386 Appendix A.1. Dense output for implicit time integration schemes

387 Thanks to their favourable stability properties implicit time integration schemes allow using
 388 large time steps. However, during time-accurate simulations, one can be interested in the numer-
 389 ical solution at some time instant $t^* = (1 - \theta)t^n + \theta t^{n+1}$, $0 < \theta < 1$, between t^n and t^{n+1} .

390 The dense output serves to provide, with a low additional computational cost, an accurate
 391 numerical approximation of the solution for any $0 \leq \theta \leq 1$, see [63, 42, 64] for an extensive
 392 review on this topic. In the following the same notation of Sec. 3.3 is used.

393 In this work we obtain an up to fifth-degree polynomial representation of any component w_h
 394 of the solution vector \mathbf{w}_h (for the sake of compactness we drop the variable index j) within the
 395 time step Δt , by considering a polynomial interpolation of the solution in time

$$w_h(\theta) = \psi_l(\theta) a_l, \quad \forall \theta \in [0, 1], \quad (\text{A.1})$$

396 where, repeated indices imply summation over the range $l = 1, \dots, N_{dof}$ and basis functions ψ_l
 397 can be chosen independently from those used for the space discretization, *e.g.* monomials. The
 398 interpolation can be applied componentwise to \mathbf{W} , thus obtaining the global vector of degrees of
 399 freedom at any intermediate time level θ . The coefficients of the expansion a_l are explicitly cal-
 400 culated by imposing the solution value and its first and second derivatives in time at the time step
 401 extrema. In fact, time derivatives are available, or easily computable, during the time integration
 402 according to

$$\frac{d\mathbf{W}}{dt} = -\widetilde{\mathbf{R}}, \quad (\text{A.2})$$

$$\frac{d^2\mathbf{W}}{dt^2} = -\frac{\partial \widetilde{\mathbf{R}}}{\partial t} = -\frac{\partial \widetilde{\mathbf{R}}}{\partial \mathbf{W}} \frac{d\mathbf{W}}{dt} = \widetilde{\mathbf{J}}\widetilde{\mathbf{R}}. \quad (\text{A.3})$$

404 While the computation of the residual $\widetilde{\mathbf{R}}$ can be considered as part of the cost of the time integra-
 405 tion scheme, this approach to dense output requires the additional evaluation of the matrix-vector
 406 product $\widetilde{\mathbf{J}}\widetilde{\mathbf{R}}$ at each time step. Finally, we remark that this simple interpolation technique can be
 407 applied to any implicit time integration scheme that relies on an exact Jacobian matrix evaluation.

408 *Appendix A.2. Spatial average in the modal DG framework*

409 In this paper we propose a rigorous yet practical approach to the spatial average that can easily
 410 be applied to a DG method that uses modal basis functions. The idea is to solve an auxiliary
 411 linear problem for computing the integral I of an arbitrary function w along a given direction
 412 $\mathbf{a} \in \mathbb{R}^d$. In the following the same notation of Sec. 3.2 is used.

413 Here the function $w \in \mathbb{R}$ is any component of the solution vector \mathbf{w} (for the sake of notation
 414 clarity we drop the variable index j), for which we solve the following equation

$$a_j \frac{\partial I}{\partial x_j} - w = 0, \quad (\text{A.4})$$

415 where $I \in \mathbb{R}$, and repeated indices imply summation over the range $j = 1, \dots, d$. By multiplying
 416 Eq. (A.4) by an arbitrary smooth test function v and integrating by parts, the weak formulation
 417 reads

$$\int_{\partial\Omega} v I (\mathbf{a} \cdot \mathbf{n}) d\sigma - \int_{\Omega} I (\mathbf{a} \cdot \nabla v) d\mathbf{x} - \int_{\Omega} v w d\mathbf{x} = 0. \quad (\text{A.5})$$

418 We substitute the solution and the test function with the finite element approximations I_h and v_h ,
 419 both belonging to the space $V_h \stackrel{\text{def}}{=} \mathbb{P}_d^k(\mathcal{T}_h)$. By using the same set of shape functions of Sec. 3.2
 420 and splitting integrals over mesh elements and faces we obtain

$$\sum_{F \in \mathcal{F}_h} \int_F \llbracket \phi_i \rrbracket_j a_j \hat{I} (I_h^\pm) d\sigma - \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial \phi_i}{\partial x_j} a_j I_h d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \phi_i w_h d\mathbf{x} = 0, \quad (\text{A.6})$$

421 for $i = 1, \dots, N_{dof}^K$, where $\hat{I} (I_h^\pm)$ denotes a simple upwind evaluation of I_h at each integration
 422 point on faces. By expressing I_h in terms of the elements of a global vector \mathbf{I} of unknown degrees
 423 of freedom, such that $I_h = \phi_l I_l$, $l = 1, \dots, N_{dof}^K$, $\forall K \in \mathcal{T}_h$, the DG solution of A.4 consists in
 424 seeking the elements of \mathbf{I} such that

$$\sum_{F \in \mathcal{F}_h} \int_F \llbracket \phi_i \rrbracket_j a_j \hat{I} ((\phi_l \mathbf{I}_l)^\pm) d\sigma - \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial \phi_i}{\partial x_j} a_j \phi_l \mathbf{I}_l d\mathbf{x} - \sum_{K \in \mathcal{T}_h} \int_K \phi_i \phi_l \mathbf{W}_l d\mathbf{x} = 0, \quad (\text{A.7})$$

425 for $i = 1, \dots, N_{dof}^K$, with homogeneous condition imposed at the inflow boundary, and where

repeated indices imply summation over the range $l = 1, \dots, N_{dof}^K$.

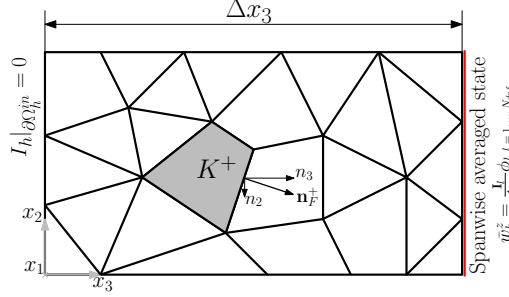


Figure A.15: Spatial average in the spanwise direction, x_3

426

427 In this paper we relied on the present approach for the computation of a spanwise averaged
 428 solution for the SD7003 testcase. According to the meshes reference frames the wing span is
 429 aligned with the x_3 direction, as represented in a simplify sketch in Figure A.15. The averaged
 430 solution \bar{w}_h^z is then computed by solving the problem of Eq. A.7 with $\mathbf{a} = [0, 0, 1]$, by dividing
 431 the resulting array of the degrees of freedom \mathbf{I} by the span, *i.e.* Δx_3 , and finally obtaining the
 432 averaged state through the scaled \mathbf{I} at the end of the wing

$$\bar{w}_h^z(x, y, \Delta x_3) = \frac{\mathbf{I}_l}{\Delta x_3} \phi_l(x, y, \Delta x_3), \quad \forall x, y \in \bar{\Omega}_h, \quad (\text{A.8})$$

433 where, for the sake of clarity, the dependence of basis functions from the coordinates has been
 434 shown.

435 References

- 436 [1] F. Bassi, A. Crivellini, S. Rebay, M. Savini, Discontinuous Galerkin solution of the Reynolds-averaged Navier-
 437 Stokes and $k-\omega$ turbulence model equations, *Comput. Fluids* 34 (2005) 507–540.
- 438 [2] P. O. Persson, N. Nguyen, J. Peraire, RANS solutions using high order discontinuous Galerkin methods, *AIAA*
 439 Paper 2007-914 (2007).
- 440 [3] T. A. Oliver, D. L. Darmofal, An unsteady adaptation algorithm for discontinuous Galerkin discretizations of the
 441 RANS equations, *AIAA Paper* 2007-3940 (2007).
- 442 [4] N. K. Burgess, D. J. Mavriplis, Robust Computation of Turbulent flows using a Discontinuous Galerkin Method,
 443 50th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition AIAA
 444 2012-0457 (2012).

- [5] B. Landman, M. Kessler, S. Wagner, E. Krämer, A parallel, high-order discontinuous Galerkin code for laminar and turbulent flows, *Comput. Fluids* 37 (4) (2008) 427–438.
- [6] R. Hartmann, J. Held, T. Leicht, Adjoint-based error estimation and adaptive mesh refinement for the RANS and k - ω turbulence model equations, *J. Comput. Phys.* 230 (2011) 4268–4284.
- [7] A. Crivellini, V. D’Alessandro, F. Bassi, A Spalart-Allmaras turbulence model implementation in a discontinuous Galerkin solver for incompressible flows, *J. Comput. Phys.* 241 (2013) 388–415. doi:<http://dx.doi.org/10.1016/j.jcp.2012.12.038>.
- [8] S. Schoenawa, R. Hartmann, Discontinuous Galerkin discretization of the Reynolds-averaged Navier-Stokes equations with the shear-stress transport model, *J. Comput. Phys.* 262 (0) (2014) 194 – 216. doi:<http://dx.doi.org/10.1016/j.jcp.2013.12.062>.
- [9] F. Bassi, A. Ghidoni, A. Perbellini, S. Rebay, A. Crivellini, N. Franchina, M. Savini, A high-order Discontinuous Galerkin solver for the incompressible RANS and k - ω turbulence model equations, *Comput. Fluids* 98 (0) (2014) 54–68, 12th USNCCM mini-symposium on High-Order Methods for Computational Fluid Dynamics - A special issue dedicated to the 80th birthday of Professor Antony Jameson. doi:<http://dx.doi.org/10.1016/j.compfluid.2014.02.028>.
- [10] S. S. Collis, Discontinuous Galerkin methods for turbulence simulation, in: *Proceedings of the 2002 Center for Turbulence Research Summer Program 2002*, 2002, pp. 115–167.
- [11] S. S. Collis, The DG/VMS method for unified turbulence simulation, 32nd AIAA Fluid Dynamics Conference and Exhibit AIAA 2002-3124 (2002).
- [12] J.-B. Chapelier, M. de la Llave Plata, F. Renac, E. Lamballais, Evaluation of a high-order discontinuous Galerkin method for the DNS of turbulent flows, *Comput. Fluids* 95 (0) (2014) 210 – 226. doi:<http://dx.doi.org/10.1016/j.compfluid.2014.02.015>.
- [13] A. Uranga, P.-O. Persson, M. Drela, J. Peraire, Implicit Large Eddy Simulation of transition to turbulence at low Reynolds numbers using a Discontinuous Galerkin method, *Int. J. Numer. Methods Eng.* 87 (1-5) (2011) 232–261. doi:10.1002/nme.3036. URL <http://dx.doi.org/10.1002/nme.3036>
- [14] A. Abbà, L. Bonaventura, M. Nini, M. Restelli, Anisotropic dynamic models for Large Eddy Simulation of compressible flows with a high order DG method, Tech. rep., eprint arXiv.org/abs/1407.6591 (2014).
- [15] A. D. Beck, T. Bolemann, D. Flad, H. Frank, G. J. Gassner, F. Hindenlang, C.-D. Munz, High-order discontinuous Galerkin spectral element methods for transitional and turbulent flow simulations, *Int. J. Numer. Methods Fluids* 76 (8) (2014) 522–548. doi:10.1002/fld.3943. URL <http://dx.doi.org/10.1002/fld.3943>
- [16] C. C. de Wiart, K. Hillewaert, L. Bricteux, G. Winckelmans, Implicit LES of free and wall-bounded turbulent flows based on the discontinuous Galerkin/symmetric interior penalty method, *Int. J. Numer. Methods Fluids* 78 (6) (2015) 335–354. doi:10.1002/fld.4021.

- 480 URL <http://dx.doi.org/10.1002/fld.4021>
- 481 [17] G. Gassner, A. Beck, On the accuracy of high-order discretizations for underresolved turbulence simulations, *Theor.*
482 *Comp. Fluid Dyn.* 27 (3-4) (2013) 221–237. doi:10.1007/s00162-011-0253-7.
483 URL <http://dx.doi.org/10.1007/s00162-011-0253-7>
- 484 [18] F. Bassi, A. Crivellini, D. A. Di Pietro, S. Rebay, An artificial compressibility flux for the discontinuous Galerkin
485 solution of the incompressible Navier-Stokes equations, *J. Comput. Phys.* 218 (2006) 794–815.
- 486 [19] F. Bassi, A. Crivellini, D. A. Di Pietro, S. Rebay, An implicit high-order discontinuous Galerkin method for steady
487 and unsteady incompressible flows, *Comput. Fluids* 36 (2007) 1529–1546.
- 488 [20] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, M. Savini, A high-order accurate discontinuous finite element method
489 for inviscid and viscous turbomachinery flows, in: R. Decuyper, G. Dibelius (Eds.), *Proceedings of the 2nd Euro-*
490 *pean Conference on Turbomachinery Fluid Dynamics and Thermodynamics*, Technologisch Instituut, Antwerpen,
491 Belgium, 1997, pp. 99–108.
- 492 [21] A. Nigro, A. Ghidoni, S. Rebay, F. Bassi, Modified Extended BDF scheme for the discontinuous Galerkin solution
493 of unsteady compressible flows, *Int. J. Numer. Methods Fluids* 76 (9) (2014) 549–574. doi:10.1002/fld.3944.
494 URL <http://dx.doi.org/10.1002/fld.3944>
- 495 [22] A. Nigro, C. De Bartolo, F. Bassi, A. Ghidoni, Up to sixth-order accurate A-stable implicit schemes applied to the
496 discontinuous Galerkin discretized Navier-Stokes equations, *J. Comput. Phys.* 276 (0) (2014) 136 – 162.
- 497 [23] F. Bassi, L. Botti, A. Colombo, A. Ghidoni, F. Massa, Linearly implicit Rosenbrock-type Runge–Kutta schemes
498 applied to the Discontinuous Galerkin solution of compressible and incompressible unsteady flows, *Comput. Fluids*
499 118 (2015) 305 – 320. doi:<http://dx.doi.org/10.1016/j.compfluid.2015.06.007>.
- 500 [24] F. Bassi, A. Colombo, C. De Bartolo, N. Franchina, A. Ghidoni, A. Nigro, Investigation of high-order temporal
501 schemes for the discontinuous Galerkin solution of the Navier-Stokes equations, in: *11th World Congress on*
502 *Computational Mechanics, WCCM 2014, 5th European Conference on Computational Mechanics, ECCM 2014*
503 *and 6th European Conference on Computational Fluid Dynamics, ECFD 2014*, 2014, pp. 5651–5662.
- 504 [25] F. Bassi, L. Botti, A. Colombo, A. Crivellini, N. Franchina, A. Ghidoni, S. Rebay, Very high-order accurate discon-
505 tinuous Galerkin computation of transonic turbulent flows on aeronautical configurations, in: N. Kroll, H. Bieler,
506 H. Deconinck, V. Couaillier, H. van der Ven, K. Sørensen (Eds.), *ADIGMA - A European Initiative on the Develop-*
507 *ment of Adaptive Higher-Order Variational Methods for Aerospace Applications*, Vol. 113 of *Notes on Numerical*
508 *Fluid Mechanics and Multidisciplinary Design*, Springer Berlin Heidelberg, 2010, pp. 25–38.
- 509 [26] F. Bassi, S. Rebay, A high order discontinuous Galerkin method for compressible turbulent flows, in: *Discontinuous*
510 *Galerkin Methods. Theory, Computation and Applications*, Vol. 11 of *Lecture Notes in Computational Science and*
511 *Engineering*, Springer-Verlag, 2000, pp. 77–88, *First International Symposium on Discontinuous Galerkin Methods*,
512 May 24–26, 1999, Newport, RI, USA.
- 513 [27] V. Dolejší, Semi-implicit interior penalty discontinuous Galerkin methods for viscous compressible flows, *Com-*
514 *munic. Comput. Phys.* 4 (2008) 231–274.

- [28] G. J. Gassner, F. Lörcher, C.-D. Munz, A discontinuous Galerkin scheme based on a space-time expansion II. Viscous flow equations in multi dimensions, *J. Sci. Comput.* 34 (2008) 260–286.
- [29] L. Botti, Influence of reference-to-physical frame mappings on approximation properties of discontinuous piece-wise polynomial spaces, *J. Sci. Comput.* 52 (3) (2012) 675–703. doi:10.1007/s10915-011-9566-3.
- [30] P. Tesini, An h -Multigrid Approach for High-Order Discontinuous Galerkin Methods, Ph.D. thesis, University of Bergamo, Bergamo, Italy (January 2008).
- [31] F. Bassi, L. Botti, A. Colombo, D. A. Di Pietro, P. Tesini, On the flexibility of agglomeration based physical space discontinuous Galerkin discretizations, *J. Comput. Phys.* 231 (1) (2012) 45 – 65. doi:10.1016/j.jcp.2011.08.018.
- [32] D. A. Di Pietro, A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods, Vol. 69 of *Mathématiques et Applications*, Springer-Verlag, 2012.
- [33] Y.-H. Choi, C. L. Merkle, The application of preconditioning in viscous flows, *J. Comput. Phys.* 105 (2) (1993) 207 – 223. doi:http://dx.doi.org/10.1006/jcph.1993.1069.
- [34] J. M. Weiss, W. A. Smith, Preconditioning applied to variable and constant density flows, *AIAA J.* 33 (11) (1995) 2050–2057.
- [35] F. Bassi, C. De Bartolo, R. Hartmann, A. Nigro, A discontinuous Galerkin method for inviscid low Mach number flows, *J. Comput. Phys.* 228 (11) (2009) 3996 – 4011. doi:http://dx.doi.org/10.1016/j.jcp.2009.02.021.
- [36] G. Hauke, T. J. R. Hughes, A unified approach to compressible and incompressible flows, *Comput. Methods Appl. Mech. Engrg.* 113 (3–4) (1994) 389 – 395. doi:http://dx.doi.org/10.1016/0045-7825(94)90055-8.
- [37] F. Bassi, L. Botti, A. Colombo, A. Crivellini, A. Ghidoni, A. Nigro, S. Rebay, Time Integration in the Discontinuous Galerkin Code MIGALE - Unsteady Problems, in: N. Kroll, C. Hirsch, F. Bassi, C. Johnston, K. Hillewaert (Eds.), *IDIHOM: Industrialization of High-Order Methods - A Top-Down Approach*, Vol. 128 of *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, Springer International Publishing, 2015, pp. 205–230. doi:10.1007/978-3-319-12886-3_11.
URL http://dx.doi.org/10.1007/978-3-319-12886-3_11
- [38] F. Brezzi, G. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous Galerkin approximations for elliptic problems, *Numer. Meth. Part. D. E.* 16 (2000) 365–378.
- [39] D. N. Arnold, F. Brezzi, B. Cockburn, L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.* 39 (5) (2002) 1749–1779.
- [40] J. J. Gottlieb, C. P. T. Groth, Assessment of Riemann solvers for unsteady one-dimensional inviscid flows of perfect gases, *J. Comput. Phys.* 78 (1988) 437–458.
- [41] D. Hänel, R. Schwane, G. Seider, On the accuracy of upwind schemes for the solution of the Navier–Stokes equations, *AIAA Paper* 87-1105 (1987).
- [42] E. Hairer, G. Wanner, Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems (2nd revised edition), Vol. 14 of *Springer Series in Computational Mathematics*, Springer Berlin Heidelberg, 1996.
- [43] S. Balay, M. F. Adams, J. Brown, P. Brune, K. Buschelman, V. Eijkhout, W. D. Gropp, K. D., M. G. Knepley, L. C.

McInnes, F. Barry, K. R. Smith, H. Zhang, PETSc Web page, <http://www.mcs.anl.gov/petsc> (2014).

URL <http://www.mcs.anl.gov/petsc>

[44] A. Crivellini, F. Bassi, An implicit matrix-free discontinuous Galerkin solver for viscous and turbulent aerodynamic simulations, *Comput. Fluids* 50 (1) (2011) 81–93.

[45] G. S. Iannelli, A. J. Baker, A stiffly-stable implicit Runge-Kutta algorithm for CFD applications, *AIAA Paper* 88-0416 (1988).

[46] J. Lang, J. Verwer, ROS3P—An accurate third-order Rosenbrock solver designed for parabolic problems, *BIT* 41 (4) (2001) 731–738.

[47] G. Di Marzo, RODAS5(4) - Méthodes de Rosenbrock d'ordre 5(4) adaptées aux problèmes différentiels-algébriques, *MSc Mathematics Thesis*; Faculty of Science, University of Geneva, Switzerland.

[48] A. Crivellini, V. D'Alessandro, F. Bassi, Assessment of a high-order discontinuous Galerkin method for incompressible three-dimensional Navier-Stokes equations: Benchmark results for the flow past a sphere up to $Re=500$, *Comput. Fluids* 86 (0) (2013) 442 – 458. doi:<http://dx.doi.org/10.1016/j.compfluid.2013.07.027>.
URL <http://www.sciencedirect.com/science/article/pii/S004579301300306X>

[49] L. Wei, A. Pollard, Direct numerical simulation of compressible turbulent channel flows using the discontinuous Galerkin method, *Comput. Fluids* 47 (1) (2011) 85 – 100. doi:<http://dx.doi.org/10.1016/j.compfluid.2011.02.015>.
URL <http://www.sciencedirect.com/science/article/pii/S0045793011000740>

[50] C. Carton de Wiart, K. Hillewaert, M. Duponcheel, G. Winckelmans, Assessment of a discontinuous Galerkin method for the simulation of vortical flows at high Reynolds number, *Int. J. Numer. Methods Fluids* 74 (7) (2014) 469–493. doi:10.1002/fld.3859.

[51] A. Tomboulides, S. Orszag, Numerical investigations of the transitional and weak turbulent flow past a sphere, *J. of Fluid Mech.* 416 (200) 45–73.

[52] J. Wu, G. Faeth, Sphere wakes in still surroundings at intermediate Reynolds numbers, *AIAA J.* 31 (1993) 1448.

[53] F. Hindenlang, G. J. Gassner, C. Altmann, A. Beck, M. Staudenmaier, C.-D. Munz, Explicit discontinuous Galerkin methods for unsteady problems, *Comput. Fluids* 61 (0) (2012) 86 – 93.

[54] P. Ploumhans, G. Winckelmans, J. Salmon, A. Leonard, M. Warren, Vortex Methods for Direct Numerical Simulation of Three-Dimensional Bluff Body Flows: Applications to the sphere at $Re=300$, 500 and 1000, *J. of Comp. Phys.* 178 (2002) 427–463.

[55] E. K. W. Poon, G. Iaccarino, A. S. H. Ooi, M. Giacobello, Numerical studies of high Reynolds number flow past a stationary and rotating sphere, in: *Seventh International Conference on CFD in the Minerals and Process Industries*, CSIRO, Melbourne, Australia, 2009.

[56] H. Sakamoto, H. Haniu, A study on vortex shedding from spheres in a uniform flow, *J. Fluid Eng.* 12 (1990) 386–392.

[57] Z. J. Wang, K. Fidkowski, R. Abgrall, F. Bassi, D. Caraeni, A. Cary, H. Deconinck, R. Hartmann, K. Hillewaert, H. T. Huynh, N. Kroll, G. May, P.-O. Persson, B. van Leer, M. Visbal, High-order CFD methods: Current status

- and perspective, *Int. J. Numer. Methods Fluids* 72 (8) (2013) 811–845.
- [58] D. J. Garmann, M. R. Visbal, C3.3: Implicit large eddy-simulations of transitional flow over the SD7003 airfoil using compact finite-differencing and filtering, in: 2nd International Workshop on High-Order CFD Methods, Cologne, Germany, 2013, http://www.dlr.de/as/Portaldata/5/Resources/dokumente/veranstaltungen/2013_05-hiocfd/contrib/C33_Garmann.pdf.
- [59] T. Bolemann, A. Beck, D. Flad, H. Frank, V. Mayer, C.-D. Munz, High-Order Discontinuous Galerkin Schemes for Large-Eddy Simulations of Moderate Reynolds Number Flows, in: N. Kroll, C. Hirsch, F. Bassi, C. Johnston, K. Hillewaert (Eds.), *IDIHOM: Industrialization of High-Order Methods - A Top-Down Approach*, Vol. 128 of Notes on Numerical Fluid Mechanics and Multidisciplinary Design, Springer International Publishing, 2015, pp. 435–456. doi:10.1007/978-3-319-12886-3_20.
URL http://dx.doi.org/10.1007/978-3-319-12886-3_20
- [60] P. Catalano, R. Tognaccini, RANS analysis of the low-Reynolds number flow around the SD7003 airfoil, *Aerospace Science and Technology* 15 (8) (2011) 615 – 626.
- [61] P. D. Boom, D. W. Zingg, Time-accurate flow simulations using an efficient Newton-Krylov-Schur approach with high-order temporal and spatial discretization, *AIAA Paper* 2012-0383 (2013).
- [62] IDIHOM, Industrialisation of High-Order Methods – A top-down approach, Specific Targeted Research Project supported by European Commission, URL: http://www.dlr.de/as/en/desktopdefault.aspx/tabid-7027/11654_read-27492/.
- [63] E. Hairer, S. P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I, Nonstiff Problems* (2nd revised edition), Vol. 8 of Springer Series in Computational Mathematics, Springer Berlin Heidelberg, 1993.
- [64] J. Dormand, *Numerical Methods for Differential Equations: A Computational Approach*, Engineering Mathematics, Taylor & Francis, 1996.